World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:12, No:9, 2018

Positive Solutions for Systems of Nonlinear Third-Order Differential Equations with p-Laplacian

Li Xiguang

Abstract**—**In this paper, by constructing a special set and utilizing fixed point theory, we study the existence and multiplicity of the positive solutions for systems of nonlinear third-order differential equations with p-laplacian, which improve and generalize the result of related paper.

Keywords**—**p-Laplacian, cone, fixed point theorem, positive solution.

I. Introduction

THE boundary value problems of differential equation with p-Laplacian arises in a variety of problems related to applied mathematics, physics and engineering. However there is still a little research about it. In recent years, some results concerning the problems have been obtained. We refer the readers to [1]-[6] and the references cited therein. In the thesis [5], the author investigated the following elliptic systems:

$$\begin{cases} \Delta u + \lambda k_1(|\ x\ |)f(u,v) = 0, \\ \Delta v + \mu k_2(|\ x\ |)g(u,v) = 0, \\ u\ |_{\partial\Omega} = v\ |_{\partial\Omega} = 0. \end{cases}$$

In the thesis [6], the author investigated the following coupled singular boundary value problems:

$$\begin{cases} (\phi_p(u''(t)))' + \omega_1(t)f_1(t,v(t)) = 0, \quad t\in(0,1), \\ (\phi_p(v''(t)))' + \omega_2(t)f_2(t,u(t)) = 0, \quad t\in(0,1), \\ \alpha_1u(0)-\beta_1u'(0)=0, \gamma_1u(1)+\delta_1u'(1)=0, u''(0)=0, \\ \alpha_2v(0)-\beta_2v'(0)=0, \gamma_2v(1)+\delta_2v'(1)=0, v''(0)=0. \end{cases}$$

Motivated by the thesis [5], [6], in this paper. We consider the following systems of third-orderdi boundary value problems:

$$\begin{cases} (\phi_p(u''(t)))' + \omega_1(t)f_1(t,u(t),v(t)) = 0, \quad t\in(0,1), \\ (\phi_p(v''(t)))' + \omega_2(t)f_2(t,u(t),v(t)) = 0, \quad t\in(0,1), \\ \alpha_1u(0)-\beta_1u'(0)=0, \gamma_1u(1)+\delta_1u'(1)=0, u''(0)=0, \\ \alpha_2v(0)-\beta_2v'(0)=0, \gamma_2v(1)+\delta_2v'(1)=0, v''(0)=0. \end{cases} \tag{1}$$

where $\phi_p(s) = |\ s\ |^{p-2}\ s, p\geq 2, \alpha_i, \gamma_i > 0, \beta_i, \delta_i \geq 0$, $f_i \in C((0,1)\times[0,+\infty)\times[0,+\infty),[0,+\infty))$, $\omega_i(t) \in C((0,1),[0,+\infty))$ and $f_i, \omega_i(t)$ may be singular at $t=0,1$. In thesis [5], the control functions need to be continuous and monotonic. In thesis [6], f_1 and f_2 are functions of one variable. Different from the works mentioned above, our purpose here is to deal with more general functions than that of thesis [6]. Moreover, the conditions that we used are weaker than that of thesis [5]. The organization of this paper is as follows, we shall first introduce some definitions and lemmas in the rest of this section. The main results will be stated and proved in Section II. In Section III, we present an example to check our result.

Li Xiguang is with the College of Mathematics and Physics Qingdao University of Science and Technology, Qingdao, 266061, China (e-mail: lxg0417@sina.com).

For the sake of convenience, we first give some conditions.

$(\mathbf{H_1})$ $f_i(t,u(t),v(t)) \leq g_i(t)h_i(u(t),v(t)), g_i(t):(0,1)\to[0,+\infty)$ may be singular at $t=0,1$, $h_i(u,v):[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous, i=1,2.

$(\mathbf{H_2})$ $\omega_i\in C((0,1),[0,+\infty)), \omega_i$ may be singular at $t=0,1$, and

$$0\leq\int_0^1\omega_i(s)g_i(s)ds<+\infty.$$

$(\mathbf{H_3})$

$$0\leq\limsup_{(u,v)\to 0}\frac{h_1(u,v)}{(u+v)^{p-1}}<\eta_1^{p-1}$$

and

$$0\leq\limsup_{(u,v)\to 0}\frac{h_2(u,v)}{(u+v)^{p-1}}<\eta_2^{p-1}.$$

$(\mathbf{H_4})$

$$(M_1^{-1}\xi_1)^{p-1}<\liminf_{(u,v)\to\infty}\frac{f_1(s,u,v)}{(u+v)^{p-1}}\leq\infty,$$

or

$$(M_2^{-1}\xi_2)^{p-1}<\liminf_{(u,v)\to\infty}\frac{f_2(s,u,v)}{(u+v)^{p-1}}\leq\infty.$$

$(\mathbf{H_5})$

$$0\leq\limsup_{(u,v)\to\infty}\frac{h_1(u,v)}{(u+v)^{p-1}}<\eta_1^{p-1}$$

and

$$0\leq\limsup_{(u,v)\to\infty}\frac{h_2(u,v)}{(u+v)^{p-1}}<\eta_2^{p-1}.$$

$(\mathbf{H_6}$

$$(M_1^{-1}\xi_1)^{p-1}<\liminf_{(u,v)\to 0}\frac{f_1(s,u,v)}{(u+v)^{p-1}}\leq\infty$$

or

$$(M_2^{-1}\xi_2)^{p-1}<\liminf_{(u,v)\to 0}\frac{f_2(s,u,v)}{(u+v)^{p-1}}\leq\infty.$$

where η_i and ξ_i $(i=1,2)$ are constants such that

$$0<\eta_i(\max_{0\leq t\leq 1}\int_0^1 G_i(t,\tau)\phi_q(\int_0^\tau\omega_i(s)g_i(s)ds)d\tau)\leq\frac{1}{2}$$

and

$$\xi_i\int_\theta^{1-\theta}G_i(\frac{1}{2},\tau)\phi_q(\int_\theta^\tau\omega_i(s)ds)d\tau\geq 1.$$

It is easy to show the systems (1) are equivalent to the following integral equations

$$\begin{cases} u(t)=\int_0^1 G_1(t,\tau)\phi_q(\int_0^\tau \omega_1(s)f_1(s,u(s),v(s))ds)d\tau, \\ v(t)=\int_0^1 G_2(t,\tau)\phi_q(\int_0^\tau \omega_2(s)f_2(s,u(s),v(s))ds)d\tau, \end{cases} \tag{2}$$

where q is a constant such that $\frac{1}{p}+\frac{1}{q}=1$, and

$$G_i(t,s)=\begin{cases} \dfrac{\varphi_i(t)\psi_i(s)}{\rho_i}, 0\le s\le t\le 1, \\ \dfrac{\varphi_i(s)\psi_i(t)}{\rho_i}, 0\le t\le s\le 1, \end{cases}$$

where

$$\rho_i=\alpha_i\gamma_i+\alpha_i\delta_i+\beta_i\gamma_i,$$
$$\varphi_i(t)=\gamma_i+\delta_i-\gamma_i t,$$
$$\psi_i(t)=\beta_i+\alpha_i t, 0\le t\le 1.$$

Let

$$A(u,v)(t)=\int_0^1 G_1(t,\tau)\phi_q(\int_0^\tau \omega_1(s)f_1(s,u(s),v(s))ds)d\tau,$$

$$B(u,v)(t)=\int_0^1 G_2(t,\tau)\phi_q(\int_0^\tau \omega_2(s)f_2(s,u(s),v(s))ds)d\tau,$$

$$F(u,v)(t)=(A(u,v)(t),B(u,v)(t)).$$

Then systems (2) are equivalent to the fixed point equation

$$F(u,v)=(u,v)$$

in the Banach space $E=X\times X$, where

$$X=\{u:u,\phi_p(u^{''})\in C^1[0,1]\}.$$

The following fixed -point theorem of cone expansion and compression type is crucial in the following argument.

Lemma 1 [7] Let K be a cone in Banach space E. Assume Ω_1,Ω_2 are open subsets of E with $0\in\Omega_1,\overline{\Omega}_1\subset\Omega_2$, and let $F:K\bigcap(\overline{\Omega}_2\setminus\Omega_1)\to K$ be a completely continuous operator such that either

$$\|Fu\|\le\|u\|,u\in K\bigcap\partial\Omega_1, \|Fu\|\ge\|u\|,u\in K\bigcap\partial\Omega_2;$$

or

$$\|Fu\|\ \ge\|u\|,u\in K\bigcap\partial\Omega_1, \|Fu\|\le\|u\|,u\in K\bigcap\partial\Omega_2,$$

then F has a fixed point in $K\bigcap(\overline{\Omega}_2\setminus\Omega_1)$.

In what follows we set

$$\|(u,v)\|=\|u\|+\|v\|,$$

where

$$\|u\|=\max_{t\in[0,1]}|u(t)|.$$

In order to apply Lemma 1, we let K be the cone defined by

$$K=\{(u,v):(u,v)\in E:u,v\ge 0, \min_{t\in[\theta,1-\theta]}(u(t)+v(t))\ge M(\|u\|+\|v\|)\},$$

where $\theta\in(0,\frac{1}{2})$, $M=\min\{M_1,M_2\}$, and

$$M_i=\min\{\frac{\delta_i+\theta\gamma_i}{\gamma_i+\delta_i},\frac{\theta\alpha_i+\beta_i}{\alpha_i+\beta_i}\}.$$

Lemma 2 [8] If $p\ge 2,\frac{1}{p}+\frac{1}{q}=1$, then

$$|\ \phi_q(x)-\phi_q(y)\ |\le\phi_q(x-y).$$

Lemma 3 Suppose that conditions $(H_1)-(H_2)$ hold, then $F:K\to K$ is completely continuous.

Proof: First we show that $F(K)\subset K$.
$\forall(u,v)\in K,t\in[0,1]$, we have

$$\begin{aligned} &A(u,v)(t)\\ &=\int_0^1 G_1(t,\tau)\phi_q(\int_0^\tau \omega_1(s)f_1(s,u(s),v(s))ds)d\tau\\ &\le\int_0^1 G_1(\tau,\tau)\phi_q(\int_0^\tau \omega_1(s)f_1(s,u(s),v(s))ds)d\tau. \end{aligned}$$

Hence

$$\begin{aligned} &\|A(u,v)\|\\ &\le\int_0^1 G_1(\tau,\tau)\phi_q(\int_0^\tau \omega_1(s)f_1(s,u(s),v(s))ds)d\tau. \end{aligned}$$

On the other hand, for $t\in[\theta,1-\theta]$, $\dfrac{G_i(t,\tau)}{G_i(\tau,\tau)}\ge M_i,\tau\in[0,1]$, we have

$$\begin{aligned} &\min_{\theta\le t\le 1-\theta}A(u,v)(t)\\ &=\min_{\theta\le t\le 1-\theta}\int_0^1 G_1(t,\tau)\phi_q(\int_0^\tau \omega_1(s)f_1(s,u(s),v(s))ds)d\tau\\ &\ge M_1\int_0^1 G_1(\tau,\tau)\phi_q(\int_0^\tau \omega_1(s)f_1(s,u(s),v(s))ds)d\tau\\ &\ge M_1\parallel A(u,v)\parallel. \end{aligned}$$

Similarly

$$\min_{\theta\le t\le 1-\theta}B(u,v)(t)\ge M_2\parallel B(u,v)\parallel.$$

Thus

$$\begin{aligned} &\min_{\theta\le t\le 1-\theta}(A(u,v)(t)+B(u,v)(t))\\ &\ge\min_{\theta\le t\le 1-\theta}A(u,v)(t)+\min_{\theta\le t\le 1-\theta}B(u,v)(t)\\ &\ge M_1\parallel A(u,v)\parallel+M_2\parallel B(u,v)\parallel\\ &\ge M\parallel(A(u,v),B(u,v))\parallel. \end{aligned}$$

We conclude that $F(K)\subset K$.

Next we show that $F:K\to K$ is completely continuous.

Let $\forall D \in K$ be a bounded set, i.e. $\exists M > 0$ such that $\forall (u,v) \in K, \|(u,v)\| \leq M$, we have

$$\begin{aligned}
&A(u,v)(t)\\
&= \int_0^1 G_1(t,\tau)\phi_q(\int_0^\tau \omega_1(s)f_1(s,u(s),v(s))ds)d\tau,\\
&\leq \int_0^1 G_1(\tau,\tau)\phi_q(\int_0^\tau \omega_1(s)g_1(s)h_1(u(s),v(s))ds)d\tau\\
&\leq \{\max \phi_q(h_1(u(s),v(s)) : 0 \leq s \leq 1\}\\
&\times \int_0^1 G_1(\tau,\tau)\phi_q(\int_0^\tau \omega_1(s)g_1(s)ds)d\tau\\
&\leq \{\max \phi_q(h_1(u(s),v(s)) : 0 \leq s \leq 1\}\\
&\times \int_0^1 G_1(\tau,\tau)d\tau\phi_q(\int_0^1 \omega_1(s)g_1(s)ds)\\
&= N_1 < \infty.
\end{aligned}$$

So $\| A(u,v) \| \leq N_1$.

Similarly,

$$\| B(u,v) \| \leq N_2 < \infty.$$

Hence,

$$\begin{aligned}
\| F(u,v) \| &= \| A(u,v) \| + \| B(u,v) \|\\
&\leq N_1 + N_2 < +\infty.
\end{aligned}$$

Correspondingly, $F : K \to K$ is bounded uniformly.

Now, we show that F is equicontinuous. The continuity of $G_i(t,s)$ on $[0,1]\times[0,1]$ implies that $G_i(t,s)$ is continuous uniformly. i.e.
$\forall \varepsilon > 0, \exists \delta > 0$, such that $\forall \xi \in [0,1], |t_1 - t_2| < \delta$, we have

$$\begin{aligned}
&|G_i(t_1,\xi) - G_i(t_2,\xi)|\\
&< \tfrac{\varepsilon}{2}(\{\max \phi_q(h_i(u(s),v(s)) : 0 \leq s \leq 1\}\\
&\times \phi_q(\int_0^1 \omega_i(s)g_i(s)ds))^{-1}.
\end{aligned}$$

Hence, $\forall D \in K$, we have

$$\begin{aligned}
&|F(u,v)(t_1) - F(u,v)(t_2)|\\
&= |(A(u,v)(t_1), B(u,v)(t_1)) - (A(u,v)(t_2), B(u,v)(t_2))|\\
&= |(A(u,v)(t_1) - A(u,v)(t_2)), (B(u,v)(t_1) - B(u,v)(t_2))|\\
&\leq |(A(u,v)(t_1) - A(u,v)(t_2))|\\
&+|(B(u,v)(t_1) - B(u,v)(t_2))|.\\
&= |\int_0^1 (G_1(t_1,\tau) - G_1(t_2,\tau))\\
&\times \phi_q(\int_0^\tau \omega_1(s)g_1(s)h_1(u(s),v(s))ds)d\tau|\\
&+|\int_0^1 (G_2(t_1,\tau) - G_2(t_2,\tau))\\
&\times \phi_q(\int_0^\tau \omega_2(s)g_2(s)h_2(u(s),v(s))ds)d\tau|\\
&\leq |\int_0^1 (G_1(t_1,\tau) - G_1(t_2,\tau))\\
&\times \max\{\phi_q(h_1(u(s),v(s)) : 0 \leq s \leq 1\}\\
&\times \phi_q(\int_0^1 \omega_1(s)g_1(s)ds)d\tau \,|\\
&+ |\int_0^1 (G_2(t_1,\tau) - G_2(t_2,\tau))\\
&\times \max\{\phi_q(h_2(u(s),v(s)) : 0 \leq s \leq 1\}\\
&\times \phi_q(\int_0^1 \omega_2(s)g_2(s)ds)d\tau \,|\\
&\leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This means that $F(K)$ is equicontinuous, so $F(K)$ is relatively compact in K.

Finally we show $F : K \to K$ is continuous.

Let $\{(u_n, v_n)\} \subset K$ be sequence such that $(u_n, v_n) \to (u_0, v_0), n \to \infty$, we have

$$\begin{aligned}
&|F(u_n,v_n)(t) - F(u_0,v_0)(t)|\\
&\leq \int_0^1 G_1(\tau,\tau)|\phi_q(\int_0^1 \omega_1(s)f_1(s,u_n(s),v_n(s))ds)\\
&- \phi_q(\int_0^1 \omega_1(s)f_1(s,u_0(s),v_0(s))ds)|d\tau\\
&+ \int_0^1 G_2(\tau,\tau)|\phi_q(\int_0^1 \omega_2(s)f_2(s,u_n(s),v_n(s))ds)\\
&- \phi_q(\int_0^1 \omega_2(s)f_2(s,u_0(s),v_0(s))ds)|d\tau\\
&\leq \int_0^1 G_1(\tau,\tau)d\tau|\phi_q(\int_0^1 \omega_1(s)(f_1(s,u_n(s),v_n(s)\\
&- f_1(s,u_0(s),v_0(s))ds)|\\
&+ \int_0^1 G_2(\tau,\tau)d\tau|\phi_q(\int_0^1 \omega_2(s)(f_2(s,u_n(s),v_n(s))\\
&- f_2(s,u_0(s),v_0(s))ds)|.
\end{aligned}$$

By (H_2), Lebesgue dominated convergence theorem and the continuity of f_1, f_2, we have

$$\|F(u_n, v_n) - F(u_0, v_0)\| \to 0, n \to \infty,$$

thus F is continuous.

By above arguments, we claim F is completely continuous.

II. Conclusions

Theorem 1 *Suppose that conditions $(H_1) - (H_4)$ hold, then systems (1) has at least one positive solution.*

Proof:

By (H_3), we may choose $r > 0$, for any $\|u\| + \|u\| \le r$ we have

$$h_1(u, v) \le \eta_1^{p-1}(u + v)^{p-1},$$

and

$$h_2(u, v) \le \eta_2^{p-1}(u + v)^{p-1}.$$

Set

$$\Omega_1 = \{(u, v) : (u, v) \in K; \|(u, v)\| < r\}$$

If $(u, v) \in K \bigcap \partial\Omega_1$ then $\|u\| + \|v\| \le r$, we have

$$\begin{aligned}
&A(u, v)(t)\\
&= \int_0^1 G_1(t, \tau)\phi_q(\int_0^\tau \omega_1(s) f_1(s, u(s), v(s))ds)d\tau,\\
&\le \int_0^1 G_1(t, \tau)\phi_q(\int_0^\tau \omega_1(s) g_1(s) h_1(u(s), v(s))ds)d\tau\\
&\le \eta_1\|(u, v)\| \max_{0\le t\le 1} \int_0^1 G_1(t, \tau)\phi_q(\int_0^\tau \omega_1(s) g_1(s)ds)d\tau\\
&\le \tfrac{1}{2}\|(u, v)\|.
\end{aligned}$$

Which implies

$$\|A(u, v)\| \le \frac{1}{2}\|(u, v)\|.$$

Similarly,

$$B(u, v)(t) \le \frac{1}{2}\|(u, v)\|.$$

Hence, for $(u, v) \in K \bigcap \partial\Omega_1$, we have

$$\|F(u, v)\| = \|A(u, v)\| + \|B(u, v)\| \le \| (u, v) \| . \qquad (3)$$

By (H_4), if we further assume

$$(M_1^{-1}\xi_1)^{p-1} < \liminf_{(u,v)\to\infty} \frac{f_1(s, u, v)}{(u + v)^{p-1}} \le \infty,$$

then there is an $\overline{R} > r$, for any $\|u\| + \|v\| > \overline{R}$, we have

$$f_1(s, u, v) \ge (M^{-1}\xi_1(u + v))^{p-1}.$$

Let $R > \max\{\overline{R}, M_1^{-1}\overline{R}, M_2^{-1}\overline{R}\}$, we set

$$\Omega_2 = \{(u, v) : (u, v) \in K; \|(u, v)\| < R\}.$$

If $(u, v) \in K \bigcap \partial\Omega_2$, then

$$\begin{aligned}
&A(u, v)(\tfrac{1}{2})\\
&= \int_0^1 G_1(\frac{1}{2}, \tau)\phi_q(\int_0^\tau \omega_1(s) f_1(s, u(s), v(s))ds)d\tau\\
&\ge M_1^{-1}\xi_1 \int_\theta^{1-\theta} G_1(\frac{1}{2}, \tau)\\
&\times\phi_q(\int_\theta^\tau \omega_1(s)(u(s) + v(s))^{p-1}ds)d\tau\\
&\ge \xi_1(\|u\| + \|v)\|) \int_\theta^{1-\theta} G_1(\frac{1}{2}, \tau)\phi_q(\int_\theta^\tau \omega_1(s)ds)d\tau\\
&\ge \|(u, v)\|.
\end{aligned}$$

which implies

$$\|A(u, v)\| \ge \|(u, v)\|.$$

Similarly

$$\|B(u, v)\| \ge \|(u, v)\|.$$

An analogous estimate holds for f_2 in condition (H_4).

Hence, for $(u, v) \in K \bigcap \partial\Omega_2$ we have

$$\|F(u, v)\| = \|A(u, v)\| + \|B(u, v)\| \ge\| (u, v) \| . \qquad (4)$$

By Lemma 1, F has at least one fixed point $(u, v) \in K \bigcap(\overline{\Omega}_2 \setminus \Omega_1)$. So, systems (1) has at least one positive solution.

Theorem 2 *Suppose that conditions (H_1), (H_2), (H_5), (H_6) hold, then systems (1) has at least one positive solution.*

Proof: By(H_5), there exist $R_0, \varepsilon_i > 0$, for $\|u\|+\|v\| \ge R_0$, we have

$$h_i(u, v) \le (\eta_i - \varepsilon_i)^{p-1}(u + v)^{p-1}.$$

Let

$$a = \max_{i=1,2} \max\{\phi_q(h_i(u, v)) : u(t) \le R_0, v(t) \le R_0\},$$

$$R > \max\{\frac{a}{\varepsilon_1}, \frac{a}{\varepsilon_2}\}.$$

We set

$$\Omega_1 = \{(u, v) : (u, v) \in K; \|(u, v)\| < R\},$$

if $(u,v)\in K\bigcap\partial\Omega_1$, then

$$\begin{aligned}
&A(u,v)(t)\\
&=\int_0^1 G_1(t,\tau)\phi_q(\int_0^\tau \omega_1(s)f_1(s,u(s),v(s))ds)d\tau\\
&\le\int_0^1 G_1(t,\tau)\phi_q(\int_0^\tau \omega_1(s)g_1(s)h_1(u(s),v(s))ds)d\tau\\
&=\int_0^1 G_1(t,\tau)\phi_q(\int_{\|u\|+\|v\|\ge R_0} \omega_1(s)g_1(s)\\
&\times h_1(u(s),v(s))ds)d\tau\\
&+\int_0^1 G_1(t,\tau)\phi_q(\int_{\|u\|+\|v\|\le R_0} \omega_1(s)g_1(s)\\
&\times h_1(u(s),v(s))ds)d\tau\\
&\le(\eta_1-\varepsilon_1)(\|u(t)\|+\|v(t)\|)\int_0^1 G_1(t,\tau)\\
&\times\phi_q(\int_{\|u\|+\|v\|\ge R_0}\omega_1(s)g_1(s)ds)d\tau\\
&+a\int_0^1 G_1(t,\tau)\phi_q(\int_{\|u\|+\|v\|\le R_0}\omega_1(s)g_1(s)ds)d\tau\\
&\le[(\eta_1-\varepsilon_1)(\|u\|+\|v\|)+a]\max_{0\le t\le 1}\int_0^1 G_1(t,\tau)\\
&\times\phi_q(\int_0^\tau\omega_1(s)g_1(s)ds)d\tau\\
&\le\frac{1}{2}\|(u,v)\|.
\end{aligned}$$

Similarly,

$$B(u,v)(t)\le\frac{1}{2}\|(u,v)\|.$$

Hence, for any $(u,v)\in K\bigcap\partial\Omega_1$, we have

$$\|F(u,v)\|=\|A(u,v)\|+\|B(u,v)\|\le\|(u,v)\|. \tag{5}$$

On the other hand, by (H_6), if we further assume

$$(M_1^{-1}\xi_1)^{p-1}<\liminf_{(u,v)\to 0}\frac{f_1(s,u,v)}{(u+v)^{p-1}}\le\infty,$$

then there is an $0<r<R$, for any $\|u\|+\|v\|\le r$, we have

$$f_1(s,u,v)\ge(M^{-1}\xi_1(u+v))^{p-1}.$$

Set

$$\Omega_2=\{(u,v):(u,v)\in K;\|(u,v)\|<r\},$$

similar to the proof of Theorem 1, for $(u,v)\in K\bigcap\partial\Omega_2$, we have

$$\|F(u,v)\|=\|A(u,v)\|+\|B(u,v)\|\ge\|(u,v)\|. \tag{6}$$

An analogous estimate holds for f_2 in condition (H_6).

By Lemma 1, F has at least one fixed point $(u,v)\in K\bigcap(\overline{\Omega}_1\setminus\Omega_2)$, so system (1) has at least one positive solution.

Theorem 3 *Suppose that conditions* $(H_1)-(H_4)$ *and* (H_6) *hold, then systems (1) has at least two positive solutions* (u_1,v_1) *and* (u_2,v_2) *with* $\|(u_1,v_1)\|<r<\|(u_2,v_2)\|$.

Proof By (H_4) if we further assume that

$$(M_1^{-1}\xi_1)^{p-1}<\liminf_{(u,v)\to\infty}\frac{f_1(s,u,v)}{(u+v)^{p-1}}\le\infty,$$

then we choose a $R_1>r$ large sufficiently, for any $\|u\|+\|v\|>R_1$, we have

$$f_1(s,u,v)\ge(M^{-1}\xi_1(u+v))^{p-1}.$$

Set

$$\Omega_{R_1}=\{(u,v):(u,v)\in K;\|(u,v)\|<R_1\},$$

similarly, for any $(u,v)\in K\bigcap\partial\Omega_{R_1}$, we have

$$\|F(u,v)\|=\|A(u,v)\|+\|B(u,v)\|\ge\|(u,v)\|. \tag{7}$$

An analogous estimate holds for f_2 in condition (H_4).

On the other hand, by (H_6) we assume that

$$(M_1^{-1}\xi_1)^{p-1}<\liminf_{(u,v)\to 0}\frac{f_1(s,u,v)}{(u+v)^{p-1}}\le\infty,$$

we choose $R_2<r$ small sufficiently, for any $\|u(t)\|+\|v(t)\|<R_2$, we have

$$f_1(s,u,v)\ge(M^{-1}\xi_1(u+v))^{p-1}.$$

Set

$$\Omega_{R_2}=\{(u,v):(u,v)\in K;\|(u,v)\|<R_2\},$$

similarly, for any $(u,v)\in K\bigcap\partial\Omega_{R_2}$, we have

$$\|F(u,v)\|=\|A(u,v)\|+\|B(u,v)\|\ge\|(u,v)\|. \tag{8}$$

An analogous estimate holds for f_2 in condition (H_6).

By Lemma 1, F has at least twice fixed points $(u_1,v_1)\in\overline{\Omega}_{R_1}\setminus\Omega_1$ and $(u_2,v_2)\in\overline{\Omega}_1\setminus\Omega_{R_2}$, so systems (1) has at least twice positive solutions with $\|(u_1,v_1)\|<r<\|(u_2,v_2)\|$.

Theorem 4 *Suppose that conditions* $(H_1)-(H_5)$ *hold, then problem (1) has at least two positive solutions* (u_1,v_1) *and* (u_2,v_2) *with* $\|(u_1,v_1)\|<R<\|(u_2,v_2)\|$.

The proof is similar to the Theorem 3, we omit it.

III. Example

As an application of Theorems 1 and 2, we give an example:

$$\begin{cases}(\phi_p(u''(t)))'+\dfrac{(u(t)+v(t))^{\alpha_1}}{6\sqrt{t(1-t)}}=0,\\(\phi_p(v''(t)))'+\dfrac{(u(t)+v(t))^{\alpha_2}}{6\sqrt{t(1-t)}}=0,\\u(0)=u(1)=0,u''(0)=0,\\v(0)=v(1)=0,v''(0)=0.\end{cases}$$

where $p \geq 2, \alpha_1 \geq \alpha_2 > p-1$. $\omega_i(t) = \frac{1}{\sqrt[4]{t(1-t)}}$, $f_i(t,u,v) = \frac{(u+v)^{\alpha_i}}{6\sqrt[4]{t(1-t)}}$.

Let

$$g_i(t) = \frac{1}{3\sqrt[4]{t(1-t)}}, h_i(u,v) = (u+v)^{\alpha_i},$$

$\omega_i(t)g_i(t) = \frac{1}{3\sqrt{t(1-t)}}$ and $\int_0^1 \omega_i(s)g_i(s)ds = \frac{\pi}{3}$, Theorem 1 holds.

If $\alpha_1 \leq \alpha_2 < p-1$, then Theorem 2 holds.

Acknowledgment

The authors would like to acknowledge the suggestions of reviewer and the financial support from the ShanDong Province Higher Educational Science and Technology Programme (Grant No. J13LI58).

References

[1] Li Z. Z.,Ge W. G. Positive Solution for p-Laplacian Singular Sturm-Liouville Boundary Value Problem Math, Appli, 2002, 15(3); 13-17.

[2] Wong F. H. Existence of Positive Solutions for m-Laplacian Boundary Value Problem. Appl, Math. Lett, 1999,12:11-17.

[3] Agarwal P. R.,ORegan D., Wong P. J. Wong F. H. Positive Solutions of Differential, Differential and Integral Equations. Singapore: Springer-Verlag,2000.

[4] Ni X. H., Ge W. G. Existence of Positive Solutions for One-dimentional p-Laplacian Coupled Boundary Value Problem. J. Math. Research and Exposition,2005, 25(3): 489-494.

[5] D. R. Dunninger, H. Y. Wang. Existence and Multiplicity of Positive Solutions for Ellipyic Systems. Nonlinear Analysis, Theory, Methods Applications, 1997, 29 (9): 1051-1060.

[6] Cai Z. X., Zhang X. Z. Positive Solutions for Third-order p-Laplacian Coupled Singular for Boundary Value Problems. ACTA Mathematicae Applicatae Sinica 2012, 35(3) 421-429.

[7] Guo D. J. Nonlinear Functional Analysis. Jinan Shandong Science Technical Publishers, 2000.

[8] Wang Y. L., Shi G. W. Positive Solutions of Fourth-order Singular Superlinear p-Laplacian BVP. ACTA Mathematicae Scientia, 2009, 29A(2): 344-352.