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On the Efficiency and Robustness of Commingle Wiener and Lévy Driven Processes for Vasciek Model

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Abstract—The driven processes of Wiener and Lévy are known self-standing Gaussian-Markov processes for fitting non-linear dynamical Vasciek model. In this paper, a coincidental Gaussian density stationarity condition and autocorrelation function of the two driven processes were established. This led to the conflation of Wiener and Lévy processes so as to investigate the efficiency of estimates incorporated into the one-dimensional Vasciek model that was estimated via the Maximum Likelihood (ML) technique. The conditional laws of drift, diffusion and stationarity process was ascertained for the individual Wiener and Lévy processes as well as the commingle of the two processes for a fixed effect and Autoregressive like Vasciek model when subjected to financial series; exchange rate of Naira-CFA Franc. In addition, the model performance error of the sub-merged driven process was miniature compared to the self-standing driven process of Wiener and Lévy.

Keywords—Wiener process, Lévy process, Vasciek model, drift, diffusion, Gaussian density stationary.

I. Introduction

DYNAMICAL systems are mathematical objects used to model physical and non-physical phenomena whose state (or instantaneous description) changes over time according to a fixed rule as discussed in [1], [2]. Dynamical system has been a vital modeling in fields such as engineering, and physics, financial statistics (e.g. stock market, exchange rates), and environmental statistics with two realistic traits that are of great interest – the stochastic term that is made-up of the observational outputs that are noisy function of the inputs such that the dynamics itself will be galvanized by some unobserved noise processes. The second distinct trait is the finite-dimensional states that are indirect observable but synopsize at any time for all information about the traits of the process relevant to prediction [3]. The understanding of physical process that evolved with time is limited by the ability to model a dynamical system. Generally, dynamic is the ability of a process or a system to change [4]. In other words, dynamic is a stochastic or random process of mathematical object usually defined as a collection of random variables associated with both linear and non-linear characteristics. The dynamics of a system may be expressed either as a continuous-time or as a discrete-time-evolutionary process, that is, dynamics is a time-evolutionary process of either a deterministic or a stochastic process. The associated random variable(s) is with or indexed by a set of random numbers, usually viewed as points in time, giving the interpretation of a stochastic process representing numerical values of some system randomly changing over time such as growth of a bacterial population, price fluctuation of a commodity etc. [5], [6].

The non-linear dynamical systems provide the mathematical language describing the time dependence of deterministic systems from the modeling point; stochasticity is paramount to the model for small number of parameters to generate a rich variety of time-series outputs. The non-linear dynamic based models such as Kalman Filter; Ornstein–Uhlenbeck (Vasicek); Laguerre stochastic processes; Gaussian Process Approximations of Stochastic Differential Equations etc. are stochastic processes that range from discrete-time to continuous-time dynamical systems with jumps and some other properties of Gaussian and Markov processes [7], [8]. All but one of these aforementioned processes are Ordinary Differential Equation (ODE) process base expect for Vasicek that is based on two other processes for low-filtered white noise as stochastic error (error term subjected to time series models). The two processes rely on the introduction of Wiener process (otherwise known as Brownian motion) and Lévy processes that are functions of Gaussian process that unveil and incorporate jumps as well as diffusion rates.

Reference [9] started the theoretical comparison of non-linear dynamic via Kullback-Leibler (KL) divergence between approximating posterior process and the exact one (i.e., between probability measures over paths) which makes the computation non-trivial. In a similar vein, [10] affirmed that system is a set of behavioral variables, namely heading direction and velocity, defines a state space in which a dynamics of robot behavior described as path planning governed by a non-linear dynamical system that generates a time course of the behavioral variables. They subjected behavioral variables of non-linear to represented the dimensional corresponds to agent behavior via Wiener process. However, in the field of data, the non-linear dynamic behavioral setting of variables has some work done for computing approximate predictions via Vasicek process [11], [12]. In extension, [13] modeled stock prices via Vasicek that was driven by Lévy process. The Bergdorf-Nielsen and Shephard (BNS) stochastic volatility model was used to cater for the volatility parameter in order to be a self–decomposable distribution that allowed flexible. For this reason, the Inverse Gaussian-OU model by calibrating moments of Lévy process was used in fitting the model to a financial as well as a simulation study. In this regard, the Wiener and Lévy driven

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processes for Vasicek model will be examined individually to ascertain their density stationary condition, autocorrelation function including their first and second density moments. The Wiener and Lévy driven processes will be meld to a single driven process to drive a one-dimensional level of the Vasicek model, then the Maximum likelihood Estimation (MLE) technique will be adopted in estimating the transition density parameters involved to investigate the robustness, efficiency and higher process estimates in comparison to the self-standing processes. The Vasicek process at one-dimensional level, which is Autoregressive of order one like will then be applied to Naira to CFA exchange rate from 2012 to June 2018.

II. Notation and Definition of Related Terms

According to [14], [5], a stochastic process, say $Y = \{Y(t)\}$ is said to be a Vasicek process if it satisfies the homogeneous linear stochastic differential equation of:

$$Y = \begin{cases} \partial Y(t) = -\delta Y(t)\partial t + s\,\partial B(t) \\ Y(0) = Y_0, \end{cases} \tag{1}$$

where δ and s are strictly positive intensity parameters and Y is a random variable independent of the Wiener process (standard Brownian motion) $B = \{B(t)\}$. The $B(t)$ is the Gaussian process such that $B(t) \sim N(0, ts)$. Additionally, $B = \{B(t)\}$ could also emerge from a Lévy process such that it could be referred to as Background Driving Lévy Process (BDLP) of the Vasicek or OU model type [15].

Reference [16] described a Weiner Process to be a macroscopic picture of a particle emerging in random system or particle of liquid suspension such that the continuous time stochastic process $\{B(t) : t \geq 0\}$ describing the particle of liquid suspension incurred by a random walk traits. These traits are for the time events $0 \leq t_1 \leq t_2 \leq ... \leq t_n$, $\beta(t_n) - \beta(t_{n-1}), \beta(t_{n-1}) - \beta_{tn-2})$; ..., $(\beta_{t2}) - \beta(t_1)$ are independent (that is independent decrements); the distribution of the increment $\beta_{(t+h)} - \beta_{(t)}$ does not depend on "t" that is stationary increments (Lévy process) and the process $\{\beta_{(t)} : t \geq 0\}$ has continuous paths. A process $\{Y_t\}$ with these mentioned traits could be represented by:

$$\beta_{(t)} = \beta_{(0)} + \mu_t + \sum \beta_{(t)}, \quad for \qquad t \geq 0 \tag{2}$$

where $\beta_{(0)}$ is the initial distribution, μ_t is the drift vector μ , and $\sum \beta_{(t)}$ been referred to as is the diffusion matrix. $\{\beta_{(t)} : t \geq 0\}$ is being referred to as Standard Brownian Motion (SBM) when the drift vector is nothing but a zero term and the diffusion matrix is the identity for a macroscopic picture emerging from random walk.

Reference [16] defined the Lévy process as a continuous time stochastic processes $(K_t)_{t\geq 0}$ with

1. $K_t = 0$
2. Stationary increments, for all $t > 0$, $K_{t+a} - K_a$ has the same distribution as K_a
3. Independent increments, for all $0 \leq t_0 \leq t_1 \leq \cdots\cdots \leq t_n$, $K_{t+a} - K_a$ $(i = 0, 1, 2, \cdots\cdots, n)$ are independent and a continuous path such that the sample paths of the Levy process are right continuous and left limits.

III. Methodology

A non-linear dynamical Gaussian Vasciek process of a single dimensional Gaussian-Markov stochastic process via Lévy and Weiner processes, respectively, can be achieved separately. Parameter estimation of the influx processes with one random effect (fixed effect) is succinctly to be focused.

A. Driven of the Vasicek Model via Weiner Process.

A one-dimensional Gaussian Vasciek process (that is an AR (1) like Markov-Gaussian could be defined as the solution to the stochastic differential equation in (1).

$$\partial Y(t) = -\delta Y(t)\partial t + s\,\partial B(t)$$

If the interest is at time "t" and "k" is a reference value for the rate,

$$\partial Y(t) = -\delta(Y(t) - k)\partial t + s\,\partial B(t); \qquad Y_0 = y_o \tag{3}$$

with $\sigma > 0$ and $\delta > 0$. Setting $Y(t) = X(t) - k$, we have

$$\partial Y(t) = \partial X(t) = -\delta Y(t)\partial t + s\,\partial B(t) \tag{4}$$

Hence,

$$\exp(\delta t)\partial Y(t) + \delta \exp(\delta t) Y(t)\partial t = s \exp(\delta t)\partial B(t) \tag{5}$$

Consequently, $\partial(\exp(\delta t)Y(t)) = s\exp(\delta t)\partial B(t)$,

By changing of variable technique, setting $Z_t = \exp(\delta t)Y(t)$ $(Z_0 = x_0 - k)$. So obtaining,

$$Z_t = (x_0 - k) + \int_0^t s\exp(\delta s)\partial B(t) \tag{6}$$

$$X(t) = Y(t) + k = \exp(-\delta t)Z_t + k \tag{7}$$

$$= \exp(-\delta t)\left((x_0 - k) + \int_0^t s\exp(\delta s)\partial B(t)\right) + k \tag{8}$$

$$= k + \exp(-\delta t)(x_0 - k) + \sigma\exp(-\delta t)\int_0^t \exp(\delta s)\partial B(t) \tag{9}$$

$X(t)$ is a unique Markov solution of (1) with

$$X(t) \sim N\left(k + \exp(-\delta t)(x_0 - k),\, \frac{s^2}{2\delta}(1 - \exp(Z\delta t)\right) \tag{10}$$

This implies that the distribution as "t" approaches infinity to the stationary distribution of $N\left(k, \frac{s^2}{2\delta}\right)$. That is, the stationary distribution has a density function. Therefore, the time change Brownian motion is

$$X(t) = k + \exp(-\delta t)(x_0 - k) + \exp(-\delta t)B_{\left(\frac{exp(\delta t)-1}{2\delta}\right)} \tag{11}$$

So the correlation of $X(t)$

$$Corr(X(t), X(t+k)) = \frac{\exp(-\delta k)}{\sqrt{(1-\exp(-\delta k))(1-\exp -2\delta(t+k))}} \tag{12}$$

As "t" approaches infinity

$$corr\left(X(t),\, X(t+k\right) = \exp(-\delta t) \tag{13}$$

B. Driven of the Vasicek Model via Lévy Process.

From (1)

$$\begin{cases} \partial Y(t) = -\delta Y(t)\partial t + s\,\partial B(t) \\ Y(0) = Y_0, \end{cases}$$

Integrating with respect to "t"

$$\int_0^t \left[\partial Y(t) = -\delta Y(t)\partial t + s\,\partial B(t)\right] \tag{14}$$

$$Y(t) = \exp(-\delta t)\left[Y_0 + \sigma \int_0^t \exp(\delta t)\partial B(s)\right] \tag{15}$$

For "s" of the standard Brownian motion that is a Markov-Gaussian process With auto-covariance function $V\left(s,\, t\right) = E\left(Y_{s,}\, Y_t\right)$.

Y is continuous in probability Markov process with mean, variance and covariance functions as:

$$E\left(Y(t)\right) = E\left[\exp(-\delta t)\left[Y_0 + s\int_0^t \exp(\delta t)\partial B(s)\right]\right] \tag{16}$$

$$E\left(Y(t)\right) = \exp(-\delta t)E\left(Y_0\right) \tag{17}$$

$$Var\left(Y(t)\right) = \frac{s^2}{2\delta} + \left(Var(Y_0) - \frac{s^2}{2\delta}\right)\exp(-2\delta t) \tag{18}$$

and

$$Cov\left(Y(s), Y(s+t)\right) =$$
$$= \left(Var(Y_0) + \frac{s^2}{2\delta}\left(\exp(-2\delta t) - 1\right)\right)\exp(-\delta(2s+t)) \tag{19}$$

If $Y_0 \sim N(0,\, \frac{s^2}{2\delta})$, Y becomes a strictly stationary Gaussian process with Covariance function and autocorrelation of Y being:

$$c(t) = Cov\left(Y(s),\, Y(s+t)\right) = \frac{s^2}{2\delta}\exp(-\delta t) \tag{20}$$

$$\rho(t) = Cov\left(Y(s),\, Y(s+t)\right) = \exp(-\delta t) \tag{21}$$

Equation (21) approximately concedes with autocorrelation of $X(t)$ of Vasciek Process driven Weiner in (13) as a process that satisfies the consistent property of a good estimator, that is as the sample size "n" increases $n \to \infty$, the true estimate approaches its true parameter.

$$\begin{cases} \partial Y(t) = -\delta Y(t)\partial t + s\,\partial B(\delta t) \\ Y(0) = Y_0, \end{cases} \tag{22}$$

IV. Parameter Estimation

From (10), $Z(t) \sim \left(0, \frac{\sigma^2}{2}\right)$ connotes that associated stationary distribution is normal with mean zero and variance $\frac{\sigma^2}{2}$. Relieving of $Z(t)$ to be a Lévy process such that $Y(t_0), Y(t_1),\, Y(t_2), \cdots\cdots, Y(t_n)$ be a sample of stationary process defined by (22) $y_0, y_1, y_2, \cdots\cdots, y_n$ their observed values and $x_t = \theta_1 y_{t-1}$. Then the transition density of the Vasciek model at one-dimensional (that is, at autoregressive of order one) is via Markovian – Gaussian probability density function as

$$f(y,t/x,s,\sigma^2) = \frac{\exp -\left(\frac{\left(y_t - xe^{-\frac{(t-s)}{\delta}}\right)^2}{\sqrt{\sigma^2\delta\left(1-e^{-\frac{2(t-s)}{\delta}}\right)}}\right)}{\sqrt{\pi^2\,\sigma^2\delta\left(1-e^{-\frac{2(t-s)}{\delta}}\right)}} \quad for\ -\infty < y_t < \infty \tag{23}$$

$$f(y,t/\theta_1, y_{t-1}, s, \sigma^2) = \frac{\exp -\left(\frac{\left(y_t - \theta_1 y_{t-1} e^{-\frac{(t-s)}{\delta}}\right)^2}{\sqrt{\sigma^2\delta\left(1-e^{-\frac{2(t-s)}{\delta}}\right)}}\right)}{\pi\sqrt{\sigma^2\delta\left(1-e^{-\frac{2(t-s)}{\delta}}\right)}} \tag{24}$$

The likelihood function of sample parameters via Markovian – Gaussian density $\Theta = \left\{\theta_1, \sigma^2, \delta\right\}$

$$L(\Theta) = fY(t_0)(y_0)\prod_{i=1}^{n} fY(t_k)(y_{tk-1}) \tag{25}$$

$$= \frac{1}{\sqrt{\pi\sigma^2}}\exp -\left(\frac{y_0^2}{\sigma^2}\right)\prod_{i=1}^{n}\frac{\exp -\left(\frac{\left(y_t - \theta_1 y_{t-1} e^{-\frac{(t-s)}{\delta}}\right)^2}{\sqrt{\sigma^2\delta\left(1-e^{-\frac{2(t-s)}{\delta}}\right)}}\right)}{\pi\sqrt{\sigma^2\delta\left(1-e^{-\frac{2(t-s)}{\delta}}\right)}} \tag{26}$$

Minimizing the observed the maximum likelihood via log-likelihood function

$$l(\Theta) = \log L(\Theta) = -\frac{(n+1)}{2}\log(\pi\sigma^2) - \frac{y_0^2}{2} -$$
$$\frac{1}{2}\sum_{i=1}^{n}\left(\delta\left(1-\exp(-\frac{2(t-s)}{\delta})\right)\right) - \sum_{i=1}^{n}\frac{\left(y_t - \theta_1 y_{t-1}e^{-\frac{(t-s)}{\delta}}\right)}{\left(\delta\sigma^2\left(1-e^{-\frac{(t-s)}{\delta}}\right)\right)} \tag{27}$$

$$\frac{\partial l(\Theta)}{\partial \theta_1} = 2y_{t-1}e^{-\frac{(t-s)}{\delta}} \sum_{i=1}^{n} \frac{\left(y_t - \theta_1 y_{t-1} e^{-\frac{(t-s)}{\delta}}\right)}{\left(\delta\sigma^2 \left(1 - e^{-\frac{(t-s)}{\delta}}\right)\right)} \tag{28}$$

Equating $\frac{\partial l(\Theta)}{\theta_1}$ to zero

$$2y_{t-1}e^{-\frac{(t-s)}{\delta}} \frac{\left[\sum_{i=1}^{n} y_t - \theta_1 e^{-\frac{(t-s)}{\delta}} \sum_{i=1}^{n} y_{t-1}\right]}{\left(\delta\sigma^2 \left(1 - e^{-\frac{(t-s)}{\delta}}\right)\right)} = 0$$

$$\widehat{\theta}_1 = \frac{\sum_{i=1}^{n} y_t}{e^{-\frac{(t-s)}{\delta}} \sum_{i=1}^{n} y_{t-1}} \tag{29}$$

$$\frac{\partial^2 l(\Theta)}{\partial \theta_1^2} = -2\left(y_{t-1}e^{-\frac{(t-s)}{\delta}}\right) \frac{e^{-\frac{(t-s)}{\delta}} \sum_{i=1}^{n} y_{t-1}}{\left(\delta\sigma^2 \left(1 - e^{-\frac{(t-s)}{\delta}}\right)\right)} \tag{30}$$

$$\frac{\partial l(\Theta)}{\partial (\sigma^2)} = -\frac{(n+1)}{2} \times \frac{\pi}{\sigma^2}$$

Equating $\frac{\partial l(\Theta)}{\partial \sigma^2}$ to zero

$$\Rightarrow \widehat{\sigma} = \sqrt{\frac{2\pi}{n+1}} \tag{31}$$

$$\frac{\partial^2 l(\Theta)}{\partial (\sigma^2)^2} = \frac{(n+1)}{2} \times \frac{\pi}{\sigma^3} \tag{32}$$

$$\frac{\partial l(\Theta)}{\partial \delta} = -\frac{n}{2}\left[1 - \left(e^{-\frac{2(t-s)}{\delta}} + \delta\left(\frac{2(t-s)}{\delta^2}\right)\right)\right]$$

$$-\left[\frac{1}{\sigma^2\delta^2}\left(1 - e^{-\frac{2(t-s)}{\delta}}\right)^{-1} + \frac{(t-s)\left(1 - e^{-\frac{(t-s)}{\delta}}\right)^{-2}}{\delta^3\sigma^2}\right.$$

$$\left.\times \left(\frac{2\theta_1 y_{t-1}(t-s)}{\delta^2}\right) \sum_{i=1}^{n} \left(y_t - \theta_1 y_{t-1} e^{-\frac{(t-s)}{\delta}}\right)\right]$$

$$= \frac{n}{2}\left[1 - e^{-\frac{2(t-s)}{\delta}} - \left(\frac{2(t-s)}{\delta}\right)\right] - \left[-\frac{1}{\sigma^2\delta^2}\left(1 - e^{-\frac{2(t-s)}{\delta}}\right)^{-1} + \right.$$

$$\left.\frac{(t-s)\left(1 - e^{-\frac{(t-s)}{\delta}}\right)^{-2}}{\delta^3\sigma^2} \left(\frac{2\theta_1 y_{t-1}(t-s)}{\delta^2}\right) \sum_{i=1}^{n} \left(y_t - \theta_1 y_{t-1} e^{-\frac{(t-s)}{\delta}}\right)\right] \tag{33}$$

Since $e^{(0)} \sim 1$ and as $\delta \to \infty$

$$\frac{\partial l(\Theta)}{\partial \delta} \approx \frac{n}{2}$$

$$\frac{\partial^2 l(\Theta)}{\partial \delta^2} = n\frac{(t-s)}{\delta}e^{-\frac{(t-s)}{\delta}} + n\frac{(t-s)}{\delta} - \left[\frac{2}{\sigma^2\delta^3}\left(1 - e^{-\frac{2(t-s)}{\delta}}\right)^{-1}\right.$$

$$+\frac{2(t-s)}{\sigma^2\delta^3}\left(1 - e^{-\frac{2(t-s)}{\delta}}\right)^{-2} + \frac{(t-s)}{\sigma^2\delta^4}\left(1 - e^{-\frac{2(t-s)}{\delta}}\right)^{-2}$$

$$\left.\times \left(\frac{2\theta_1 y_{t-1}(t-s)}{\delta^2}\right) \frac{2\theta_1 (t-s)^2 \sum_{i=1}^{n} y_{t-1} e^{-\frac{(t-s)}{\delta}}}{\delta^3}\right] \tag{34}$$

Constructing a gradient vector of the partial derivatives with respect to each parameter and adopting the Newton-Raphson iterative technique:

$$\gamma^{(m+1)} = \gamma^{(m)} + \left[-E\begin{pmatrix} \frac{\partial^2 l(\Theta)}{\partial \theta_1^2} & & \\ & \frac{\partial^2 l(\Theta)}{\partial \delta^2} & \\ & & \frac{\partial^2 l(\Theta)}{\partial (\sigma^2)^2} \end{pmatrix}\right]^{-1} \times$$

$$\begin{pmatrix} 2y_{t-1}e^{-\frac{(t-s)}{\delta}} \sum_{i=1}^{n} \frac{\left(y_t - \theta_1 y_{t-1} e^{-\frac{(t-s)}{\delta}}\right)}{\left(\delta\sigma^2\left(1 - e^{-\frac{(t-s)}{\delta}}\right)\right)} \\ \frac{\partial l(\Theta)}{\partial \sigma^2} \\ -\frac{(n+1)}{2}\frac{\pi}{\sigma^2} \end{pmatrix} \tag{35}$$

$$\gamma^{(m+1)} = \gamma^{(m)} + \left[I_{(n)}^{(m)}\right]^{-1} S^{(m)} \tag{36}$$

where $I_{(n)}^{(m)}$ and $S^{(m)}$ are the Fisher information and Score matrixes respectively to be evaluated by via iterative procedure with model performance criteria of Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC).

$$AIC = 2l(\Theta) + 2\dim(\Theta) \tag{37}$$

$$BIC = 2l(\Theta)\log(n-p) + 2\dim(\Theta) \tag{38}$$

where "n" is the sample size and "p" is the number of estimated parameters.

V. Application to Financial Market

The monthly exchange rate of Nigeria-Naira to CFA franc was subjected to the solution of the conflated processes. The CFA franc is the ISO currency of eight francophone West African countries, such that the currency does maintain the same value in each of those countries. The exchange rate was between January 2012 and June 2018.

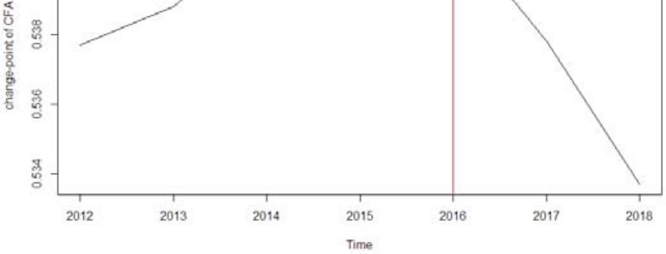

Fig. 1 Change-Point of the Naira to CFA exchange rate

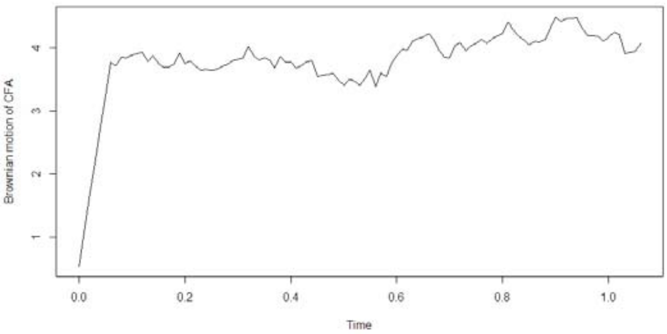

Fig. 2 Weiner and Lévy processes of the Naira to CFA exchange rate

From Fig. 1, the time change-point plot unfolded the evolution of a steady rising in the rate from 2012 to late 2013 before a constant rate was maintained from early 2014 until late 2015. A drastic time change-point occurred in at the beginning of first quarter of 2016, the red strike line unveiled 2016 as the year of change point. The change-point led to a continuous downward rate of exchange (jumps) from 2016 to June 2018.

TABLE I
ASCERTAINED CONDITIONS OF THE COMMINGLE LÉVY AND WEINER PROCESSES FOR VASCIEK MODEL

Conditional law	Lévy driven	Weiner driven	Commingle driven
Drift coefficient	-0.3011	-0.3012	-0.30010
Diffusion coefficient $\sqrt{2s}$	0.0790	0.0787	0.0748
Stationary coefficient $\frac{s}{r}$	0.0006	0.0006	0.0006
Distribution function	0.0003	0.0004	0.0037
Quantile function	0.0024	0.0028	0.0046

TABLE II
ESTIMATED PARAMETERS OF THE COMMINGLE LÉVY AND WEINER PROCESSES OF VASCIEK MODEL

Parameters	Lévy driven	Weiner driven	Commingle driven
θ_1	0.04971	0.04972	0.05620
δ	0.00167	0.00165	0.00169
s	0.00312	0.00310	0.0028
r	5	5	5
AIC	78.4756	78.2095	76.7209
BIC	74.0264	74.3090	73.0264

Comparing the variational parameters in the varied processes in Tables I and II, the diffusion rates of 0.0789, 0.0787 and 0.0748 were estimated for the Lévy, Weiner and commingle driven processes for the non-linear dynamical Vasciek model respectively with fixed effect. Among the diffusion rates, the merged driven processes of the Lévy and Weiner rates yielded a robust and accommodated lower rate. The evolutions of solution $y(t)$ descent continuous after five years move at 2016 after a steady undulated exchange rate from 2013 to 2016. Additionally, the three driven processes ascertained the density stationary condition of the Vasciek model via an approximated value that is $0.0006 < 1$ as well as approximately the distributional functions of 0.0004. The Autoregressive like coefficients (one-dimensional Vasciek model) of θ_1 s (0.04971, 0.04972 and $0.05620 < 1$) for the three driven process of the Vasciek respectively collaborate the stationary condition of the density stationary with fixed effect. However, the model performance (AIC & BIC) variability among the driven processes of the non-linear dynamical Vasciek model was not far apart but lesser in the meld driven process of the Vasciek model with 76.7209 and 73.0264 respectively with indication of a lesser performance error.

VI. CONCLUSION

Having introduced a commingle driven process of Lévy and Weiner to carve-out the Vasciek model, individuality and conflate driven processes were subjected to the conditional laws of the non-linear dynamical Vasciek model. The Vasciek models of the three driven processes ascertained density stationary process, diffusion rate estimate and drift coefficient via application to Naira-CFA exchange rate. The applicability of the driven processes of the non-linear dynamical Vasciek models with Gaussian-Markov as the white noise revealed that the influx driven processes of the Lévy and Weiner possessed a lesser miniature performance error.

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