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Characterizations of Γ-Semirings by Their Cubic Ideals

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Abstract—Cubic ideals, cubic bi-ideals and cubic quasi-ideals of a Γ-semiring are introduced and various properties of these ideals are investigated. Among all other results, some characterizations of regular Γ-semirings are achieved.

Keywords—Γ-semiring, cubic ideal, normal cubic ideal, cubic bi-ideal, cubic quasi-ideal, cartesian product, regular, intra-regular.

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I. Introduction

SEMIRINGS which is a common generalization of rings and distributive lattices, was introduced by Vandiver [8]. It has been found very useful for solving problems in different areas of pure and applied mathematics, information sciences, etc., since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. Ideals of semiring play a central role in the structure theory and useful for many purposes. The theory of Γ-semirings was introduced by [6]. Since then many researchers enriched this field.

The theory of fuzzy sets was proposed by Zadeh [9] and used as a mathematical tool for describing the behavior of the systems that are too complex or illdefined to admit precise mathematical analysis by classical methods and tools. The study of fuzzy algebraic structure was started by Rosenfeld [7]. The learning of cubic sets and cubic subgroups were initiated by Jun et al. [3], [4]. Khan et al. [5] applied this in case of cubic *h*-ideals of hemirings. Chinnadurai et al. [1], [2] used this notion to study cubic bi-ideals and cubic lateral ideals in near-ring and ternary near-ring respectively. As an extension of these results, in this paper we have presented cubic ideal in Γ-semiring and studied various properties of this ideal. Also, we have defined cubic bi-ideals and cubic quasi-ideals of Γ-semiring and used these to obtain some characterizations of regular and intra-regular Γ-semiring.

II. Preliminaries

We recall the following preliminaries for subsequent use.

Definition 1. Let S and Γ be two additive commutative semigroups with zero. Then S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \to S$ ((a,α,b) ↦ aαb) satisfying the following conditions:
(i) $(a+b)\alpha c = a\alpha c + b\alpha c$,
(ii) $a\alpha(b+c) = a\alpha b + a\alpha c$,
(iii) $a(\alpha+\beta)b = a\alpha b + a\beta b$,
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.
(v) $0_S\alpha a = 0_S = a\alpha 0_S$,
(vi) $a0_\Gamma b = 0_S = b0_\Gamma a$
for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$. For simplification we write 0 instead of 0_S and 0_Γ.

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Example 1. Let S be the set of all m×n matrices over $\mathbf{Z_0^-}$ (the set of all non-positive integers) and Γ be the set of all n×m matrices over $\mathbf{Z_0^-}$, then S forms a Γ-semiring with usual addition and multiplication of matrices.

A subset A of a Γ-semiring S is called a left(resp. right) ideal of S if A is closed under addition and $S\Gamma A \subseteq A$ (resp.$A\Gamma S \subseteq A$). A subset A of a Γ-semiring S is called an ideal if it is both left and right ideal of S. A subset A of a Γ-semiring S is called a quasi-ideal of S if A is closed under addition and $S\Gamma A \cap A\Gamma S \subseteq A$. A subset A of a Γ-semiring S is called a bi-ideal if A is closed under addition and $A\Gamma S\Gamma A \subseteq A$.

Definition 2. A fuzzy subset of a nonempty set X is defined as a function μ : X → [0,1].

Definition 3. Let X be a non-empty set. A cubic set A in X is a structure $A = \{< x, \widetilde{\mu}, f >: x \in X\}$ which briefly denoted as $A =< \widetilde{\mu}, f >$ where $\widetilde{\mu} = [\mu^-, \mu^+]$ is an interval valued fuzzy set (briefly, IVF) in X and f is a fuzzy set in X.

Definition 4. For any non-empty set G of a set X, the characteristic cubic set of G is defined to be the structure $\chi_G(x) =< x, \widetilde{\zeta}_{\chi_G}(x), \eta_{\chi_G}(x) : x \in X >$ where

$$\begin{aligned} \widetilde{\zeta}_{\chi_G}(x) &= [1,1] \approx \widetilde{1}\ if\ x \in G \\ &= [0,0] \approx \widetilde{0}\ otherwise. \end{aligned}$$

and

$$\begin{aligned} \eta_{\chi_G}(x) &= 0\ if\ x \in G \\ &= 1\ otherwise. \end{aligned}$$

Throughout this paper, unless otherwise mentioned S denotes the Γ-semiring and for any two set P and Q, we use the following notation:

$$\cap\{P, Q\} = P \cap Q \text{ and } \cup\{P, Q\} = P \cup Q.$$

III. Basic Definitions and Results of Cubic Ideals

In this section, the notions of cubic ideals in Γ-semiring are introduced and some of their basic properties are investigated.

Definition 5. Let $< \widetilde{\mu}, f >$ be a non empty cubic subset of a Γ-semiring S. Then $< \widetilde{\mu}, f >$ is called a cubic left ideal [cubic right ideal] of S if

(i) $\widetilde{\mu}(x+y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(x+y) \leq \max\{f(x), f(y)\}$ and
(ii) $\widetilde{\mu}(x\gamma y) \supseteq \widetilde{\mu}(y), f(x\gamma y) \leq f(y)$ [respectively $\widetilde{\mu}(x\gamma y) \supseteq \widetilde{\mu}(x), f(x\gamma y) \leq f(x)$].

for all $x, y \in S$, $\gamma \in \Gamma$.

A cubic ideal of a Γ-semiring S is a non empty cubic subset of S which is a cubic left ideal as well as a cubic right ideal of S. Note that if $<\widetilde{\mu}, f>$ is a cubic left or right ideal of a Γ-semiring S, then $\widetilde{\mu}(0) \supseteq \widetilde{\mu}(x)$ and $f(0) \leq f(x)$ for all $x \in S$.

A cubic right ideal is defined similarly. By a cubic ideal $<\widetilde{\mu}, f>$, we mean that $<\widetilde{\mu}, f>$ is both cubic left and cubic right ideal.

Example 2. Consider S be the additive commutative semigroup of all non positive integers and Γ be the additive commutative semigroup of all non positive even integers. Then S is a Γ-semiring if $a\gamma b$ denotes the usual multiplication of integers a, γ, b where $a, b \in S$ and $\gamma \in \Gamma$. Let $<\widetilde{\mu}, f>$ be a cubic subset of S, defined as follows

$$\begin{aligned} \widetilde{\mu}(x) &= [1,1] \quad \text{if } x = 0 \\ &= [0.6, 0.7] \quad \text{if x is even} \\ &= [0.1, 0.2] \quad \text{if x is odd} \end{aligned}$$

and

$$\begin{aligned} f(x) &= 0 \quad \text{if } x = 0 \\ & 0.4 \quad \text{if x is even} \\ & 0.9 \quad \text{if x is odd} \end{aligned}$$

The cubic subset $<\widetilde{\mu}, f>$ of S is a cubic ideal S.

Throughout this section, we prove results only for cubic left ideals. Similar results can be obtained for cubic right ideals and cubic ideals.

Theorem 1. A cubic set $C =<\widetilde{\mu}, f>$ of S is a cubic left ideal of S if and only if any level subset $C_t =<\widetilde{\mu}_t, f_t>:= \{x \in S : \widetilde{\mu}(x) \supseteq [t,t] \text{ and } f(x) \leq t, t \in [0,1]\}$ is a left ideal of S, provided it is non-empty.

Proof: Let $<\widetilde{\mu}, f>$ be a cubic left ideal of S and assume that $<\widetilde{\mu}_t, f_t> \neq \phi$ for $t \in$ [0,1]. Let $x, z \in S$ and $a, b \in <\widetilde{\mu}_t, f_t>$. Then $\widetilde{\mu}(a+b) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\} \supseteq [t,t]$ and $f(a+b) \leq \max\{f(a), f(b)\} \leq t$; implies that $a+b \in <\widetilde{\mu}_t, f_t>$. Also, in addition for $\gamma \in \Gamma$, $\widetilde{\mu}(x\gamma a) \supseteq \widetilde{\mu}(a) \supseteq [t,t]$ and $f(x\gamma a) \leq f(a) \leq t$ which implies $x\gamma a \in <\widetilde{\mu}_t, f_t>$. So, $<\widetilde{\mu}_t, f_t>$ is a left ideal of S. Conversely, suppose $<\widetilde{\mu}_t, f_t>$ is a left ideal of S. If possible, suppose $<\widetilde{\mu}, f>$ is not a cubic left ideal of S. Then there exist $a, b \in S$ such that $\widetilde{\mu}(a+b) \subset \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\}$ or $f(a+b) > \max\{f(a), f(b)\}$. Taking $t_0 = \frac{1}{2}[f(a+b) + \max\{f(a), f(b)\}]$, we see that $t_0 \in$ [0,1] and $f(a+b) > t_0 > \max\{f(a), f(b)\}$ whence $a, b \in f_{t_0}$ but $a+b \notin f_{t_0}$ - which is a contradiction. Therefore, $\widetilde{\mu}(a+b) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\}$ and $f(a+b) \leq \max\{f(a), f(b)\}$for all $a, b \in S$. The other property can be proved similarly. Consequently, $<\widetilde{\mu}, f>$ is a cubic left ideal of S. ■

Theorem 2. Let A be a non-empty subset of a Γ-semiring S. Then A is a left ideal of S if and only if the characteristic function $\chi_A =<\widetilde{\mu}_{\chi_A}, f_{\chi_A}>$ is a cubic left ideal of S.

Proof: Assume that A is a left ideal of S and $x, y \in S$ and $\gamma \in \Gamma$. Suppose $\widetilde{\mu}_{\chi_A}(x+y) \subset \cap\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(y)\}$ and $f_{\chi_A}(x+y) > \max\{f_{\chi_A}(x), f_{\chi_A}(y)\}$. It follows that $\widetilde{\mu}_{\chi_A}(x+y) = \widetilde{0}, \cap\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(y)\} = \widetilde{1}$ and $f_{\chi_A}(x+y) = 1$, $\max\{f_{\chi_A}(x), f_{\chi_A}(y)\} = 0$. This imply that $x, y \in A$ but $x+y \notin A$ $-$ a contradiction. So, $\widetilde{\mu}_{\chi_A}(x+y) \supseteq \cap\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(y)\}$ and $f_{\chi_A}(x+y) \leq \max\{f_{\chi_A}(x), f_{\chi_A}(y)\}$. Similarly we can show that $\widetilde{\mu}(x\gamma y) \supseteq \widetilde{\mu}(y)$, $f(x\gamma y) \leq f(y)$. Therefore $\chi_A =<\widetilde{\mu}_{\chi_A}, f_{\chi_A}>$ is a cubic left ideal of S. Conversely, assume that $\chi_A =<\widetilde{\mu}_{\chi_A}, f_{\chi_A}>$ is a cubic left ideal of S for any subset A of S. Let $x, y \in A$, $a, b \in S$ and $\gamma \in \Gamma$. Then $\widetilde{\mu}_{\chi_A}(x) = \widetilde{\mu}_{\chi_A}(y) = \widetilde{1}$ and $f_{\chi_A}(x) = f_{\chi_A}(y) = 0$. Now $\widetilde{\mu}(x+y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\} = \widetilde{1}$, $f(x+y) \leq \max\{f(x), f(y)\} = 0$ and $\widetilde{\mu}(x\gamma y) \supseteq \widetilde{\mu}(y) = \widetilde{1}$, $f(x\gamma y) \leq f(y) = 0$. This implies $x+y, x\gamma y \in A$. Hence A is a left ideal of S. ■

Definition 6. Let $A =<\widetilde{\mu}, f>$ and $B =<\widetilde{\theta}, g>$ be two cubic sets of a Γ-semiring S. Define intersection of A and B by

$$A \cap B =<\widetilde{\mu}, f> \cap <\widetilde{\theta}, g>=<\widetilde{\mu} \cap \widetilde{\theta}, f \cup g> .$$

Proposition 1. Intersection of a non-empty collection of cubic left ideals is a cubic left ideal of S.

Proof: Let $A_i = \{<\widetilde{\mu}_i, f_i>: i \in I\}$ be a non-empty family of ideals of S. Let $a, b, x, y, z \in S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} (\underset{i\in I}{\cap}\widetilde{\mu}_i)(x+y) &= \underset{i\in I}{\cap}\{\mu_i(x+y)\} \supseteq \underset{i\in I}{\cap}\{\cap\{\widetilde{\mu}_i(x), \widetilde{\mu}_i(y)\}\} \\ &= \cap\{\underset{i\in I}{\cap}\widetilde{\mu}_i(x), \underset{i\in I}{\cap}\widetilde{\mu}_i(y)\} \\ &= \cap\{(\underset{i\in I}{\cap}\widetilde{\mu}_i)(x), (\underset{i\in I}{\cap}\widetilde{\mu}_i)(y)\}. \end{aligned}$$

$$\begin{aligned} (\underset{i\in I}{\cup}f_i)(x+y) &= \underset{i\in I}{\sup}\{f(x+y)\} \leq \underset{i\in I}{\sup}\{\max\{f_i(x), f_i(y)\}\} \\ &= \max\{\underset{i\in I}{\sup}f_i(x), \underset{i\in I}{\sup}f_i(y)\} \\ &= \max\{(\underset{i\in I}{\cup}f_i)(x), (\underset{i\in I}{\cup}f_i)(y)\}. \end{aligned}$$

Again

$$(\underset{i\in I}{\cap}\widetilde{\mu}_i)(x\gamma y) = \underset{i\in I}{\cap}\{\widetilde{\mu}_i(x\gamma y)\} \supseteq \underset{i\in I}{\cap}\{\widetilde{\mu}_i(y)\} = (\underset{i\in I}{\cap}\widetilde{\mu}_i)(y).$$

$$(\underset{i\in I}{\cup}f_i)(x\gamma y) = \underset{i\in I}{\sup}\{f_i(x\gamma y)\} \leq \underset{i\in I}{\sup}\{f_i(y)\} = (\underset{i\in I}{\cap}f_i)(y).$$

$$\begin{aligned} (\underset{i\in I}{\cup}f_i)(x) &= \underset{x\in I}{\sup}\{f_i(x)\} \leq \underset{i\in I}{\sup}\{\max\{f_i(a), f_i(b)\}\} \\ &= \max\{\underset{i\in I}{\sup}f_i(a), \underset{i\in I}{\sup}f_i(b)\} \\ &= \max\{(\underset{i\in I}{\cup}f_i)(a), (\underset{i\in I}{\cup}f_i)(b)\}. \end{aligned}$$

Hence, $\underset{i\in I}{\cap}A_i = \{<\underset{i\in I}{\cap}\widetilde{\mu}_i, \underset{i\in I}{\cup}f_i>: i \in I\}$ is a cubic left ideal of S. ■

Proposition 2. Let $f : R \rightarrow S$ be a morphism of $\Gamma-$hemirings and $A =<\widetilde{\phi}, g>$ be a cubic left ideal of S, then $f^{-1}(A)$ is a cubic left ideal of R where $f^{-1}(A)(x) =< f^{-1}(\widetilde{\phi})(x), f^{-1}(g)(x) >=<\widetilde{\phi}(f(x)), g(f(x))>$

Proof: Let $f : R \rightarrow S$ be a morphism of Γ-semirings. Let $A =<\widetilde{\phi}, g>$ be a cubic left ideal of S and $r, s \in R$, $\gamma \in \Gamma$. Then

$$\begin{aligned} f^{-1}(\widetilde{\phi})(r+s) &= \widetilde{\phi}(f(r+s)) = \widetilde{\phi}(f(r) + f(s)) \\ &\supseteq \cap\{\widetilde{\phi}(f(r)), \widetilde{\phi}(f(s))\} \\ &= \cap\{(f^{-1}(\widetilde{\phi}))(r), (f^{-1}(\widetilde{\phi}))(s)\} \end{aligned}$$

$$\begin{aligned} f^{-1}(g)(r+s) &= g(f(r+s)) = g(f(r)+f(s)) \\ &\leq \max\{g(f(r)), g(f(s))\} \\ &= \max\{(f^{-1}(g))(r), (f^{-1}(g))(s)\} \end{aligned}$$

Again $(f^{-1}(\widetilde{\phi}))(r\gamma s) = \widetilde{\phi}(f(r\gamma s)) = \widetilde{\phi}(f(r)\gamma f(s)) \supseteq \widetilde{\phi}(f(s)) = (f^{-1}(\widetilde{\phi}))(s)$.
$(f^{-1}(g))(r\gamma s) = g(f(r\gamma s)) = g(f(r)\gamma f(s)) \leq g(f(s)) = (f^{-1}(g))(s)$.
Thus $< f^{-1}(\widetilde{\phi})(x), f^{-1}(g)(x) >$ is a cubic left ideal of R. ■

Definition 7. A cubic left ideal $< \widetilde{\mu}, f >$ of a Γ-semiring S, is said to be normal cubic left ideal if $\widetilde{\mu}(0) = \widetilde{1},\ f(0) = 0$.
Proposition 3. Given a cubic left ideal $< \widetilde{\mu}, f >$ of a Γ-semiring S, let $< \widetilde{\mu}_+, f_+ >$ be a cubic set in S obtained by $\widetilde{\mu}_+(x) = \widetilde{\mu}(x) + \widetilde{1} - \widetilde{\mu}(0)$, $f_+(x) = f(x) - f(0)$ for all $x \in S$. Then $< \widetilde{\mu}_+, f_+ >$ is a normal cubic left ideal of S.

Proof: For all $x, y \in S$ and $\gamma \in \Gamma$, we have $\widetilde{\mu}_+(0) = \widetilde{\mu}(0) + \widetilde{1} - \widetilde{\mu}(0) = \widetilde{1}$, $f_+(0) = f(0) - f(0) = 0$ Now,

$$\begin{aligned} \widetilde{\mu}_+(x+y) &= \widetilde{\mu}(x+y) + \widetilde{1} - \widetilde{\mu}(0) \\ &\supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\} + \widetilde{1} - \widetilde{\mu}(0) \\ &= \cap\{\{\widetilde{\mu}(x) + \widetilde{1} - \widetilde{\mu}(0)\}, \{\widetilde{\mu}(y) + \widetilde{1} - \widetilde{\mu}(0)\}\} \\ &= \cap\{\widetilde{\mu}_+(x), \widetilde{\mu}_+(y)\} \end{aligned}$$

$$\begin{aligned} f_+(x+y) &= f(x+y) - f(0) \\ &\leq \max\{f(x), f(y)\} - f(0) \\ &= \max\{\{f(x) - f(0)\}, \{f(y) - f(0)\}\} \\ &= \max\{f_+(x), f_+(y)\} \end{aligned}$$

and

$$\widetilde{\mu}_+(x\gamma y) = \widetilde{\mu}(x\gamma y) + \widetilde{1} - \widetilde{\mu}(0) \supseteq \widetilde{\mu}(y) + \widetilde{1} - \widetilde{\mu}(0) = \widetilde{\mu}_+(y).$$

$$f_+(x\gamma y) = f(x\gamma y) - f(0) \leq f(y) - f(0) = f_+(y).$$

Hence, $< \widetilde{\mu}_+, f_+ >$ is a normal cubic left ideal of S. ■

Definition 8. Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\nu}, g >$ be cubic subsets of X. The cartesian product of A and B is defined by $(A \times B)(x, y) = (< \widetilde{\mu}, f > \times < \widetilde{\nu}, g >)(x, y) = (< \widetilde{\mu} \times \widetilde{\nu}, f \times g >)(x, y) = [\cap\{\widetilde{\mu}(x), \widetilde{\nu}(y)\}, \max\{f(x), g(y)\}]$ for all $x, y \in X$.
Theorem 3. Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\nu}, g >$ be cubic left ideals of a Γ-semiring S. Then $A \times B$ is a cubic left ideal of the Γ-semiring $S \times S$.

Proof: Let $(x_1, x_2), (y_1, y_2) \in S \times S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} &(\widetilde{\mu} \times \widetilde{\nu})((x_1, x_2) + (y_1, y_2)) \\ &= (\widetilde{\mu} \times \widetilde{\nu})(x_1 + y_1, x_2 + y_2) \\ &= \cap\{\widetilde{\mu}(x_1 + y_1), \widetilde{\nu}(x_2 + y_2)\} \\ &\supseteq \cap\{\cap\{\widetilde{\mu}(x_1), \widetilde{\mu}(y_1)\}, \cap\{\widetilde{\nu}(x_2), \widetilde{\nu}(y_2)\}\} \\ &= \cap\{\cap\{\widetilde{\mu}(x_1), \widetilde{\nu}(x_2)\}, \cap\{\widetilde{\mu}(y_1), \widetilde{\nu}(y_2)\}\} \\ &= \cap\{(\widetilde{\mu} \times \widetilde{\nu})(x_1, x_2), (\widetilde{\mu} \times \widetilde{\nu})(y_1, y_2)\} \end{aligned}$$

$$\begin{aligned} &(f \times g)((x_1, x_2) + (y_1, y_2)) \\ &= (f \times g)(x_1 + y_1, x_2 + y_2) \\ &= \max\{f(x_1 + y_1), g(x_2 + y_2)\} \\ &\leq \max\{\max\{f(x_1), f(y_1)\}, \max\{g(x_2), g(y_2)\}\} \\ &= \max\{\max\{f(x_1), g(x_2)\}, \max\{f(y_1), f(y_2)\}\} \\ &= \max\{(f \times g)(x_1, x_2), (f \times g)(y_1, y_2)\} \end{aligned}$$

and

$$\begin{aligned} &(\widetilde{\mu} \times \widetilde{\nu})((x_1, x_2)\gamma(y_1, y_2)) \\ &= (\widetilde{\mu} \times \widetilde{\nu})(x_1\gamma y_1, x_2\gamma y_2) \\ &= \cap\{\widetilde{\mu}(x_1\gamma y_1), \widetilde{\nu}(x_2\gamma y_2)\} \\ &\supseteq \cap\{\widetilde{\mu}(y_1), \widetilde{\nu}(y_2)\} \\ &= (\widetilde{\mu} \times \widetilde{\nu})(y_1, y_2). \end{aligned}$$

$$\begin{aligned} &(f \times g)((x_1, x_2)\gamma(y_1, y_2)) \\ &= (f \times g)(x_1\gamma y_1, x_2\gamma y_2) \\ &= \max\{f(x_1\gamma y_1), g(x_2\gamma y_2)\} \\ &\leq \max\{f(y_1), g(y_2)\} \\ &= (f \times g)(y_1, y_2). \end{aligned}$$

Hence, $A \times B$ is a cubic left ideal of $S \times S$. ■

IV. Cubic Bi-Ideals and Cubic Quasi-Ideals

Definition 9. Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\theta}, g >$ be two cubic sets of a Γ-semiring S. Define composition of A and B by

$$A\Gamma_c B =< \widetilde{\mu}, f > \Gamma_c < \widetilde{\theta}, g >=< \widetilde{\mu}\Gamma_c\widetilde{\theta}, f\Gamma_c g >$$

where

$$\begin{aligned} \widetilde{\mu}\Gamma_c\widetilde{\theta}(x) &= \underset{x=a_1\gamma b_1}{\cup}[\cap\{\widetilde{\mu}(a_1), \widetilde{\theta}(b_1)\}] \\ &= \widetilde{0}, \text{if x cannot be expressed as above} \end{aligned}$$

and

$$\begin{aligned} f\Gamma_c g(x) &= \inf_{x=a_1\gamma b_1}\{\max\{f(a_1), g(b_1)\}\} \\ &= 1, \text{if x cannot be expressed as above} \end{aligned}$$

for $x, a_1, b_1 \in S$ and $\gamma \in \Gamma$.
Definition 10. Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\theta}, g >$ be two cubic sets of a Γ-semiring S. Define generalized composition of A and B by

$$Ao_cB =< \widetilde{\mu}, f > o_c < \widetilde{\theta}, g >=< \widetilde{\mu}o_c\widetilde{\theta}, fo_cg >$$

where

$$\begin{aligned} \widetilde{\mu}o_c\widetilde{\theta}(x) &= \underset{x=\sum_{i=1}^{n} a_i\gamma_i b_i}{\cup}[\cap_i\{\cap\{\widetilde{\mu}(a_i), \widetilde{\theta}(b_i)\}\}] \\ &= \widetilde{0}, \text{if x cannot be expressed as above} \end{aligned}$$

and

$$\begin{aligned} fo_cg(x) &= \inf_{x=\sum_{i=1}^{n} a_i\gamma_i b_i}[\max_i\{\max\{f(a_i), g(b_i)\}\}] \\ &= 1, \text{if x cannot be expressed as above} \end{aligned}$$

where $x, a_i, b_i \in S$ and $\gamma_i \in \Gamma$, for i=1,...,n.
Proposition 4. Let $A =< \widetilde{\mu}_1, f >$, $B =< \widetilde{\mu}_2, g >$ be

two cubic ideal of a Γ-semiring S. Then $A\Gamma_c B \subseteq A o_c B \subseteq A \cap B \subseteq A, B$, where $A\Gamma_c B =< \widetilde{\mu}_1 \Gamma_c \widetilde{\mu}_2, f\Gamma_c g >$ and $A o_c B =< \widetilde{\mu}_1 o_c \widetilde{\mu}_2, f o_c g >$.

Proof: Suppose $A =< \widetilde{\mu}_1, f >$, $B =< \widetilde{\mu}_2, g >$ be two cubic ideals of a Γ-semiring S. Then

$$\begin{aligned}(\widetilde{\mu}_1 o_c \widetilde{\mu}_2)(x) &= \bigcup_{x=\sum_{i=1}^{n} a_i\gamma_i b_i}\{\bigcap_i\{\cap\{\widetilde{\mu}_1(a_i), , \widetilde{\mu}_2(b_i),\}\}\} \\ &\text{where } x, a_i, b_i \in S \text{ and } \gamma_i \in \Gamma \\ &\supseteq \bigcup_{x=a_1\gamma b_1}\{\cap\{\widetilde{\mu}_1(a_1), \widetilde{\mu}_2(b_1))\}\} \\ &\text{where } x, a_1, b_1 \in S \text{ and } \gamma \in \Gamma \\ &= (\widetilde{\mu}_1 \Gamma_c \widetilde{\mu}_2)(x)\end{aligned}$$

$$\begin{aligned}(f o_c g)(x) &= \inf_{x=\sum_{i=1}^{n} a_i\gamma_i b_i}\{\max_i\{\max\{f(a_i), g(b_i)\}\}\} \\ &\text{where } x, a_i, b_i \in S \text{ and } \gamma_i \in \Gamma \\ &\leq \inf_{x=a_1\gamma b_1}\{\max\{f(a_1), g(b_1)\}\} \\ &\text{where } x, a_1, b_1 \in S \text{ and } \gamma \in \Gamma \\ &= (f\Gamma_c g)(x)\end{aligned}$$

Therefore, $A\Gamma_c B \subseteq A o_c B$.

$$\begin{aligned}(\widetilde{\mu}_1 o_c \widetilde{\mu}_2)(x) &= \bigcup_{x=\sum_{i=1}^{n} a_i\gamma_i b_i}\{\bigcap_i\{\cap\{\widetilde{\mu}_1(a_i), \widetilde{\mu}_2(b_i)\}\}\} \\ &\text{where } x, a_i, b_i \in S \text{ and } \gamma_i \in \Gamma \\ &\subseteq \bigcup\{\bigcap_i\{\widetilde{\mu}_1(a_i)\}\} \\ &\subseteq \bigcup_{x=\sum_{i=1}^{n} a_i\gamma_i b_i}\{\cap\{\widetilde{\mu}_1(\sum_{i=1}^{n} a_i\gamma_i b_i)\}\} = \widetilde{\mu}_1(x)\end{aligned}$$

$$\begin{aligned}(f o_c g)(x) &= \inf_{x=\sum_{i=1}^{n} a_i\gamma_i b_i}\{\max_i\{\max\{f(a_i), g(b_i)\}\}\} \\ &\text{where } x, a_i, b_i \in S \text{ and } \gamma_i \in \Gamma \\ &= \inf\{\max_i f(a_i)\} \\ &\geq \inf_{x=\sum_{i=1}^{n} a_i\gamma_i b_i}\{\max\{f(\sum_{i=1}^{n} a_i\gamma_i b_i)\}\} = f(x)\end{aligned}$$

Since this is true for every representation of x, $A o_c B \subseteq A$. Similarly we can prove that $A o_c B \subseteq B$. Therefore, $A o_c B \subseteq A \cap B$. Hence the Proposition. ■

Definition 11. A cubic subset $< \widetilde{\mu}, f >$ of a Γ-semiring S is called cubic bi-ideal if for all $x, y \in S$ and $\alpha, \beta \in \Gamma$ we have
(i) $\widetilde{\mu}(x + y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(x + y) \leq \max\{f(x), f(y)\}$
(ii) $\widetilde{\mu}(x\alpha y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(x\alpha y) \leq \max\{f(x), f(y)\}$
(iii) $\widetilde{\mu}(x\alpha y\beta z) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(z)\}$, $f(x\alpha y\beta z) \leq \max\{f(x), f(z)\}$

Definition 12. A cubic subset $< \widetilde{\mu}, f >$ of a Γ-semiring S is called cubic quasi-ideal if for all $x, y \in S$ we have
(i) $\widetilde{\mu}(x + y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(x + y) \leq \max\{f(x), f(y)\}$
(ii) $(\widetilde{\mu} o_c \widetilde{\zeta}_{\chi_S}) \cap (\widetilde{\zeta}_{\chi_S} o_c \widetilde{\mu}) \subseteq \widetilde{\mu}$, $(f o_c \eta_{\chi_S}) \cup (\eta_{\chi_S} o_c f) \supseteq f$.

Theorem 4. A cubic subset $< \widetilde{\mu}, f >$ of a Γ-semiring S is a cubic left ideal of S if and only if for all $x, y \in S$, we have
(i) $\widetilde{\mu}(x + y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(x + y) \leq \max\{f(x), f(y)\}$
(ii) $\widetilde{\zeta}_{\chi_S} o_c \widetilde{\mu} \subseteq \widetilde{\mu}$, $\eta_{\chi_S} o_c f \supseteq f$.

Proof: Assume that $< \widetilde{\mu}, f >$ is a cubic left ideal of S. Then it is sufficient to show that the condition (ii) is satisfied. Let $x \in S$. If x can be expressed as $x = \sum_{i=1}^{n} a_i\gamma_i b_i$, for $a_i, b_i \in S$ and $\gamma_i \in \Gamma$; i=1,...,n, then we have

$$\begin{aligned}(\widetilde{\zeta}_{\chi_S} o_c \widetilde{\mu})(x) &= \bigcup_{x=\sum_{i=1}^{n} a_i\gamma_i b_i}[\bigcap_i\{\cap\{\widetilde{\zeta}_{\chi_S}(a_i), \widetilde{\mu}(b_i)\}\}] \\ &\subseteq \bigcup_{x=\sum_{i=1}^{n} a_i\gamma_i b_i}[\bigcap_i\{\cap\{\widetilde{\mu}(a_i\gamma_i b_i)\}\}] \\ &\subseteq \bigcup_{x=\sum_{i=1}^{n} a_i\gamma_i b_i}[\cap\{\widetilde{\mu}(\sum_{i=1}^{n} a_i\gamma_i b_i)\}] = \widetilde{\mu}(x).\end{aligned}$$

$$\begin{aligned}(\eta_{\chi_S} o_c f)(x) &= \inf_{x=\sum_{i=1}^{n} a_i\gamma_i b_i}[\max_i\{\max\{\eta_{\chi_S}(a_i) f(b_i)\}\}] \\ &= \inf_{x=\sum_{i=1}^{n} a_i\gamma_i b_i}[\max_i\{\max\{f(b_i\}\}] \\ &\geq \inf_{x=\sum_{i=1}^{n} a_i\gamma_i b_i}[\max_i\{\max\{f(a_i\gamma_i b_i)\}\}] \\ &\geq \inf_{x=\sum_{i=1}^{n} a_i\gamma_i b_i}[\max\{f(\sum_{i=1}^{n} a_i\gamma_i b_i)\}] = f(x).\end{aligned}$$

This implies that $\widetilde{\zeta}_{\chi_S} o_c \widetilde{\mu} \subseteq \widetilde{\mu}$, $\eta_{\chi_S} o_c f \supseteq f$. Conversely, assume that the given conditions hold. Then it is sufficient to show the second condition of the definition of cubic ideal.

Let $x, y \in S$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned}\widetilde{\mu}(x\gamma y) &\supseteq (\widetilde{\zeta}_{\chi_S} o_c \widetilde{\mu})(x\gamma y) = \cup[\bigcap_i \{\cap\{\widetilde{\zeta}_{\chi_S}(a_i), \widetilde{\mu}(b_i)\}\}] \\ &\qquad\qquad x\gamma y = \sum_{i=1}^{n} a_i\gamma_i b_i \\ &\supseteq \widetilde{\mu}(y)(\text{since } x\gamma y = x\gamma y).\end{aligned}$$

$$\begin{aligned}f(x\gamma y) &\leq (\eta_{\chi_S} o_c f)(x\gamma y) \\ &= \inf[\max_i\{\max\{\eta_{\chi_S}(a_i), f(b_i)\}\}] \\ &\qquad x\gamma y = \sum_{i=1}^{n} a_i\gamma_i b_i \\ &\leq f(y)(\text{since } x\gamma y = x\gamma y).\end{aligned}$$

Hence $< \widetilde{\mu}, f >$ is a cubic left ideal of S. ■

Theorem 5. Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\nu}, g >$ be a cubic right ideal and a cubic left ideal of a Γ-semiring S, respectively. Then $A \cap B$ is a cubic quasi-ideal of S.

Proof: Let x, y be any element of S. Then

$$\begin{aligned}&(\widetilde{\mu} \cap \widetilde{\nu})(x + y) \\ &= \cap\{\widetilde{\mu}(x + y), \widetilde{\nu}(x + y)\} \\ &\supseteq \cap\{\cap\}\widetilde{\mu}(x), \widetilde{\mu}(y)\}, \cap\{\widetilde{\nu}(x), \widetilde{\nu}(y)\}\} \\ &= \cap\{\cap\{\widetilde{\mu}(x), \widetilde{\nu}(x)\}, \cap\{\widetilde{\mu}(y), \widetilde{\nu}(y)\}\} \\ &= \cap\{(\widetilde{\mu} \cap \widetilde{\nu})(x), (\widetilde{\mu} \cap \widetilde{\nu})(y)\}.\end{aligned}$$

$$\begin{aligned}&(f \cup g)(x + y) \\ &= \max\{f(x + y), g(x + y)\} \\ &\leq \max\{\max\{f(x), f(y)\}, \max\{g(x), g(y)\}\} \\ &= \max\{\max\{f(x), g(x)\}, \cap\{f(y), g(y)\}\} \\ &= \max\{(f \cup g)(x), (f \cup g)(y)\}.\end{aligned}$$

On the other hand, we have

$$((A \cap B)o_c\chi_S) \cap (\chi_S o_c(A \cap B))$$

$$\subseteq (Ao_c\chi_S) \cap (\chi_S o_c B) \subseteq (A \cap B).$$

This completes the proof. ■

Lemma 1. Any cubic quasi-ideal of S is a cubic bi-ideal of S.

Proof: Let $< \widetilde{\mu}, f >$ be any cubic quasi-ideal of S. It is sufficient to show that $\widetilde{\mu}(x\alpha y\beta z) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(z)\}$, $f(x\alpha y\beta z) \leq \max\{f(x), f(z)\}$ and $\widetilde{\mu}(x\alpha y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(x\alpha y) \leq \max\{f(x), f(y)\}$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.
In fact, by the assumption, we have

$$\begin{aligned}&\widetilde{\mu}(x\alpha y\beta z) \\ &\supseteq ((\widetilde{\mu}o_c\widetilde{\zeta}_{\chi_S}) \cap (\widetilde{\zeta}_{\chi_S} o_c\widetilde{\mu}))(x\alpha y\beta z) \\ &= \cap\{(\widetilde{\mu}o_c\widetilde{\zeta}_{\chi_S})(x\alpha y\beta z), (\widetilde{\zeta}_{\chi_S} o_c\widetilde{\mu})(x\alpha y\beta z)\} \\ &= \cap\{\cup(\cap(\widetilde{\mu}(a_i), \widetilde{\zeta}_{\chi_S}(b_i))), \cup(\cap(\widetilde{\zeta}_{\chi_S}(a_i), \widetilde{\mu}(b_i)))\} \\ &\qquad x\alpha y\beta z = \sum_{i=1}^{n} a_i\gamma_i b_i \qquad x\alpha y\beta z = \sum_{i=1}^{n} a_i\gamma_i b_i \\ &\supseteq \cap\{\cap(\widetilde{\mu}(x), \widetilde{\zeta}_{\chi_S}(z)), \cap(\widetilde{\zeta}_{\chi_S}(x), \widetilde{\mu}(z))\}(\text{since } x\alpha y\beta z = x\alpha y\beta z) \\ &= \cap\{\widetilde{\mu}(x), \widetilde{\mu}(z)\}\end{aligned}$$

$$\begin{aligned}&f(x\alpha y\beta z) \\ &\leq (fo_c\eta_{\chi_S}) \cup (\eta_{\chi_S} o_c f))(x\alpha y\beta z) \\ &= \max\{(fo_c\eta_{\chi_S})(x\alpha y\beta z), (\eta_{\chi_S} o_c f)(x\alpha y\beta z)\} \\ &= \max\{\inf(\max(f(a_i), \eta_{\chi_S}(b_i))), \inf(\max(\eta_{\chi_S}(a_i), f(b_i)))\} \\ &\qquad x\alpha y\beta z = \sum_{i=1}^{n} a_i\gamma_i b_i \qquad x\alpha y\beta z = \sum_{i=1}^{n} a_i\gamma_i b_i \\ &\leq \max\{\max(\eta_{\chi_S}(x), f(z)), \max(f(x), \eta_{\chi_S}(z))\} \\ &\quad (\text{since } x\alpha y\beta z + 0\gamma = x\alpha y\beta z) \\ &= \max\{f(x), f(z)\}\end{aligned}$$

Similarly, we can show that $\widetilde{\mu}(x\alpha y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(x\alpha y) \leq \max\{f(x), f(y)\}$ for all $x, y \in S$ and $\alpha \in \Gamma$. ■

V. Regularity and Intra-Regularity

In this section, the concept of regularity in Γ-semiring are studied by using cubic ideal, cubic bi-ideal, cubic quasi-ideal.

Definition 13. A Γ-semiring S is said to be regular if for each $x \in S$, there exist $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x = x\alpha a\beta x$.

Theorem 6. Let S be a regular Γ-semiring. Then for any cubic right ideal $A =< \widetilde{\mu}, f >$ and any cubic left ideal $B =< \widetilde{\nu}, g >$ of S we have $A\Gamma_c B = A \cap B$.

Proof: Let S be a regular Γ-semiring. By Proposition 4, we have $A\Gamma_c B \subseteq A \cap B$. For any $a \in S$, there exist $x_1 \in S$ and $\alpha_1, \beta_1 \in \Gamma$ such that $a = a\alpha_1 x_1\beta_1 a$.
Then

$$\begin{aligned}(\widetilde{\mu}\Gamma_c\widetilde{\nu})(a) &= \bigcup_{a=e\gamma b}\{\cap\{\widetilde{\mu}(e),\widetilde{\nu}(b)\}\} \supseteq \cap\{\widetilde{\mu}(a\alpha_1 x_1), \widetilde{\nu}(a)\} \\ &\supseteq \cap\{\widetilde{\mu}(a), \widetilde{\nu}(a)\} = (\widetilde{\mu} \cap \widetilde{\nu})(a).\end{aligned}$$

$$\begin{aligned}(f\Gamma_c g)(a) &= \inf_{a=e\gamma b}\{\max\{f(e),g(b)\}\} \\ &\leq \max\{f(a\alpha_1 x_1), g(a)\} \\ &\leq \max\{f(a), g(a)\} = (f \cup g)(a).\end{aligned}$$

Therefore $(A \cap B) \subseteq (A\Gamma_c B)$. Hence $A\Gamma_c B = A \cap B$. ■

Corollary 1. If S be a regular Γ-semiring, then for any cubic right ideal $A =< \widetilde{\mu}, f >$ and any cubic left ideal $B =< \widetilde{\nu}, g >$ of S we have $Ao_c B = A \cap B$.

Theorem 7. Let S be a regular Γ-semiring. Then
(i) $A \subseteq Ao_c\chi_S o_c A$ for every cubic bi-ideal $A =< \widetilde{\mu}, f >$ of S.
(ii) $A \subseteq Ao_c\chi_S o_c A$ for every cubic quasi-ideal $A =< \widetilde{\mu}, f >$ of S.

Proof: Suppose that $A =< \widetilde{\mu}, f >$ be any cubic bi-ideal of S and x be any element of S. Since S is regular there exist

$a \in S$ and $\alpha, \beta \in \Gamma$ such that $x = x\alpha a\beta x$. Now

$$\begin{aligned}
&(\widetilde{\mu} o_c \widetilde{\zeta}_{\chi_S} o_c \widetilde{\mu})(x) \\
&= \cup(\cap\{(\widetilde{\mu} o_c \widetilde{\zeta}_{\chi_S})(a_i), \widetilde{\mu}(b_i)\}) \\
&\qquad x = \sum_{i=1}^{n} a_i \gamma_i b_i \\
&\supseteq \cap\{(\widetilde{\mu} o_c \widetilde{\zeta}_{\chi_S})(x\alpha a), \widetilde{\mu}(x)\} \\
&= \cap\{\cup(\cap\{\widetilde{\mu}(a_i), (\widetilde{\zeta}_{\chi_S})(b_i)\}), \widetilde{\mu}(x)\}\} \\
&\qquad x\alpha a = \sum_{i=1}^{n} a_i \gamma_i b_i \\
&\supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(x)\}(since\ x\alpha a = x\alpha a\beta x\alpha a). \\
&= \widetilde{\mu}(x)
\end{aligned}$$

$$\begin{aligned}
&(f o_c \eta_{\chi_S} o_c f)(x) \\
&= \inf(\max\{(f o_c \eta_{\chi_S})(a_i), f(b_i)\}) \\
&\qquad x = \sum_{i=1}^{n} a_i \gamma_i b_i \\
&\leq \max\{(f o_c \eta_{\chi_S})(x\alpha a), f(x)\} \\
&= \max\{\inf(\max\{(f(a_i), \eta_{\chi_S}(b_i))\}), f(x)\} \\
&\qquad x\alpha a = \sum_{i=1}^{n} a_i \gamma_i b_i \\
&\leq \max\{f(x), f(x)\}(since\ x\alpha a = x\alpha a\beta x\alpha a) \\
&= f(x)
\end{aligned}$$

This implies that $A \subseteq A o_c \chi_S o_c A$.
(i)⇒(ii) This is straight forward from Lemma 1. ■

Theorem 8. Let S be a regular Γ-semiring. Then
(i) $A \cap B \subseteq A o_c B o_c A$ for every cubic bi-ideal $A =< \widetilde{\mu}, f >$ and every cubic ideal $B =< \widetilde{\nu}, g >$ of S.
(ii) $A \cap B \subseteq A o_c B o_c A$ for every cubic quasi-ideal $A =< \widetilde{\mu}, f >$ and every cubic ideal $B =< \widetilde{\nu}, g >$ of S.

Proof: Suppose S is a regular Γ-semiring and $A =< \widetilde{\mu}, f >$, $B =< \widetilde{\nu}, g >$ be any cubic bi-ideal and cubic ideal of S, respectively and x be any element of S. Since S is regular, there exist $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x = x\alpha a\beta x$. Now

$$\begin{aligned}
&(\widetilde{\mu} o_c \widetilde{\nu} o_c \widetilde{\mu})(x) \\
&= \cup(\cap\{(\widetilde{\mu} o_c \widetilde{\nu})(a_i), \widetilde{\mu}(b_i)\}) \\
&\qquad x = \sum_{i=1}^{n} a_i \gamma_i b_i \\
&\supseteq \cap\{(\widetilde{\mu} o_c \widetilde{\nu})(x\alpha a), \widetilde{\mu}(x)\} \\
&= \cap\{\cup(\cap\{(\widetilde{\mu}(a_i), \widetilde{\nu}(b_i))\}, \widetilde{\mu}(x)\} \\
&\qquad x\alpha a = \sum_{i=1}^{n} a_i \gamma_i b_i \\
&\supseteq \cap\{\cap\{\widetilde{\mu}(x), \widetilde{\nu}(a\beta x\alpha a), \widetilde{\mu}(x)\}(since\ x\alpha a = x\alpha a\beta x\alpha a) \\
&\supseteq \cap\{\widetilde{\mu}(x), \widetilde{\nu}(x)\} = (\widetilde{\mu} \cap \widetilde{\nu})(x).
\end{aligned}$$

$$\begin{aligned}
&(f o_c g o_c f)(x) \\
&= \inf(\max\{(f o_c g)(a_i), f(b_i)\}) \\
&\qquad x = \sum_{i=1}^{n} a_i \gamma_i b_i \\
&\leq \max\{(f o_c g)(x\alpha a), f(x)\} \\
&= \max\{\inf(\max\{(f(a_i), g(b_i))\}), f(x)\} \\
&\qquad x\alpha a = \sum_{i=1}^{n} a_i \gamma_i b_i \\
&\leq \max\{\max\{f(x), g(a\beta x\alpha a), f(x)\}(since\ x\alpha a = x\alpha a\beta x\alpha a) \\
&\supseteq \max\{f(x), g(x)\} = (f \cup g)(x).
\end{aligned}$$

(i)⇒(ii) This is straight forward from Lemma 1. ■

Definition 14. A Γ-semiring S is said to be intra-regular if for each x∈S, there exist $a_i, a_i^{'}$ ∈S, and $\eta, \alpha_i, \beta_i \in \Gamma$, i∈ **N**, the set of natural numbers, such that $x = \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a_i^{'}$.

Theorem 9. Let S be a intra-regular Γ-semiring. Then $A \cap B \subseteq A o_c B$ for every cubic left ideal $A =< \widetilde{\mu}, f >$ and every cubic right ideal $A =< \widetilde{\nu}, g >$ of S.

Proof: Suppose S is intra-hemiregular. Let $A =< \widetilde{\mu}, f >$ and $A =< \widetilde{\nu}, g >$ be any cubic left ideal and cubic right ideal of S respectively. Now let $x \in S$. Then by hypothesis there exist $a_i, a_i^{'}$ ∈S, and $\alpha_i, \beta_i, \eta \in \Gamma$, $i \in$ **N**, the set of natural numbers, such that $x = \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a_i^{'}$. Therefore

$$\begin{aligned}
(\widetilde{\mu} o_c \widetilde{\nu})(x) \quad &= \cup[\bigcap_i\{\cap\{\widetilde{\mu}(a_i), \widetilde{\nu}(b_i)\}\}] \\
&\qquad x = \sum_{i=1}^{n} a_i \gamma_i b_i \\
&\supseteq \bigcap_i[\cap\{\widetilde{\mu}(a_i \alpha_i x), \widetilde{\nu}(x \beta_i a_i^{'})\}] \\
&\supseteq \cap\{\widetilde{\mu}(x), \widetilde{\nu}(x)\} = (\widetilde{\mu} \cap \widetilde{\nu})(x).
\end{aligned}$$

$$\begin{aligned}
(f o_c g)(x) \quad &= \inf[\max_i\{\max\{f(a_i), g(b_i)\}\}] \\
&\qquad x = \sum_{i=1}^{n} a_i \gamma_i b_i \\
&\leq \max_i[\max\{f(a_i \alpha_i x), g(x \beta_i a_i^{'})\}] \\
&\leq \max\{f(x), g(x)\} = (f \cup g)(x).
\end{aligned}$$

Hence the proof. ■

Theorem 10. Let S be both regular and intra-regular Γ-semiring. Then
(i) $A = A o_c A$ for every cubic bi-ideal $A =< \widetilde{\mu}, f >$ of S.
(ii) $\mu = \widetilde{\mu} o_h \mu$ for every cubic quasi-ideal $A =< \widetilde{\mu}, f >$ of S.

Proof: Suppose S be both regular and intra-regular Γ-semiring. Let $x \in S$ and $A =< \widetilde{\mu}, f >$ be any cubic bi-ideal of S. Since S is both regular and intra-regular, there exist $a_i, b_i \in S$ and $\alpha_i, \beta_i, \alpha_i^{'}, \beta_i^{'}, \eta \in \Gamma$, $i \in$ **N** such that $x = \sum_{i=1}^{n} x \alpha_i a_i \alpha_i^{'} x \eta x \beta_i^{'} b_i \beta_i x$. Therefore

$$\begin{aligned}
(\widetilde{\mu} o_c \widetilde{\mu})(x) &= \cup[\bigcap_i\{\cap\{\widetilde{\mu}(a_i), \widetilde{\mu}(b_i)\}\}] \\
&\qquad x=\sum_{i=1}^{n} a_i\gamma_i b_i \\
&\supseteq \bigcap_i[\cap\{\widetilde{\mu}(x\alpha_i a_i\alpha_i^{'}x), \widetilde{\mu}(x\beta_i^{'}b_i\beta_i x)\}] \\
&\qquad x=\sum_{i=1}^{n} x\alpha_i a_i\alpha_i^{'}x\eta x\beta_i^{'}b_i\beta_i x \\
&\supseteq \widetilde{\mu}(x).
\end{aligned}$$

$$\begin{aligned}
(fo_c f)(x) &= \inf[\max_i\{\max\{f(a_i), f(b_i)\}\}] \\
&\qquad x=\sum_{i=1}^{n} a_i\gamma_i b_i \\
&\leq \max_i[\max\{f(x\alpha_i a_i\alpha_i^{'}x), f(x\beta_i^{'}b_i\beta_i x)\}] \\
&\qquad x=\sum_{i=1}^{n} x\alpha_i a_i\alpha_i^{'}x\eta x\beta_i^{'}b_i\beta_i x \\
&\leq f(x).
\end{aligned}$$

Now $Ao_cA \subseteq Ao_c\chi_S \subseteq A$. Hence $Ao_cA = A$ for every cubic bi-ideal $A$of S.
(i)⇒(ii) This is straightforward from the Lemma 1. ■

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