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Analytical and Numerical Results for Free Vibration of Laminated Composites Plates

Mohamed Amine Ben Henni, Taher Hassaine Daouadji, Boussad Abbes, Yu Ming Li, Fazilay Abbes

***Abstract*—**The reinforcement and repair of concrete structures by bonding composite materials have become relatively common operations. Different types of composite materials can be used: carbon fiber reinforced polymer (CFRP), glass fiber reinforced polymer (GFRP) as well as functionally graded material (FGM). The development of analytical and numerical models describing the mechanical behavior of structures in civil engineering reinforced by composite materials is necessary. These models will enable engineers to select, design, and size adequate reinforcements for the various types of damaged structures. This study focuses on the free vibration behavior of orthotropic laminated composite plates using a refined shear deformation theory. In these models, the distribution of transverse shear stresses is considered as parabolic satisfying the zero-shear stress condition on the top and bottom surfaces of the plates without using shear correction factors. In this analysis, the equation of motion for simply supported thick laminated rectangular plates is obtained by using the Hamilton's principle. The accuracy of the developed model is demonstrated by comparing our results with solutions derived from other higher order models and with data found in the literature. Besides, a finite-element analysis is used to calculate the natural frequencies of laminated composite plates and is compared with those obtained by the analytical approach.

***Keywords*—**Composites materials, laminated composite plate, shear deformation theory of plates, finite element analysis, free vibration.

I. INTRODUCTION

LAMINATED composite plates are widely used in different engineering and industrial domains, thanks to their lightweight and sustainability. Many researchers studied the behavior of composite structures in civil and mechanical engineering applications.

The classical laminated plate theory (CLPT) is the first theory that was used to determine the stresses and deformations in thin plates. In this theory, transverse shear effects are neglected which limits its use to thin plates only. The first-order shear deformation theory proposed by Mindlin [1] is the first theory that takes into account for transverse shear effects in the determination of deformations and stresses for relatively thick plates. To overcome the limitations of the first order shear deformation theory, higher-order shear deformation theories were proposed [2]-[6]. These theories involve higher-order terms in Taylor's expansions of the displacements in the thickness coordinate. A review of these theories applied to the analysis of laminated composite plates is available in the works of Mantari et al. [6] and Daouadji et al. [7]. High-order shear deformation theories were also used to determine the natural frequencies of laminated composite plates [8]-[10].

In this paper, the free vibration behavior of antisymmetric cross-ply laminated composite plates is studied using a refined shear deformation theory. The obtained results are compared to those obtained using the commercial finite element code ABAQUS.

Mohamed Amine Ben Henni is with the Université de Reims, France (e-mail: benhennia8@gmail.com).

II. THEORETICAL FORMULATION

A. Main Assumptions

A rectangular *n* layered orthotropic plate of total thickness *h* is considered with the coordinate system defined in Fig. 1. The main assumptions of the present plate's theory are:

– The infinitesimal strains are considered since the displacements are small in comparison with the plate thickness.
– The transverse displacement *w* is decomposed in bending w_b and shear w_s components. These components are functions of coordinates *x*, *y* and time *t* only.

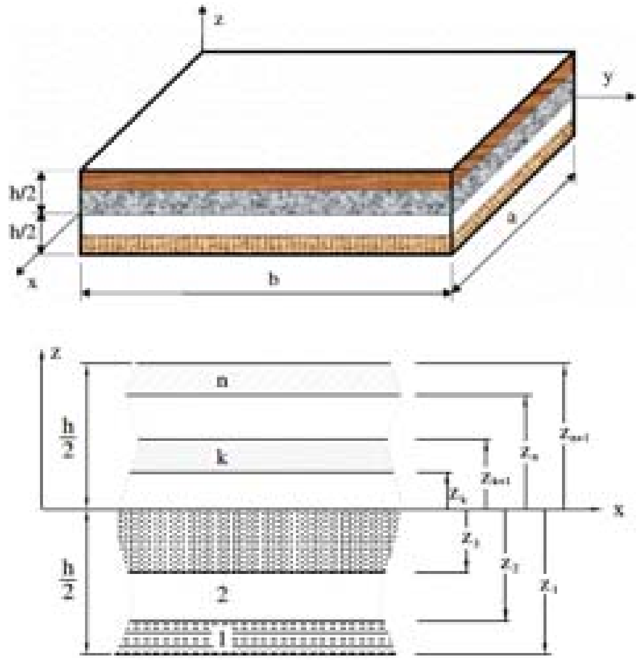

Fig. 1 Cross-ply laminated composite plate

B. Kinematics

The displacements of a material point at (*x*, *y*, *z*) of the plate can be written as:

$$
\begin{aligned}
u(x,y,z,t) &= u_0(x,y,t) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \\
v(x,y,z,t) &= v_0(x,y,t) - z\frac{\partial w_b}{\partial y} - f(z)\frac{\partial w_s}{\partial y} \\
w(x,y,z,t) &= w_b(x,y,t) + w_s(x,y,t)
\end{aligned} \tag{1}
$$

where u_0 and v_0 represents the plate mid-plane displacements in the x and y directions respectively; and $f(z)$ represents a shape function determining the transverse shear strains and stresses distribution through the thickness of the plate.

The proposed function verifies the zero transverse shear stresses at the top and the bottom of the plate. In the present model, the through thickness distribution of the transverse shear stresses is taken into account by means of the hyperbolic and exponential function of the assumed displacement field:

$$
f(z) = z\left[1 + \frac{3\pi}{2}\sec h^2(0.5)\right] - \frac{3\pi}{2}h\tanh(z/h) \tag{2}
$$

The constitutive equations for an orthotropic rectangular plate can be expressed in its axis of symmetry as:

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix} + f(z)\begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = g(z)\begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix} \tag{3}
$$

where

$$
\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix}, \begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_b}{\partial x^2} \\ -\dfrac{\partial^2 w_b}{\partial y^2} \\ -2\dfrac{\partial^2 w_b}{\partial x \partial y} \end{Bmatrix} \tag{4}
$$

$$
\begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_s}{\partial x^2} \\ -\dfrac{\partial^2 w_s}{\partial y^2} \\ -2\dfrac{\partial^2 w_s}{\partial x \partial y} \end{Bmatrix}, \begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial w_s}{\partial y} \\ \dfrac{\partial w_s}{\partial x} \end{Bmatrix} \tag{5}
$$

$$
g(z) = 1 - f'(z), \; f'(z) = \frac{df(z)}{dz} \tag{6}
$$

The generalized Hooke's law represents the stress state in each layer as:

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{7}
$$

where Q_{ij} are the stiffnesses defined in terms of engineering constants in the material axes of the layer by:

$$
\begin{aligned}
&Q_{11} = \frac{E_{11}}{1-\nu_{12}\nu_{21}}, \; Q_{12} = \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}}, \; Q_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}} \\
&Q_{66} = G_{12}, \; Q_{44} = G_{23}, \; Q_{55} = G_{13}
\end{aligned} \tag{8}
$$

As each layer of the laminate is oriented arbitrarily with respect to its local coordinates, its constitutive equation must be transformed to the global laminate coordinates (x, y, z). The constitutive law in the laminate coordinates of the k^{th} layer is:

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^{(k)} \tag{9}
$$

where $\bar{Q}_{ij}$ are the transformed material constants:

$$
\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\cos^2\theta\sin^2\theta + Q_{22}\sin^4\theta \\
\bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})\cos^2\theta\sin^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta) \\
\bar{Q}_{22} &= Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\cos^2\theta\sin^2\theta + Q_{22}\cos^4\theta \\
\bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})\cos^3\theta\sin\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\cos\theta \\
\bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})\cos\theta\sin^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\cos^3\theta \\
\bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\cos^2\theta\sin^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta) \\
\bar{Q}_{44} &= Q_{44}\cos^2\theta + Q_{55}\sin^2\theta \\
\bar{Q}_{45} &= (Q_{55} - Q_{44})\cos\theta\sin\theta \\
\bar{Q}_{55} &= Q_{55}\cos^2\theta + Q_{44}\sin^2\theta
\end{aligned} \tag{10}
$$

C. Governing Equations of Motion of the Composite Plate

The equation of motion of the laminated composite plate is derived using Hamilton's energy principle:

$$\delta\int_{t_1}^{t_2}\left(U - V - T\right)dt = 0 \quad (11)$$

where U and T are the strain and the kinetic energies the plate, and V is the work of external forces.

Using the principle of minimum total energy leads to the general equation of motion and boundary conditions. Taking the variation of (11) and integrating by parts, we obtain:

$$\int_{t_1}^{t_2}\left[\int_V\begin{pmatrix}\left(\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{xz}\right)\\ -\rho\left(\ddot{u}_0\delta u_0 + \ddot{v}_0\delta v_0 + (\ddot{w}_b + \ddot{w}_s)\delta(w_b + w_s)\right)\end{pmatrix}dV\right]dt = 0 \quad (12)$$

Rectangular laminated composites plates are generally classified in accordance with the type of support used. We are concerned in this study by simply supported composite plate with the following boundary conditions:

$$x = \pm\frac{a}{2}:\begin{cases}v_0 = w_b = w_s = 0,\ \dfrac{\partial w_b}{\partial y} = \dfrac{\partial w_s}{\partial y} = 0\\ N_x = 0,\ M_x^b = M_x^s = 0\end{cases} \quad (13)$$

$$y = \pm\frac{b}{2}:\begin{cases}u_0 = w_b = w_s = 0,\ \dfrac{\partial w_b}{\partial x} = \dfrac{\partial w_s}{\partial x} = 0\\ N_y = 0,\ M_y^b = M_y^s = 0\end{cases} \quad (14)$$

The displacement functions that satisfy the previous boundary conditions are taken in the form of Fourier series:

$$\begin{Bmatrix}u_0\\ v_0\\ w_b\\ w_s\end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\begin{Bmatrix}U_{mn}\cos(\lambda\, x)\sin(\mu\, y)e^{i\omega t}\\ V_{mn}\sin(\lambda\, x)\cos(\mu\, y)e^{i\omega t}\\ W_{bmn}\sin(\lambda\, x)\sin(\mu\, y)e^{i\omega t}\\ W_{smn}\sin(\lambda\, x)\sin(\mu\, y)e^{i\omega t}\end{Bmatrix} \quad (15)$$

where U_{mn}, V_{mn}, W_{bmn} and W_{smn} are arbitrary parameters to be determined, ω is the eigen frequency associated with a mn eigen mode $\lambda = m\pi/a$ and $\mu = n\pi/b$. Using (15) with (13) and (14) in (12), we obtain the following eigenvalue equations for the free vibration problem for each fixed value of m and n:

$$\left(\left[K\right] - \omega^2\left[M\right]\right)\left\{\Delta\right\} = \left\{0\right\} \quad (16)$$

where $\{\Delta\} = \{U_{mn}, V_{mn}, W_{bmn}, W_{smn}\}$ is the column vector and $[K]$ and $[M]$ are the stiffness and mass matrices, respectively. The terms of these matrices can be found in [10].

III. Results

In this study, free vibration analysis of anti-symmetric cross-ply laminate composite plates by using the present shear deformation theory for laminated plates is proposed. The fundamental frequencies are obtained by solving the eigenvalue system (16). For validation purposes, the results obtained by the present model are compared with those obtained by Reddy [2], Adim et al. [10] and our finite element simulations. The following non-dimensional fundamental frequency is used in presenting our results:

$$\bar{\omega} = \omega\frac{a^2}{h}\sqrt{\frac{\rho}{E_2}} \quad (17)$$

The lamina properties used in this study are given in Table I.

TABLE I
LAMINA PROPERTIES

Material	Properties
1	$E_1 = (3 \rightarrow 40)E_2$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5G_2$ $\nu_{12} = 0.25$
2	$E_1 = (3 \rightarrow 40)E_2$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.6G_2$ $\nu_{12} = 0.25$

A. Validation of the Proposed Theory

The fundamental frequencies of a simply supported antisymmetric cross-ply $(0/90)_n$ plates were calculated by varying the number of plies n and E_1/E_2 ratio.

Table II shows the comparison between our results and the solutions given by Reddy [2] for different values of orthotropic ratio. The non-dimensional fundamental frequencies obtained by using the proposed shear deformation theory are comparable to those obtained by Reddy [2].

TABLE II
NON-DIMENSIONAL FUNDAMENTAL FREQUENCIES OF ANTISYMMETRIC SQUARE PLATES AT VARIOUS VALUES OF ORTHOTROPIC RATIO WITH $a/h = 5$

# layers	Model	E_1/E_2 3	10	20	30	40
$(0^0/90^0)_1$	Reddy [2]	6.2169	6.9887	7.8210	8.5050	9.0871
	Present	6.2181	6.9939	7.8327	8.5234	9.1124
$(0^0/90^0)_2$	Reddy [2]	6.5008	8.1954	9.6265	10.5348	11.1716
	Present	6.5009	8.1932	9.6215	10.5282	11.1642
$(0^0/90^0)_3$	Reddy [2]	6.5558	8.4052	9.9181	10.8547	11.5012
	Present	6.5562	8.4057	9.9195	10.8577	11.5064
$(0^0/90^0)_5$	Reddy [2]	6.5842	8.5126	10.0674	11.0197	11.6730
	Present	6.5848	8.5143	10.0713	11.0266	11.6833

B. Parametric Study and Finite Element Simulations

To assess the validity of the proposed theory, we carried out a parametric study to compute the fundamental frequencies of a simply supported antisymmetric cross-ply $(0/90)_n$ plates by varying the number of plies n, the E_1/E_2, a/h and a/b ratios. The obtained results are compared with the numerical results obtained by the finite element code ABAQUS.

A four-node linear (doubly curved thin or thick shell) S4R type element was used. A preliminary study was conducted to define the optimal finite element mesh (Fig. 2). The relevant geometrical and mechanical properties used in the finite element analysis were the same as that used in the analytical method to simulate correctly the free vibration behavior of the plate.

Fig. 3 shows the fundamental frequencies of a simply supported antisymmetric cross-ply $(0/90)_n$ plates with a/h=20. The frequencies increase with increasing number of plies and E_1/E_2 ratio, which is coherent with the increasing stiffness of the plate. We can also notice the very good agreement between the proposed analytical model and the numerical finite element model.

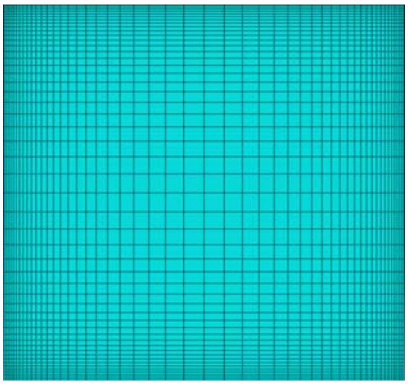

Fig. 2 Finite element mesh for the square plate

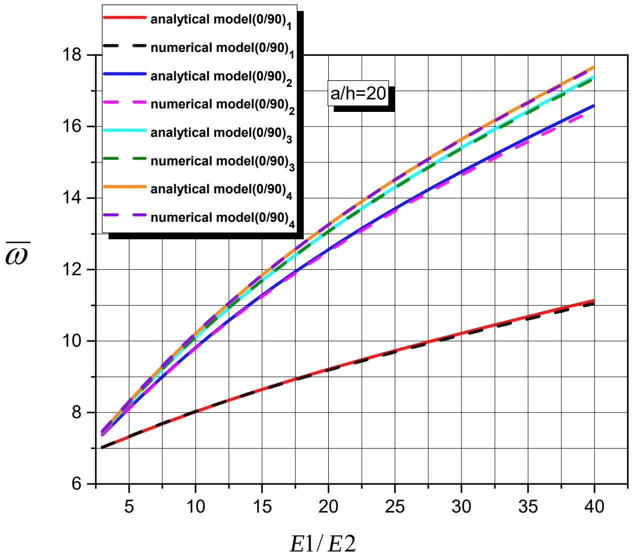

Fig. 3 Non-dimensional fundamental frequencies of antisymmetric square plates at different values of orthotropic ratio E_1/E_2

Fig. 4 shows the effect of the thickness ratio a/h on the natural frequencies of a simply supported antisymmetric cross-ply $(0/90)_n$ and orthotropic ratio E_1/E_2=40. The natural frequencies increase with increasing thickness ratio. The numerical results compare very well with the analytical ones for high thickness ratios (i.e. thin plates). However, we notice slight difference when the thickness ratio decreases (i.e. thick plates). For thick plates, the transverse shear is more important and the finite element S4R uses a first-order shear deformation theory.

Fig. 5 shows the effect of the geometry ratio a/b on the natural frequencies of a simply supported antisymmetric cross-ply $(0/90)_2$ and orthotropic ratio E_1/E_2=40. The natural frequencies increase with increasing geometry ratio. The numerical results compare very well with the analytical ones for thin plates (a/h>30) and slightly deviates for thicker plates due to the first-order shear deformation theory used in the finite element solution.

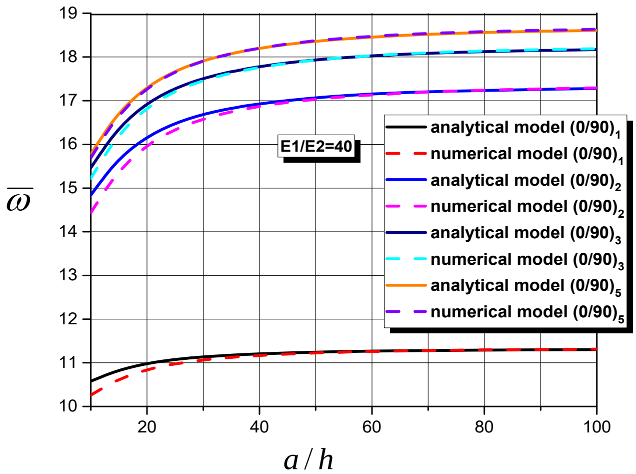

Fig. 4 Non-dimensional fundamental frequencies of antisymmetric square plates at various values of a/h

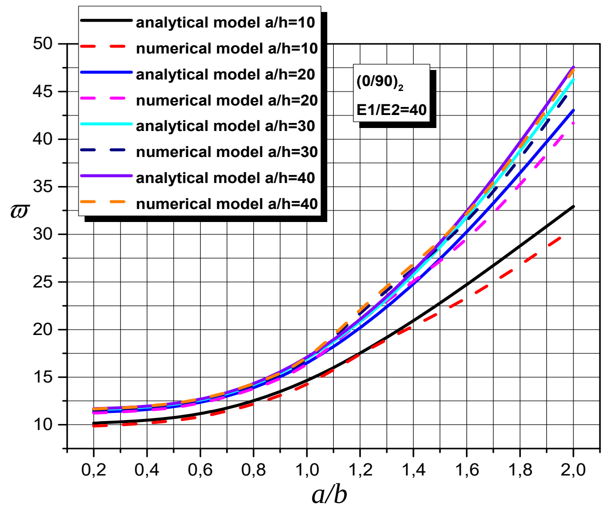

Fig. 5 Non-dimensional fundamental frequencies of antisymmetric square plates at various values of a/b

IV. Conclusion

In this study, the FEM and a refined shear deformation theory have been successfully used to analyze the free vibration of simply supported antisymmetric cross-ply laminated composite plates. A parametric study shows that the natural frequencies obtained by the FEM are very close to those obtained by the analytical proposed theory for thin plates; and slightly deviates for thicker plates. The present analytical theory allows for parabolic variation in terms of the transverse shear strains across the plate thickness and satisfies the zero-shear stress on the top and bottom surfaces of the plate without needing shear correction factors. However, the finite elements available in the commercial codes usually use a first-order shear deformation theory. The proposed analytical model is more suitable to capture the transverse shear leading to accurate natural frequencies results in free vibration of antisymmetric cross-ply laminated composite plates.

ACKNOWLEDGMENT

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