# A Study about the Distribution of the Spanning Ratios of Yao Graphs

Maryam Hsaini, Mostafa Nouri-Baygi

***Abstract*—**A critical problem in wireless sensor networks is limited battery and memory of nodes. Therefore, each node in the network could maintain only a subset of its neighbors to communicate with. This will increase the battery usage in the network because each packet should take more hops to reach its destination. In order to tackle these problems, spanner graphs are defined. Since each node has a small degree in a spanner graph and the distance in the graph is not much greater than its actual geographical distance, spanner graphs are suitable candidates to be used for the topology of a wireless sensor network. In this paper, we study Yao graphs and their behavior for a randomly selected set of points. We generate several random point sets and compare the properties of their Yao graphs with the complete graph. Based on our data sets, we obtain several charts demonstrating how Yao graphs behave for a set of randomly chosen point set. As the results show, the stretch factor of a Yao graph follows a normal distribution. Furthermore, the stretch factor is in average far less than the worst case stretch factor proved for Yao graphs in previous results. Furthermore, we use Yao graph for a realistic point set and study its stretch factor in real world.

***Keywords*—**Wireless sensor network, spanner graph, Yao Graph.

## I. Introduction

Wireless networks are modeled as graphs in which wireless devices are represented by nodes and connections between them are represented by edges. As the number of edges in a wireless network increases, so does the energy consumption. If two nodes have no connection in the ranges of each other in a wireless network, they need to rely on multi-hop transmission. As a result, data routing becomes necessary because the value of the transmission field greatly affects the network topology and energy consumption [1]. On the other hand, decreasing the number of edges increases the number of hubs which induces high cost and energy consumption. Therefore, it is preferred to use constructions with small sizes in designing routing tables where the shortest paths closely approximate direct Euclidean distances [2].

During the past years, researchers have studied spanners intensively [7]-[15]. Let $\boldsymbol{G} = (\boldsymbol{V}.\boldsymbol{E})$ be a connected weighted graph where $\boldsymbol{V}$ and $\boldsymbol{E}$ refer to the sets of nodes and edges, respectively. Moreover, let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vertices in $\boldsymbol{G}$; i.e. $\boldsymbol{u}.\boldsymbol{v} \,\boldsymbol{\epsilon}\, \boldsymbol{V}$. We denote the distance in $\boldsymbol{G}$ between $\boldsymbol{u}$ and $\boldsymbol{v}$ by $\boldsymbol{d_G}\,(\boldsymbol{u}.\boldsymbol{v})$. This distance is the length of the shortest path between $\boldsymbol{u}$ and $\boldsymbol{v}$. For a sub-graph $\boldsymbol{G} = (\boldsymbol{V}.\boldsymbol{E}')$ of $\boldsymbol{G}$, where $\boldsymbol{E}' \subseteq \boldsymbol{E}$, we call $\boldsymbol{G}'$a spanner graph if for every $(\boldsymbol{u}.\boldsymbol{v}) \,\boldsymbol{\epsilon}\, \boldsymbol{V}$, $\boldsymbol{d_{G'}}\,(\boldsymbol{u}.\boldsymbol{v}) \leq \boldsymbol{t}.\boldsymbol{d_G}\,(\boldsymbol{u}.\boldsymbol{v})$. The value of $\boldsymbol{t}$ is referred to as the stretch factor or the spanning ratio of $\boldsymbol{G}'$. A geometric graph is a complete graph where for every pair of points $(\boldsymbol{u}, \boldsymbol{v})$, $\boldsymbol{d_G}(\boldsymbol{u}.\boldsymbol{v})$ is the Euclidean distance between the points [3].

In Yao graph $\boldsymbol{Y_k}\,(\boldsymbol{P})$ where $\boldsymbol{k}$ is an integer larger than $\boldsymbol{2}$, and $\boldsymbol{P}$ is a set of nodes in the plane, the area around node $\boldsymbol{u}$ can be divided into $\boldsymbol{k}$ cones. In each non-empty cone $\boldsymbol{C_u}$ we choose the shortest edge $(\boldsymbol{u}.\boldsymbol{v}) \in \boldsymbol{C_u}$ to reach the nearest neighbor node $\boldsymbol{v} \in \boldsymbol{P}$ and add $(\boldsymbol{u}.\boldsymbol{v})$ to $\boldsymbol{Y_k}$. The Yao graph can be classified as a spanner graph for $\boldsymbol{k} \geq \boldsymbol{4}$. Therefore, the shortest path from u to v for all pairs of nodes $\boldsymbol{u}.\boldsymbol{v} \in \boldsymbol{P}$ is not greater than $\boldsymbol{t} \cdot |\boldsymbol{uv}|$ where $\boldsymbol{t} \geq \boldsymbol{1}$ is referred to as the stretch factor of $\boldsymbol{Y_k}(\boldsymbol{P})$ and $|\boldsymbol{uv}|$ is the Euclidean distance between $\boldsymbol{u}$ and $\boldsymbol{v}$. The stretch factor previously calculated gave us a very big value which is not good enough so we have to work on resolving this problem. In this paper, we try to obtain a better understanding of the stretch factor for the Yao graph [4].

The rest of the paper is organized as follows: Related works are discussed in Section II; Section III explains the proposed method; the experimental results are reported in Section IV and Section V concludes the paper.

## II. Related Work

Molla has proved that $\boldsymbol{Y_2}$ and $\boldsymbol{Y_3}$ are not spanners but Yao graph for $\boldsymbol{k = 4}$ is a spanner with a stretch factor of $\boldsymbol{8\sqrt{2}\,(29 + 23\sqrt{2})}$ [4]. Keng and Xia [5] showed that the stretch factor of $\boldsymbol{Y_5}$ is in the range of [2.87, 3.74]. While the gap between the lower and the upper bound of the above interval is small, the bounds of the stretch factor of $\boldsymbol{Y_5}$ are very tight. Furthermore, they have proved that Yao graph with five cones is a spanner with a maximum stretch factor of $\boldsymbol{2 + \sqrt{3} \approx 3.74}$; and Yao graph $\boldsymbol{Y_k}$ is a spanner if and only if $\boldsymbol{k \geq 4}$. Similarly, it is important to focus on the tight bounds of other Yao graphs $\boldsymbol{Y_k}$ for $\boldsymbol{k \geq 4}$ and investigate them. Finally, they showed that the Yao-Yao graph for $\boldsymbol{k = 5}$ is not a sapanner.

Authors in [6] improved the bounds for odd values of $k \geq 5$ where the spanning ratio is at most $1/(1 - 2\,\sin(3\theta/4k))$ and they made an improvement over the former bound of $1/(1 - 2\,\sin(\theta\,/2))$ for even values of $k$ where $k \geq 7$. The lower bound of $Y_5$ spanning ratio of is 2.87. In addition, they reduced the upper bound of $Y_6$ spanning ratio from 17.6 to 5.8. The results obtained for the stretch factors of Yao graph with different values of parameter $k$ are shown in Table I [5].

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TABLE I
THE UPPER BOUND FOR THE STRETCH FACTOR OF THE YAO GRAPH

| Parameter $k$ | Spanning ratio of $Y_k$ |
|---|---|
| $k = 2, k = 3$ | not a spanner |
| $k = 4$ | $t = 8\sqrt{2}(26 + 23\sqrt{2})$ |
| $k = 5$ | $t = 2 + \sqrt{3}$ |
| $k = 6$ | $t = 5.8$ |
| $k \geq 7$ | $t = \frac{1}{1-2\,sin(\frac{\pi}{k})}$ for even $k$, <br> $t = \frac{1}{1-2\,sin(\frac{3\pi}{4k})}$ for odd $k$ |

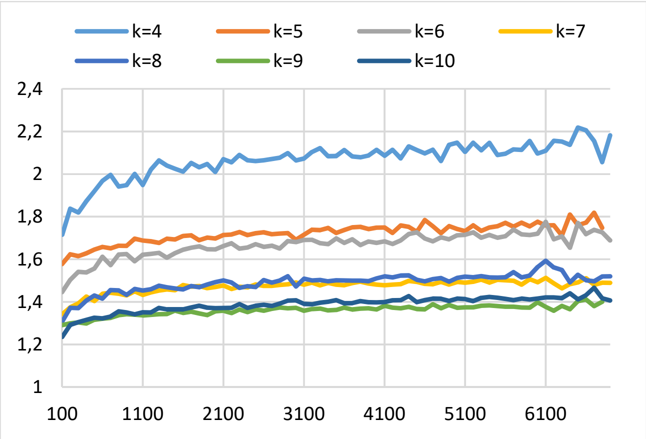


Fig. 1 The stretch factor for different values of k in different graph sizes

Nag and Roy [2] mainly dealt with the various graphs used as spanners and also interpreted and reviewed some of the relevant algorithms that are concerned with the Yao graph. They discussed the modifications to these graphs and also explained how this graph could be used to save energy and improve performance. The Yao-Yao graph, for example, can be used as a spanner for certain values of the stretch factor. In Yao graph, perpendicular projections are made from each node within a sector to the anticlockwise wall and the source node is then connected to its nearest nodes in each cone.

## III. THE PROPOSED METHOD

As described in Section II, there have been many studies about the maximum stretch factor for specific networks. However, as long as we know, none of them worked on finding the average stretch factor and/or described the distribution of the stretch factor of a network for a random set of points. This paper is intended to obtain the distribution of the stretch factor in order to discuss what it really is. If we show that the distribution of the stretch factor is close to its upper bound, we realize that the Yao graph is not appropriate for wireless networks. On the other hand, if the distribution is far from the upper bound, the Yao graph is a candidate for using in wireless network topologies. For this purpose, we generate different random graphs with different sizes and draw Yao graphs for them. Besides, we calculate the stretch factors for these graphs. Furthermore, we compare the above stretch factors with the results reported in the literature. The researchers have failed to provide the mean stretch factor.

## IV. EXPERIMENTS

This paper has used random graphs in order to obtain the spread of the stretch factor. To this aim, we have created a number of complete graphs by generating the random coordinates of each node using the Java language. When the user enters an integer that represents the number of nodes, both coordinates will be double-precision numbers. We also conducted two experiments to evaluate our method. In the first experiment, the goal was to obtain a stretch factor for each graph. As seen in Fig. 1, we have generated 1213 graphs randomly where the number of nodes ranges from 100 to 6900. We have considered the average stretch factors of the sets of graphs with the same numbers of nodes. As depicted in Fig. 1, increasing the value of variable $\boldsymbol{k}$ leads to a decrease in the stretch factor.

We considered 1213 graphs. Then, we calculated their stretch factor. The results of this experiment for different values of $\boldsymbol{k}$ demonstrate that the spanning ratio changes in a given range, as shown in Fig. 1. When $\boldsymbol{k} = \mathbf{4}$, the stretch factors for different graphs are in the range [1.8, 2.2] while the maximum value of the stretch factor is $\mathbf{8}\sqrt{\mathbf{2}}\,(\mathbf{29} + \mathbf{23}\sqrt{\mathbf{2}})$ which is a large number approximately equal to 696.

For $\boldsymbol{k} = \mathbf{5}$, the results show that the stretch factor varies between 1.6 and 1.8. As a result of the first experiment, the spanning ratio varies in a predefined range. In addition, as the value of variable $k$ increases, the average stretch factor decreases.

The second experiment aims to calculate the distribution of the stretch factor in the given range. As illustrated in Fig. 2, the horizontal axis shows the stretch factor, and the vertical axis shows the number of graphs. For example, for the range of 2 to 2.1, there are 404 graphs. Strictly speaking, the stretch factor is in this range as Fig. 2 shows. The same experiment is performed for $\boldsymbol{k} = \mathbf{5}$. As the results in Fig. 2 show, the resulting distributions resemble the normal distribution. As illustrated in Fig. 3 both mean and standard deviation decrease as $k$ increases.

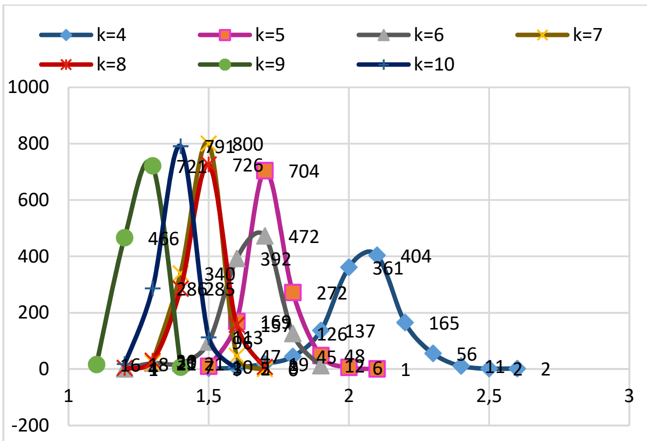


Fig. 2 Distribution of the stretch factor

In order to show the applicability of spanner graphs for real networks, we use the cell phone network in Thi Qar, Iraq

(Zain Network) as a case study. As illustrated in Fig. 5, there are about 146 communication towers represented by red points. We scanned a map of the communication towers in Thi Qar in order to obtain the coordinates of each point and to show the distribution of the towers at the points, as shown in Fig. 6 (a).

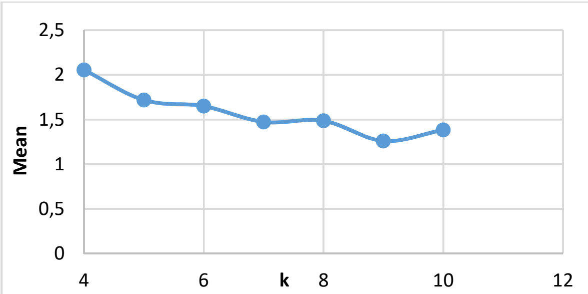


Fig. 3 Mean of the stretch factors for different values of $k$

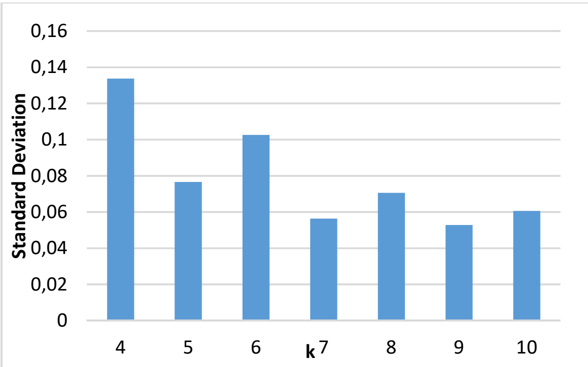


Fig. 4 Standard deviation of the stretch factors for different values of $k$

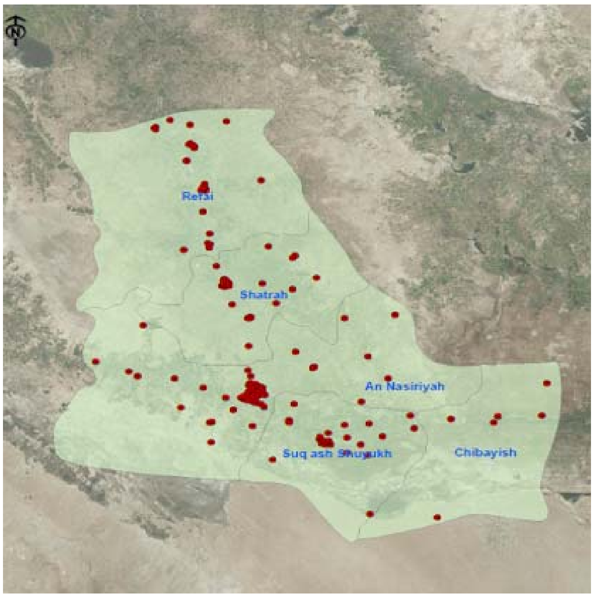


Fig. 5 The distribution of the mobile network towers in Thi Qar, Iraq

Then, the Yao graph for $k = 4$ was generated, as illustrated in Fig. 6 (b). Subsequently, the stretch factor for $k = 4$ was calculated which is close to 1.65. Therefore, the distance between any two nodes in the generated graph was less than or equal to 1.65 times the actual distance between the nodes. In the same way, the Yao graphs for $k = 6$ and $k = 8$ were generated.

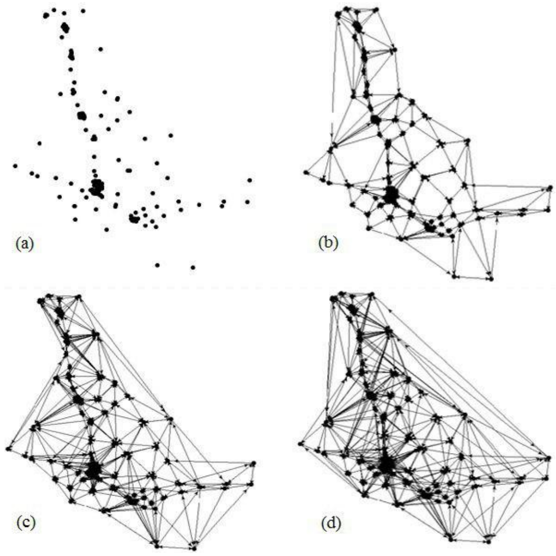


Fig. 6 (a) The distribution of the mobile network towers in Thi Qar, Iraq, (b) the Yao graph for $k = 4$, (c) the Yao graph for $k = 6$, (d) the Yao graph for $k = 8$

Then, the stretch factors for $k = 6$ and $k = 8$ were generated which are depicted in Figs. 6 (c) and (d), respectively. It was found that the stretch factor was 1.40 for $k = 6$ and 1.43 for $k = 8$.

In this real network, the number of edges between Yao graph nodes was calculated as 384 for Fig. 6 (b), 574 for Fig. 6 (c), and 740 for Fig. 6 (d). Mean and standard deviation of the obtained stretch factors were calculated for each $k$.

## V. Conclusion

During the past years, researchers have studied spanner graphs intensively [7]-[15]. However, as long as we know, none of them worked on finding the average stretch factor and/or described the distribution of the stretch factor of a network for a random set of points. In this paper we tried to find the distribution of the stretch factor of Yao graphs for a random set of points. We showed that the stretch factor of Yao graphs follows a normal distribution with mean value usually less than 2, which is far from the upper bound proved in the literature for different values of $k$, especially for small values. These results proved that Yao graphs are good candidates for using in wireless network topologies to reduce the complexity of the network and power consumption.

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