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The Relationship between the Energy of Gravitational Field and the Representative Pseudotensor

R. I. Khrapko

***Abstract*—**As is known, the role of the energy-momentum pseudotensors of the gravitational field is to extend the conservation law to the gravitational interaction by taking into account the energy and momentum of the gravitational field. We calculated the contribution of the Einstein pseudotensor to the total mass of a stationary material body and its gravitational field. It turned out that this contribution is positive, despite the fact that the mass-energy of a stationary gravitational field is negative. We concluded that the pseudotensor incorrectly describes the energy of the gravitational field. Nevertheless, this pseudotensor has been used in a large number of scientific works for 100 years. We explain this by the fact that the covariant component of the pseudotensor was regarded as the mass-energy. Besides, we prove the advantage of the covariant energy-momentum conservation law for matter in the Minkowski space-time.

***Keywords*—**Conservation law, covariant integration, gravitation field, isolated system.

I. Introduction. Energy of the Gravitation Field Is Negative

When an electron is attracted to a proton from infinity by the force of the electric field, the energy of the electric field is converted into the kinetic energy, and the corresponding part of the electric field is eliminated. As a result, in harmony with the mass-energy conservation law, we have the excitation energy of the atom instead of the energy of the part of the electric field. This means that the total mass-energy of the system, “electron + proton + their field”, is conserved and the mass-energy of the matter is increased due to the disappearance of the part of the field. Then 13.6 eV are radiated, and we get the neutral atom without an external electric field.

Now consider compression of a dilute dust cloud under the action of own gravitational attraction forces. If we adhere to the concept of gravitational field, the cloud is surrounded by its gravitational field. In the compression process, the kinetic energy appears. And this kinetic energy will be converted into the thermal energy when the compression is stopped by the pressure forces. As a result, the mass-energy of the matter is increased in the same way as it is increased at the electron-proton attraction. But in the case of gravitational attraction, the gravitational field of the dust is not eliminated. Instead, the gravitational field strengthened and extends to the volume that has become free from the cloud. So, if we trust that the mass-energy of the system “cloud + its gravitational field” is conserved in the process of compression, we have to ascribe a negative mass-energy to the gravitational field.

A. Guth demonstrates the negativity of the gravitational energy by the example of a thin sperical shell of mass in Apperdix A of [1].

The standard method to calculate the mass-energy of matter, M, implies the use of the energy-momentum tensor T^{ν}_{μ}. The standard method to calculate the mass-energy of gravitational field, G, implies the use of the Einstein pseudotensor [2]-[4] or Landau-Lifshitz pseudotensor [5] t^{ν}_{μ}. The standard method to calculate the total mass-energy of a system “matter + its gravitational field”, $J = M + G$, implies the use of the sum $T^{\nu}_{\mu} + t^{\nu}_{\mu}$. The contribution of the pseudotensor to the mass of the system “cloud + its gravitational field” must be negative, $G < 0$, in order to compensate for the growth of the mass of the matter under compression if we trust that $J = M + G = \text{Const}$.

II. Calculation of the Mass-Energy of Matter

An infinitesimal space element dV_{ν} contains the 4-momentum element of the matter:

$$dP_{\mu} = T^{\nu}_{\mu}\sqrt{-g}\,dV_{\nu}, \ (\mu, \nu, ... = 0, ..., 3) \tag{1}$$

here dP_{μ} is an infinitesimal covariant vector (this is the totality of its components), $T^{\nu}_{\mu}\sqrt{-g}$ is the energy-momentum *tensor density* of the weight +1, dV_{ν} is the covariant vector density of the weight -1, and g is the determinant of the metric tensor of the coordinate system (see e.g. § 96 in [5]). For example, $dV_t = dxdydz$, or $dV_t = drd\theta d\varphi$ Note that the time component dP_t is not the mass of the element of the matter.

R. I. Khrapko is with the Moscow Aviation Institute - Volokolamskoe shosse 4, 125993 Moscow, Russia (e-mail: khrapko_ri@hotmail.com, khrapko_ri@mai.ru).

The mass is the module of 4-momentum (1), $dM = dP_t / \sqrt{g_{tt}}$.

A calculation of the *total* 4-momentum of an object is not possible, in general, because we cannot sum up vectors (1), which components are defined by curvilinear coordinates in different points of a curved space-time. However, in the case of a static body, the infinitesimal 4-momentums (1) are parallel to each other and have only time components dP_t in a static coordinate system. So, in this case, it is possible to calculate the total mass-energy of a body M by integrating of modules of 4-momentums (1) over space using this coordinate system:

$$.M = \int T_t^t (\sqrt{-g} / \sqrt{g_{tt}}) dV_t \, . \tag{2}$$

Note, that $T_t^t = \rho$ is the mass-energy *volume* density (see (95.10) in [3] or § 100 in [5]). et us use (2) to calculate the mass-energy of a sphere of perfect fluid. The space-time of such a body is discribed by the Schwarzschild's internal and external solutions of the Einstein equation. The internal solution [6] (see also § 96 in [3]) depends on two parameters, R and r_1:

$$ds^2 = \left(\frac{3}{2}\sqrt{1 - \frac{r_1^2}{R^2}} - \frac{1}{2}\sqrt{1 - \frac{r^2}{R^2}} \right)^2 dt^2$$
$$- \left(1 - \frac{r^2}{R^2} \right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\varphi^2, \tag{3}$$

$$\sqrt{-g} = \sqrt{-g_{tt} g_{rr}} r^2 \sin\theta \, . \tag{4}$$

here $0 \le r \le r_1 \le R$, r_1 is the coordinate of the surface where the external Schwarzschild space-time is attached, and R is the curvature radius of the inner space determined by the fluid volume density:

$$.T_t^t = \rho = \frac{3}{8\pi R^2} \, . \tag{5}$$

The external solution:

$$\begin{aligned} ds^2 &= (1 - 2m/r) dt^2 - (1 - 2m/r)^{-1} dr^2 \\ &- r^2 d\theta^2 - r^2 \sin^2\theta \, d\varphi^2 \end{aligned} \tag{6}$$

depends on the single (Schwarzschild) parameter $m = r_g / 2$, which describes external gravitational field. It is seen that at smooth sewing the inner solution together with the external solution:

$$m = r_1^3 / 2R^2 \, . \tag{7}$$

In view of the Birkhoff's theorem, the parameter m remains constant when compressing the sphere. Now the mass-energy of the sphere, M , can be calculated by using of (2):

$$\begin{aligned} M &= \int T_t^t (\sqrt{-g} / \sqrt{g_{tt}}) dr d\theta d\varphi \\ &= \int_0^{r_1} T_t^t \sqrt{-g_{rr}} r^2 dr 4\pi \end{aligned} \tag{8}$$

Using (5) and $g_{rr} = -\left(1 - \frac{r^2}{R^2} \right)^{-1}$ yields:

$$\begin{aligned} M = \int_0^{r_1} \frac{3}{2R} \frac{r^2 dr}{\sqrt{R^2 - r^2}} &= \frac{3R}{4} (\sin^{-1}\xi - \xi\sqrt{1 - \xi^2}), \\ \xi &= r_1 / R. \end{aligned} \tag{9}$$

The series expansions:

$$\sin^{-1}\xi = \xi + \xi^3/6 + 3\xi^5/40 + \ldots,$$
$$\sqrt{1 - \xi^2} = 1 - \xi^2/2 - \xi^4/8 + \ldots \text{ yields:}$$
$$M = R\xi^3/2 + 3R\xi^5/20 + .. = m + 3m^2/(5r_1) + .. \tag{10}$$

So, $M > m$. This excess of the matter mass M over the Schwarzschild parameter m was named the (positive) gravitational mass defect (§ 100 in [5]). Thus, the parameter m equals the initial mass of the fluid when the gravitation field was negligible.

Because of $J = M + G$, the energy of the gravitation field G must be negative if we wish the total mass, $m = J = M + G$, to be conserved. [In fact, it turned out that $G > 0$ and $m < J$, see (17)].

The calculation of the gravitational energy G must use the formula of the type (2):

$$.G = \int t_t^t (\sqrt{-g} / \sqrt{g_{tt}}) dV_t \, , \tag{11}$$

where $t_t^t \sqrt{-g}$ is the component of the Einstein [2]-[4], or Landau-Lifshitz [5] *pseudotensor density.* And it must be:

$$.G = \int t_t^t (\sqrt{-g} / \sqrt{g_{tt}}) dV_t = -3m^2/(5r_1) + \ldots < 0 \, ,.. \tag{12}$$

But the pseudotensor density is very complicated. So, physicists calculated neither (11) nor:

$$J = M + G = \int (T_t^t + t_t^t)(\sqrt{-g} / \sqrt{g_{tt}}) dV_t \, ,$$

Instead of the total mass J , an integral without $1/\sqrt{g_{tt}}$

was calculated:

$$U = \int (T_t^t + t_t^t)\sqrt{-g}\,dV_t \,.$$

This integral was chosen because the pseudotensor property provided a nontensor equality:

$$\partial_\nu [(T_\mu^\nu + t_\mu^\nu)\sqrt{-g}] = 0 \qquad (13)$$

and, consequently, the conservation of the nontensor integral U.

III. The Standard Calculation of the "Total Mass-Energy"

The Einstein pseudotensor density is [2]-[4]:

$$t_\mu^\nu \sqrt{-g} = \frac{1}{16\pi}\left(-\partial_\mu (g^{\alpha\beta}\sqrt{-g}) \frac{\partial \mathfrak{L}}{\partial(\partial_\nu (g^{\alpha\beta}\sqrt{-g}))} + \delta_\mu^\nu \mathfrak{L} \right),$$

$$\mathfrak{L} = g^{\alpha\beta}\sqrt{-g}\,(\Gamma_{\alpha\mu}^\nu \Gamma_{\beta\nu}^\mu - \Gamma_{\alpha\beta}^\mu \Gamma_{\mu\nu}^\nu)\,. \qquad (14)$$

The expression (14) is awkward. So, nobody calculated the mass of the gravitational field (11). But luckily, we can use a result of a calculation that was performed by Tolman. Tolman integrated the sum of the covariant components:

$$dJ_t = (T_t^t + t_t^t)\sqrt{-g}\,dV_t \qquad (15)$$

for a quasi-static isolated system with respect to quasi-Galilean coordinates. The integration gives (see [7], or (91.1), (91.3), (92.4), (97.5) in [3]):

$$U = \int dJ_t = \int (T_t^t + t_t^t)\sqrt{-g}\,dV_t = \int (T_t^t - T_1^1 - T_2^2 - T_3^3)\sqrt{-g}\,dV_t = \int (\rho + 3p)\sqrt{-g}\,dV_t = m \qquad (16)$$

where p is pressure. This integral quantity (16) is named U. It cannot be named J_t because it is not a component of a vector: it is obtained by adding up the components dJ_t belonging to different points where the coordinate bases differ from each other. And there is no coordinate basis to which the quantity (16) could be long as a component. The quantity U is deprived of a geometrical meaning and is senseless. It is not the mass of the system. Nevertheless, the result (16) is very useful! It shows that the integral.

$$\int t_t^t \sqrt{-g}\,dV_t = \int 3p\sqrt{-g}\,dV_t > 0$$

is *positive* because pressure p is positive. And this means a very important fact: the contribution G (11) of the Einstein pseudotensor to the mass of the system, is also positive [8] because $\sqrt{g_{tt}} > 0$:

$$.G = \int t_t^t (\sqrt{-g} / \sqrt{g_{tt}})\,dV_t > 0\,. \qquad (17)$$

The Landau-Lifshitz pseudotensor gives the same result; it is stated by formula (105.23) in [5]. So, the pseudotensors give positive value for the mass-energy of the gravitational field. The pseudotensor does not provide the conservation law:

$$M > m,\; G > 0,\; \text{so},\; J = M + G > m\,.$$

This means that, in reality, the pseudotensors do not represent mass-energy of the gravitational field, which is negative. (If we adhere to the concept of gravitational field and trust that the mass-energy of the system is conserved).

Tolman wrote: "This satisfactory result (16) can serve to increase our confidence in the practical advantages of Einstein's procedure in introducing the pseudo-tensor densities of potential gravitational energy and momentum" [3, p. 250]. Unfortunately, this is an illusion. The conserved quantity (16):

$$U = \int (T_t^t + t_t^t)\sqrt{-g}\,dV_t$$

has no geometrical and physical sense.

IV. The Conservation Law for Senseless Quantities

The idea about the energy-momentum pseudotensor of the gravitation field arose from the belief that the conservation of an integral quantity in time is ensured by the vanishing of the partial divergence. Landau & Lifshitz wrote: "The integral $\int T_\mu^\nu \sqrt{-g}\,dV_\nu$ is conserved only if the condition:

$$\partial_\nu (\sqrt{-g}T_\mu^\nu) = 0 \qquad (18)$$

is fulfilled" [5 § 96]. The covariant local conservation law:

$$\nabla_\nu (\sqrt{-g}T_\mu^\nu) = 0 \qquad (19)$$

was discredited: "In this form, however, this equation does not generally express any conservation law whatever". To understand the situation, let us consider two hypersurfaces V_1 and V_2 which have a closed surface as their common boundary. In any space and in any coordinate system, two quadruples of numbers:

$$P_{1\mu} = \int_{V_1} \sqrt{-g}T_\mu^\nu dV_\nu\,,\; P_{2\mu} = \int_{V_2} \sqrt{-g}T_\mu^\nu dV_\nu\,, \qquad (20)$$

will be equal to each other, $P_{1\mu} = P_{2\mu}$, if (18) is fulfilled, because of the Gauss' theorem:

$$\int_{V_2} (\sqrt{-g}T_\mu^{\ \nu})dV_\nu - \int_{V_1} (\sqrt{-g}T_\mu^{\ \nu})dV_\nu$$
$$= \oint_{\partial\Omega} (\sqrt{-g}T_\mu^{\ \nu})dV_\nu = \int_\Omega \partial_\nu (\sqrt{-g}T_\mu^{\ \nu})d\Omega = 0 \quad (21)$$

here $\partial\Omega = V_2 \bigcup (-V_1)$ is the boundary of 4-volume Ω. But this constant quadruple P_μ (20) has no geometric meaning if a curvilinear coordinate system is in use. We repeat, the integrals (20) are not applied to any specific point in space and is therefore deprived of a geometric meaning since we do not know the basis to which the components P_μ refer, and there is no law of their transformation at a coordinate change.

To feel the wantonness of the nontensor equations (18), (20) in curvilinear coordinates, consider a plane with mechanical tensions. Let the tension tensor density be nonzero in the right halfplane and be presented in polar coordinates r, φ at $-\pi/2 < \varphi < \pi/2$ by the equations ($\sqrt{-g}$ is included):

$$T^{rr} = r\sin\varphi,\ T^{r\varphi} = T^{\varphi r} = \cos\varphi,\ T^{\varphi\varphi} = 0. \quad (22)$$

The condition (18) is fulfilled:

$$\begin{aligned} &\partial_r T^{rr} + \partial_\varphi T^{r\varphi} = \sin\varphi - \sin\varphi = 0, \\ &\partial_r T^{\varphi r} + \partial_\varphi T^{\varphi\varphi} = 0. \end{aligned} \quad (23)$$

So, integrals of the type (20) obtained by integration over the circle $r = Const$,

$$F^r = \int T^{rr} dl_r = \int_{-\pi/2}^{\pi/2} r\sin\varphi d\varphi = 0, \quad (24)$$
$$F^\varphi = \int T^{\varphi r} dl_r = \int_{-\pi/2}^{\pi/2} \cos\varphi d\varphi = 2 \quad (25)$$

do not depend on the circle radius r. These integrals pretend to be the components of the (constant) force F^i acting on an arc of radius r in this the strained halfplane. The component (25), however, does not determine any *direction* since it is not applied to any specific point, while a vector with the component $F^\varphi = 2$ has different directions in space, depending on its application point. Therefore, the pseudovector integrals (24), (25) are meaningless.

V. True Calculation of Integral Quantities

Counting the mass-energy of matter when using curvilinear coordinates requires parallel transfer of the 4-momentum elements of the matter (1):

$$dP_\mu = T_\mu^{\ \nu}\sqrt{-g}dV_\nu$$

to the point x' of counting using a special two-point tensor function called a translator [9],

$$\Psi_{\mu'}^{\mu}(x', x). \quad (26)$$

Before the integration, one must transfer the elementary vectors dP_μ by the formula:

$$dP_{\mu'}(x') = \Psi_{\mu'}^{\mu}(x', x)dP_\mu(x). \quad (27)$$

Then the integral 4-momentum can be obtained:

$$P_{\mu'}(x') = \int \Psi_{\mu'}^{\mu}(x', x)T_\mu^{\ \nu}(x)\sqrt{-g(x)}dV_\nu. \quad (28)$$

The integration (28) is integration of the elements $\Psi_{\mu'}^{\mu}(x', x)T_\mu^{\ \nu}(x)\sqrt{-g(x)}dV_\nu$ that are scalar at points x, and curvilinear coordinates do not make problems.

Let us consider two hypersurfaces V_1 and V_2 which have a closed surface as their common boundary as in (21). Then the two 4-momentums occur at the point x':

$$P_{1\mu'}(x') = \int_{V_1} \Psi_{\mu'}^{\mu}(x', x)T_\mu^{\ \nu}(x)\sqrt{-g(x)}dV_\nu. \quad (29)$$
$$P_{2\mu'}(x') = \int_{V_2} \Psi_{\mu'}^{\mu}(x', x)T_\mu^{\ \nu}(x)\sqrt{-g(x)}dV_\nu. \quad (30)$$

Their difference equals the integral over the closed hypersurface $\partial\Omega = V_2 \bigcup (-V_1)$:

$$P_{2\mu'} - P_{1\mu'} = \oint_{\partial\Omega} \Psi_{\mu'}^{\mu}(x', x)T_\mu^{\ \nu}(x)\sqrt{-g(x)}dV_\nu. \quad (31)$$

Now the Gauss theorem (which certainly implies partial differentiation) gives:

$$\oint_{\partial\Omega} \Psi_{\mu'}^{\mu}(x', x)T_\mu^{\ \nu}(x)\sqrt{-g(x)}dV_\nu$$
$$= \int_\Omega \partial_\nu [\Psi_{\mu'}^{\mu}(x', x)T_\mu^{\ \nu}(x)\sqrt{-g(x)}]d\Omega. \quad (32)$$

But $\Psi_{\mu'}^{\mu}(x', x)T_\mu^{\ \nu}(x)\sqrt{-g(x)}$ is a vector density at x. So, we can change: $\partial_\nu \to \nabla_\nu$, and as a result we have:

$$P_{2\mu'} - P_{1\mu'}$$
$$= \int_\Omega \{\nabla_\nu [\Psi_{\mu'}^{\mu}(x', x)]T_\mu^{\ \nu}(x)\sqrt{-g(x)}\}d\Omega.$$
$$+ \int_\Omega \{\Psi_{\mu'}^{\mu}(x', x)\nabla_\nu [T_\mu^{\ \nu}(x)\sqrt{-g(x)}]\}d\Omega. \quad (33)$$

$\nabla_\nu[\Psi^\mu_{\mu'}(x',x)]=0$ in the Minkowski space in any coordinate system [9]. Thus, the covariant equation (19):

$$\nabla_\nu(T^\nu_\mu\sqrt{-g})=0,$$

ensures the conservation of the energy-momentum P_μ in Minkowski space. In a space-time of General Relativity, the energy-momentum P_μ of matter does not conserve, in general, because of the term:

$$\nabla_\nu[\Psi^\mu_{\mu'}(x',x)]$$

in (33). An example of creation of matter in a gravitation field was presented in [10]. But, even worse, the energy-momentum of matter P_μ (28) does not have a single-valued meaning in a space-time of General Relativity because the translator $\Psi^\mu_{\mu'}(x',x)$ (26) depends on the path of the transfer.

VI. Conclusion

We have to admit that we do not know how to describe the energy of a stationary gravitational field in the frame of the General Relativity. The pseudotensors are not suitable for this.

The nontensor equation (13) [11, (35)]:

$$\partial_\nu[(T^\nu_\mu+t^\nu_\mu)\sqrt{-g}]=0$$

means the conservation of the quantity (16):

$$U=\int(T^t_t+t^t_t)\sqrt{-g}\,dV_t=m=\text{Const},$$

which has no geometrical and physical sense. And here we do not touch the problem of the energy of gravitational waves.

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