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Comparing the Efficiency of Simpson's 1/3 and 3/8 Rules for the Numerical Solution of First Order Volterra Integro-Differential Equations

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***Abstract*---**This paper compared the efficiency of Simpson's 1/3 and 3/8 rules for the numerical solution of first order Volterra integro-differential equations. In developing the solution, collocation approximation method was adopted using the shifted Legendre polynomial as basis function. A block method approach is preferred to the predictor corrector method for being self-starting. Experimental results confirmed that the Simpson's 3/8 rule is more efficient than the Simpson's 1/3 rule.

***Keywords*---**Collocation shifted Legendre polynomials, Simpson's rule and Volterra integro-differential equations.

I. Introduction

MOST problems in the areas of mechanics, mathematical biology, physics and economics involve a combination of differential and integral equations otherwise called Integro Differential Equations (IDEs). Many branches of linear and nonlinear functional analysis involves IDEs most especially the Volterra type which are found to be useful in the theory of engineering, mechanics, physics, chemistry, astronomy, biology and economics (see [5]-[7], [12], [14], [16], [17], [19]). Unfortunately many of these problems cannot be solved analytically.

Numerical solutions of Volterra IDEs of the discrete types have been extensively studied by many researchers. The numerical method for nonlinear Volterra IDEs was introduced in [15], [17]. The implicit Runge-Kutta methods of optimal order for Volterra IDEs were suggested in [10]. The mixed interpolation and collocation methods for first and second order Volterra IDEs with periodic solution were introduced in [11]. Approximate solution of high order linear Volterra-Fredholm IDEs in terms of Taylor polynomials was considered in [8]. The numerical solution of the system of nonlinear Volterra IDEs with nonlinear differential part by the operational Tau method and error estimation was considered by [18]. Reference [1] developed approximate solution of second-order IDE of Volterra type in reproducing kernel Hilbert space (RKHS) method. The quadrature rules to find the numerical solutions of the initial value problems for Volterra IDEs of the second kind appeared in [9]. Reference [4] devoted their time in the development of the Taylors expansion approach for nonlinear IDEs. The linear multistep method for Volterra IDEs was constructed by [2] and [13]. In this work, the continuous type is considered.

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II. Derivation of the Method

Applying methods of solution for first order initial value problems of ordinary differential equation of the following form

$$y'(x) = f(x, y(x)), y(x_0) = y_0 \tag{1}$$

as discussed in [20] can be used to solve systems of equations arising from the discretization of first order initial value problems of the Volterra type of the following form

$$y'(x) = f(x, y(x), z(x)), y(x_0) = y_0 \tag{2}$$

where

$$z(x) = \int_{x_0}^{x} K(x, t, y(t))\,dt$$

$y(x)$ is an unknown function to be determined as in [20] by the shifted Legendre polynomial of the form;

$$y(x) = \sum_{i=0}^{m} c_i p_i(\psi) \tag{3}$$

where $c_i \in \mathbb{R}$, $y \epsilon C^1(a,b)$ and $\psi = (x - x_n)$

The first derivative of (3) is substituted into (2) to obtain a differential system of the form

$$y' = \sum_{i=0}^{m} c_i p_i'(\psi) \tag{4}$$

Interpolating (3) at $x_{n+r}, r = 0$ and $k-1$ and collocating (4) at $x_{n+r}, r = 1, \dots, k$ and after some substitutions and manipulations, we obtain the continuous scheme of the form;

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h\sum_{j=0}^{k} \beta_j(x) f\left(x_{n+j}, y(x_{n+j}), z(x_{n+j})\right) \tag{5}$$

where

$$z_n(x) = h\sum_{j=0}^{n} \alpha_j w_{nj} K(x_n, x_j, y_j), n > j, z_0 = 0 \quad (6)$$

and the weights w_{nj}depend on the choice of quadrature rule. In this work, the Simpson's 1/3 and 3/8 rules shall be used in evaluating the integral part. Evaluating (5) at some grid points, leads to a discrete scheme.

III. Specification of the Method

Consider a four step method, that is when $k = 4$ is substituted in (3), we get

$$y(x) = \sum_{i=0}^{5} c_i p_i(\psi)$$

Collocating (4) at $x_{n+r}, r = 1,2, \ldots, 4$ and interpolating (3) at x_n and x_{n+3}, we obtain the continuous linear multistep method after some substitutions and manipulations of the form;

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \quad (7)$$

where

$$\begin{bmatrix} \alpha_0(x) = 1 - \frac{80}{27h}\psi + \frac{250}{81h^2} - \frac{350}{243h^3}\psi^3 + \frac{25}{81h^4}\psi^4 - \frac{2}{81h^5}\psi^5 \\ \alpha_3(x) = \frac{80}{27h}\psi - \frac{250}{81h^2} + \frac{350}{243h^3}\psi^3 - \frac{25}{81h^4}\psi^4 + \frac{2}{81h^5}\psi^5 \\ \beta_1(x) = -\frac{34}{8}\psi + \frac{641}{108h}\psi^2 - \frac{1063}{324h^2}\psi^3 + \frac{83}{108h^3}\psi^3 - \frac{7}{108h^4}\psi^5 \\ \beta_2(x) = -\frac{8}{3}\psi + \frac{23}{18h}\psi^2 + \frac{31}{108h^2}\psi^3 - \frac{2}{9h^3}\psi^4 + \frac{1}{36h^4}\psi^5 \\ \beta_3(x) = -\frac{14}{9}\psi + \frac{247}{108h}\psi^2 - \frac{497}{324h^2}\psi^3 + \frac{49}{108h^3}\psi^4 - \frac{5}{108h^4}\psi^5 \\ \beta_4(x) = \frac{1}{9}\psi - \frac{13}{54h}\psi^2 + \frac{67}{324h^2}\psi^3 - \frac{2}{27h^3}\psi^4 + \frac{1}{108h^4}\psi^5 \end{bmatrix}$$

Evaluating (7) and its first derivative at x_{n+4}, x_{n+2}, x_{n+1} and x_n respectively with $\psi = (x - x_n)$ and substituting in (6), we obtain the following block discrete method;

$$\begin{bmatrix} y_{n+4} = \frac{224}{243}y_{n+3} + \frac{19}{243}y_n + \frac{20}{81}hf_{n+1} - \frac{8}{27}hf_{n+2} + \frac{76}{81}hf_{n+3} + \frac{28}{81}hf_{n+4} \\ y_{n+2} = \frac{232}{243}y_{n+3} + \frac{11}{243}y_n + \frac{13}{81}hf_{n+1} - \frac{16}{27}hf_{n+2} - \frac{37}{81}hf_{n+3} + \frac{2}{81}hf_{n+4} \\ y_{n+1} = \frac{251}{243}y_{n+3} - \frac{8}{243}y_n - \frac{34}{81}hf_{n+1} - \frac{35}{27}hf_{n+2} - \frac{32}{81}hf_{n+3} + \frac{1}{81}hf_{n+4} \\ y_{n+3} = y_n + \frac{27}{80}hf_n + \frac{51}{40}hf_{n+1} + \frac{9}{10}hf_{n+2} + \frac{21}{40}hf_{n+3} - \frac{3}{80}hf_{n+4} \end{bmatrix} \quad (8)$$

IV. Analysis of the Method

Order and Error Constant

Expanding the block (8) in Taylor's series and collecting like terms in powers of$h$, we have;

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = (0,0,0,0,)^T$$

and

$$C_6 = \left(-\frac{1}{1620}, -\frac{1}{1620}, \frac{3}{160}, -\frac{7}{405}\right)^T$$

The method has order $p = (5,5,5,5)^T$ and error constant

$$C_6 = -\frac{1}{1620}, -\frac{1}{1620}, \frac{3}{160}, -\frac{1}{405}.$$

Consistency

Following [3], the block method is said to be consistent since $p = 5 > 1$.Hence, the method is consistent.

Zero Stability

The block solution of (8) is said to be zero stable if the roots z_r; $r = 1, \ldots, n$ of the first characteristic polynomial $\breve{\rho}(z)$, defined by

$$p(z) = det|zQ - T|$$

satisfies $|z_r| \le 1$and every root with $|z_r| = 1$ has multiplicity not exceeding the order of the differential equation in the limit as $h \to 0$

From the block solution of (1.7), we have

$$p(z) = det|zQ - T|$$
$$z = (0,0,0,1)$$

This shows that the block method is zero stable, since all roots with modulus one do not have multiplicity exceeding the order of the differential equation in the limit as $h \to 0$.

Convergence

According [3], the method is convergent, since it is consistent and zero stable.

V.Numerical Illustration

In order to support our theoretical discussion of the proposed method; the computations, associated with the examples, are performed using MAPLE 18. Furthermore, the performance of the methods is tested on some numerical examples contained in the literature ranging from linear to nonlinear first order Volterra integro-differential equations (VIDEs).

Examples

The method was demonstrated using the following examples

(a) $y' = 1 - \int_0^x y(t)\,dt$

$y(0) = 0, 0 \le x \le 1$

The exact solution $y(x) = \sin(x)$ with results shown in Tables I and II.

(b) $y' = 1 + 2x - y + \int_0^x x(1+2x)e^{t(x-t)} y(t)\,dt$

$y(0) = 1, 0 \le x \le 1$

With exact solution $y(x) = e^{-x^2}$ Tables III and IV show the results.

(c) $y' = (1 + x^2 + 2x)e^{-x} - y + 5x^2 + 8 - \int_0^x t y(t)\,dt$

$y(0) = 10.0, 0 \le x \le 1$

The exact solution is $y(x) = 10 - xe^{-x}$ and results are shown in Tables V and VI.

TABLE I
COMPARING RESULTS OBTAINED FOR STEP SIZE $h = 0.1$

x	Exact Solution	Result by Simpson's 1/3	Result by Simpson's 3/8	Absolute Error by Simpson's 1/3	Absolute Error by Simpson's 3/8
0.1	0.099833416	0.1003003205	0.1002356986	0.00046690385	0.00040228195
0.2	0.198669330	0.1989905635	0.1989737085	0.0003212327	0.0003043777
0.3	0.295520206	0.2968489675	0.2969643986	0.0013287608	0.0014441919
0.4	0.389418342	0.3926808396	0.3927336692	0.0032624973	0.0033153269
0.5	0.479425538	0.4871615245	0.4871709427	0.0077359859	0.0077454041
0.6	0.564642473	0.5736036793	0.5728604405	0.0089612059	0.0082179671
0.7	0.644217687	0.6553532197	0.6534519115	0.0111355325	0.0092342243
0.8	0.717356090	0.7340234430	0.7306365462	0.0166673521	0.0132804553
0.9	0.783326909	0.8074782946	0.8021183186	0.0241513850	0.0187914090
1.0	0.841470984	0.8675629174	0.8588828875	0.0260919326	0.0174119027

TABLE II
COMPARING RESULTS USING STEP SIZE $h = 0.01$

x	Exact Solution	Result by Simpson's 1/3	Result by Simpson's 3/8	Absolute Error by Simpson's 1/3	Absolute Error by Simpson's 3/8
0.1	0.099833416	0.09986309733	0.09984865043	0.00002968068	0.00001523378
0.2	0.198669330	0.1987989523	0.1986543557	0.0001296215	0.0000149751
0.3	0.295520206	0.2958184292	0.2953057539	0.0002982225	0.0002144528
0.4	0.389418342	0.3899501457	0.3887140323	0.0005318034	0.0007043100
0.5	0.479425538	0.4802515722	0.4778267362	0.0008260336	0.0015988024
0.6	0.564642473	0.5658171315	0.5616381278	0.0011746581	0.0030043456
0.7	0.644217687	0.6457885666	0.6392019729	0.0015708794	0.0050157143
0.8	0.717356090	0.7193623248	0.7096408640	0.0020062339	0.0077152269
0.9	0.783326909	0.7857987802	0.7721574111	0.0024718706	0.0111694985
1.0	0.841470984	0.8444286060	0.8260421081	0.0029576212	0.0154288767

TABLE III
COMPARING RESULTS OF EXAMPLE **(B)** OBTAINED WITH STEP SIZE $h = 0.1$

x	Exact Solution	Result by Simpson's 1/3	Result by Simpson's 3/8	Absolute Error by Simpson's 1/3	Absolute Error by Simpson's 3/8
0.1	1.010050167	1.008344622	1.008167532	0.001705545	0.001882635
0.2	1.040810774	1.039207888	1.039419918	0.001602886	0.001390856
0.3	1.094174284	1.090462513	1.091534056	0.003711771	0.002640228
0.4	1.173510871	1.165565212	1.168611045	0.007945659	0.004899826
0.5	1.284025417	1.262892013	1.268763150	0.021133404	0.015262267
0.6	1.433329415	1.410674256	1.424613033	0.022655159	0.008716382
0.7	1.632316220	1.600308029	1.626000003	0.032008191	0.006316217
0.8	1.896480879	1.842124958	1.885083735	0.054355921	0.011397144
0.9	2.247907987	2.137938029	2.202315886	0.109969958	0.045592101
1.0	2.718281828	2.597820716	2.709484149	0.120461112	0.008797679

TABLE IV
COMPARING RESULTS OF EXAMPLE **(b)** OBTAINED WITH $h = 0.01$

x	Exact Solution	Result by Simpson's 1/3	Result by Simpson's 3/8	Absolute Error by Simpson's 1/3	Absolute Error by Simpson's 3/8
0.1	1.010050167	1.010010960	1.010052969	0.000039207	0.000002802
0.2	1.040810774	1.040638070	1.041035206	0.000172704	0.000224432
0.3	1.094174284	1.093744368	1.094539458	0.000429916	0.000365174
0.4	1.173510871	1.172659813	1.175248980	0.000851058	0.001738109
0.5	1.284025417	1.282530749	1.288772270	0.001494668	0.004746853
0.6	1.433329415	1.430880700	1.443666502	0.002448715	0.010337087
0.7	1.632316220	1.628470649	1.652145803	0.003845571	0.019829583
0.8	1.896480879	1.890590331	1.931598071	0.005890548	0.035117192
0.9	2.247907987	2.239009786	2.306888493	0.008898201	0.058980506
1.0	2.718281828	2.704916403	2.813875703	0.013365425	0.095593875

TABLE V
RESULTS OF EXAMPLE (C) OBTAINED WITH SIZE $h = 0.1$

x	Exact Solution	Result by Simpson's 1/3	Result by Simpson's 3/8	Absolute Error by Simpson's 1/3	Absolute Error by Simpson's 3/8
0.1	9.909516258	9.913760264	9.917745274	0.004244006	0.008229016
0.2	9.836253849	9.838817708	9.842204649	0.002563859	0.005950800
0.3	9.777754534	9.789749559	9.791584780	0.011995025	0.013830246
0.4	9.731871982	9.761103357	9.761467493	0.029231375	0.029595511
0.5	9.696734670	9.765684645	9.763019952	0.068949975	0.066285282
0.6	9.670713018	9.745268230	9.728056527	0.074555212	0.057343509
0.7	9.652390287	9.742611455	9.706664649	0.090221168	0.054274362
0.8	9.640536829	9.777917620	9.723631156	0.137380791	0.083094327
0.9	9.634087306	9.837248350	9.760106932	0.203161044	0.126019626
1.0	9.632120559	9.838211245	9.721971039	0.206090686	0.089850480

TABLE VI
RESULTS OF EXAMPLE (C) OBTAINED WITH SIZE $h = 0.01$

x	Exact Solution	Result by Simpson's 1/3	Result by Simpson's 3/8	Absolute Error by Simpson's 1/3	Absolute Error by Simpson's 3/8
0.1	9.909516258	9.909802455	9.909661135	0.000286197	0.000144877
0.2	9.836253849	9.837463876	9.836090653	0.001210027	0.000163196
0.3	9.777754534	9.780455634	9.775703965	0.002701100	0.002050569
0.4	9.731871982	9.736567069	9.725356410	0.004695087	0.006515572
0.5	9.696734670	9.703862806	9.682294561	0.007128136	0.014440109
0.6	9.670713018	9.680657899	9.644132171	0.009944881	0.026580847
0.7	9.652390287	9.665470424	9.608803421	0.013080137	0.043586866
0.8	9.640536829	9.657018317	9.574568313	0.016481488	0.065968516
0.9	9.634087306	9.654165500	9.539961903	0.020078194	0.094125403
1.0	9.632120559	9.655938603	9.503823165	0.023818044	0.128297394

VI. CONCLUSION

In this work, information about solving VIDEs using Simpson's rules is presented. Collocation approximation method of the shifted Legendre polynomial as basis function was used to obtain the approximate solution for solving first order VIDEs. The approximate solution was obtained in block mode which have the advantage of being self-starting hence eliminating the use of predictor corrector method. Unlike the approach in predictor corrector method, all additional equations are obtained from the same continuous formulation which shows the beauty of the method. Experimental results confirmed that the Simpson's 3/8 rule is efficient than Simpson's 1/3 rule.

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