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On Chvátal's Conjecture for the Hamiltonicity of 1-Tough Graphs and Their Complements

Shin-Shin Kao, Yuan-Kang Shih, Hsun Su

Abstract**—**In this paper, we show that the conjecture of Chvátal, which states that any 1-tough graph is either a Hamiltonian graph or its complement contains a specific graph denoted by F, does not hold in general. More precisely, it is true only for graphs with six or seven vertices, and is false for graphs with eight or more vertices. A theorem is derived as a correction for the conjecture.

Keywords—Complement, degree sum, Hamiltonian, tough.

I. INTRODUCTION

EVER since Chvátal introduced the concept of toughness of graphs, numerous studies have been done, see [1] for a survey. In [2], which was originally published in 1973, Chvátal posted seven conjectures. Five of the conjectures regard the existence of a minimum toughness that guarantees a certain cycle structure in any graph, one of them is about the Hamiltonicity of 2-tough neighborhood-connected graphs, and the other one relates the existence of a Hamiltonian cycle of any 1-tough graph with its complement graph. These conjectures are inspiring and have led to a bountiful harvest of results. So far, the minimum toughness t_0 which makes the conjecture "there exists t_0 such that every t_0-tough graph is hamiltonian" hold has not been found. The best result by now is published by Bauer et al. [3], who showed that if such a t_0 exists, it must be $t_0 \geq \frac{9}{4}$. For Chvátal's conjecture regarding the Hamiltonicity of any 1-tough graph and its complement, which is presented below, much fewer researches are done.

Conjecture 1. (see [2]) If G is 1-tough, then either G is Hamiltonian or its complement $\overline{G}$ contains the graph F in Fig. 1 (a).

In this paper, we are devoted to the study of the above conjecture. Since F has six vertices, it is obvious that Conjecture 1 deals with graphs with at least six vertices. We shall give graphic examples showing that Conjecture 1 is not true when $|G| = n \geq 8$, and a proof that the conjecture holds for $|G| = n \in \{6,7\}$. Our corrections of Chvátal's conjecture will be presented as Theorem 5 and 6. This paper is organized as follows. Notations, terminologies, and some known theorems are given in Section II, and our main results are shown in Section III.

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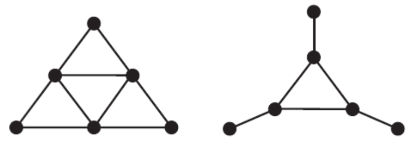

Fig. 1 (a) The graph F. (b) The complement graph of F, denoted by $\bar{F}$

II. TERMINOLOGY AND KNOWN RESULTS

Let G = (V, E) be a finite and simple graph with its vertex set V and edge set E. Two vertices u and v are *adjacent* in G if $(u,v) \in E$. For any $u \in V$, the *neighborhood* of u in G is defined by $N_G(u) = \{v|(u,v) \in E\} \subset V$. The *degree* of u in G, denoted by $\deg_G(u)$, is the number $|N_G(u)|$. The *minimum degree* $\delta(G)$ of G is defined as $\delta(G) = \min\{\deg_G(u) \mid u \in V\}$. $\sigma_k(G)$ denotes the minimum degree sum taken over all independent sets of k vertices of G. The *complement* graph $\bar{G} = (V', E')$ of a graph G = (V, E) is defined as $V = V'$ and $E' = \{(u,v) \mid (u,v)$ does not belong to $E\ \forall u,v \in V\}$. For undefined notations and terminologies, we follow [4].

A path P between two vertices v_0 and v_k is represented by $P = \langle v_0, v_1, \ldots, v_k \rangle$, where all vertices are different and every two consecutive vertices are adjacent. We also write the path $P = \langle v_0, v_1, \ldots, v_k \rangle$ as $\langle v_0, v_1, \ldots, v_i, P', v_j, v_{j+1}, \ldots, v_k \rangle$, where P' denotes the path $\langle v_i, v_{i+1}, \ldots, v_j \rangle$. A path of G is called a *Hamiltonian path* if it traverses all vertices of V exactly once. A cycle of G is called a *Hamiltonian cycle* if the cycle traverses all vertices of V exactly once except the beginning vertex and the end vertex. We say that a graph G is *Hamiltonian* if there exists a Hamiltonian cycle in G. The *circumference* c(G) of a graph G is defined as the length of the longest cycle in G. We define k as the *vertex connectivity* of G, and k(G) the number of components of G. Suppose G is not a complete graph. We say G is t-*tough* if t is a nonnegative real number and $t \leq |S|/k(G-S)$, where S is a vertexcut of G. The maximum real number t for which G is t-tough is called the *toughness* of G, and the toughness of any complete graph is ∞. It is known that every Hamiltonian graph is 1-tough, and every 1-tough graph is 2-connected.

Let G_1 and G_2 be two graphs. G_1 and G_2 are called *disjoint* if G_1 and G_2 have no vertex in common. The *union* of two disjoint graphs, G_1 and G_2, denoted by $G_1 + G_2$, is a graph with $V(G_1+G_2) = V(G_1) \cup V(G_2)$ and $E(G_1+G_2) = E(G_1) \cup E(G_2)$. The union of n copies of a graph G is written as nG. Obviously, $\overline{K_n} = nK_1$. The *join* of two disjoint subgraphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph obtained from $G_1 + G_2$ by joining each vertex of G_1 to each vertex of G_2.

Here, we list some known theorems, which will be used in the following sections.

Theorem 1. (see [5], [6].) If G is a 1-tough graph with |G| =n ≥ 11 such that $\sigma_2(G) \geq n-4$, then G is hamiltonian.

Theorem 2. (see [6], [7].) If G is a 1-tough graph with |G| = n ≥ 3, then $c(G) \geq \min\{n, \sigma_2(G)+2\}$.

We have an immediate result from the above theorem.

Corollary 1. If G is a 1-tough graph with |G| = n ≥ 3 such that $\sigma_2(G) \geq n-2$, then G is Hamiltonian.

Theorem 3. (see [8].) Let G be a 1-tough graph on |G| = n ≥ 3 vertices with $\delta(G) \geq n/3$. Then $c(G) \geq 5n/6+1$.

Theorem 4. (see [1].) If G is a 1-tough graph with |G| = n ≥ 3 and $\sigma_3(G) \geq n+k-2$, then G is Hamiltonian.

III. Main Results

It is easy to see that the complete bipartite graph $K_{m,m}$, where m ≥ 6, is 1-tough, Hamiltonian, and its complement $\overline{K_{m,m}} = K_m \cup K_m$ contains F in Fig. 1 (a). Thus, $K_{m,m}$ provides a family of bipartite graphs which are counterexamples to Conjecture 1. For nonbipartite cases, let n ≥ 8, and $D_n = \{\overline{K_3} \vee K_2 \vee K_{n-5}\} \cup \{(a,x),(b,y),(c,z)\}$, where $\{a,b,c\}$ are the three isolated vertices of $\overline{K_3}$ and $\{x,y,z\} \in V(K_{n-5})$. See Fig. 2 (a) for an illustration.

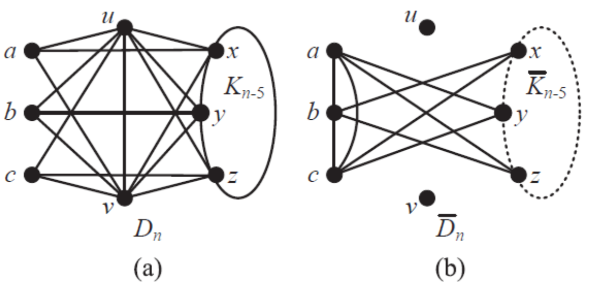

Fig. 2 (a) The graph D_n. (b) The complement graph of D_n, denoted by $\overline{D_n}$

We have the following lemma.

Lemma 1. For n ≥ 8, D_n is 1-tough, hamiltonian, and its complement graph $\overline{D_n}$ contains the graph F.

Proof. By brute force, D_n is 1-tough. (In fact, D_8 is $\frac{4}{3}$-tough and D_n is $\frac{5}{4}$-tough for n ≥ 9.) Next, we will show that D_n is Hamiltonian. Because K_{n-5} is a complete graph, there exists a Hamiltonian path P in K_{n-5} between x and y. Thus D_n has a Hamiltonian cycle $\langle x,a,u,c,v,b,y,P,x\rangle$. On the other hand, $\overline{D_n}$ contains the edges, $\{(a,b),(b,c),(c,a),(a,y),(y,c),(b,x),(x,c),(a,z),(z,b)\}$, which implies that $\overline{D_n}$ contains the graph F. Therefore, D_n serves to illustrate that Conjecture 1 is false.

Theorem 5 affirms that Conjecture 1 is true for graphs with six or seven vertices.

Theorem 5. Let |G| = n ∈ {6,7}. If G is 1-tough, then either G is Hamiltonian or its complement $\bar{G}$ contains the graph F.

Proof. We consider G with |G| = 6 first. In this case, we want to show that G is Hamiltonian and its complement $\bar{G}$ does not contain F. By Theorem 3, c(G) = 6, so G is a Hamiltonian graph. Assume that $\bar{G}$ contains F, then G must contain fewer edges then $\bar{F}$, the complement of F. See Fig. 1 (b) for an illustration of $\bar{F}$. Since the graph $\bar{F}$ is $\frac{1}{2}$-tough, G cannot be better than $\frac{1}{2}$-tough, which violates the known condition that G is 1-tough. Therefore, $\bar{G}$ does not contain F. Next, we consider G with |G| = 7. Note that G being 1-tough implies that k ≥ 2. There are two cases.

Case 1. $\sigma_3(G) \geq 7$. With Theorem 4, G contains a Hamiltonian cycle, denoted by $C_G = \langle 1,2,3,4,5,6,7,1\rangle$. Obviously, E(G) consists of all edges in C_G and possibly more. Let C_7 be a cycle with length 7 and $\overline{C_7}$ the complement of C_7. It is easy to see that $\overline{C_7}$ does not contain F, so $\bar{G}$ cannot contain F. As a result, Conjecture 1 holds in this case.

Case 2. $\sigma_3(G) \leq 6$. Since k ≥ 2, this case occurs only when there exists an independent set of three vertices $\{x,y,z\}$ such that $deg_G(x) = deg_G(y) = deg_G(z) = 2$, and $\sigma_3(G) = 6$. We shall let $V(G) = \{x,y,z,a,b,c,d\}$. Under Case 2, there are three major subcases and totally five possibilities for which we must provide rigorous proofs. Table I gives an illustration for these possible situations. For simplicity, we shall label these subcases by (a), (b), (c) and so on.

TABLE I
Case Analysis in the Proof Case 2 in Theorem 5

(a) $\lvert N_G(x) \cup N_G(y) \cup N_G(z)\rvert = 2$
(b) $\lvert N_G(x) \cup N_G(y) \cup N_G(z)\rvert = 3$
(c) $\lvert N_G(x) \cup N_G(y) \cup N_G(z)\rvert = 4$
(d) Two of $N_G(x)$, $N_G(y)$ and $N_G(z)$ are identical.
(e) All of $N_G(x)$, $N_G(y)$ and $N_G(z)$ are different.
(f) None of $N_G(a)$, $N_G(b)$, $N_G(c)$ and $N_G(d)$ covers $\{x,y,z\}$.
(g) One of $N_G(a)$, $N_G(b)$, $N_G(c)$ and $N_G(d)$ covers $\{x,y,z\}$.

a) $|N_G(x) \cup N_G(y) \cup N_G(z)| = 2$. Let $N_G(x) \cup N_G(y) \cup N_G(z) = \{a,b\}$. Thus, the subgraph induced by $\{x,y,z,a,b\}$ is $K_{3,2}$, and G is not Hamiltonian. Removing $\{a,b\}$ results in a graph with at least four components, so G is $\frac{1}{2}$-tough or weaker. It violates the assumption that G is 1-tough, so this case cannot happen.

b) $|N_G(x) \cup N_G(y) \cup N_G(z)| = 3$. Let $N_G(x) \cup N_G(y) \cup N_G(z) = \{a,b,c\}$. Removing $\{a,b,c\}$ results in a graph with at least four components, so G is $\frac{3}{4}$--tough or weaker. Again, it contradicts the known fact that G is 1-tough, so this case should not occur.

c) $|N_G(x) \cup N_G(y) \cup N_G(z)| = 4$.There are two possibilities: (d) and (e).

d) Two of $N_G(x)$, $N_G(y)$ and $N_G(z)$ are identical. W.L.O.G., let $N_G(x) = N_G(y) = \{a,b\}$ and $N_G(z) = \{c,d\}$. In this case, removing $\{a,b\}$ results in a graph with at least three components, so G is $\frac{2}{3}$-tough or weaker, which violates the condition that G is 1-tough, so this case will not happen.

e) All of $N_G(x)$, $N_G(y)$ and $N_G(z)$ are different. There are two subcases. There are two subcases: (f) and (g).

f) None of $N_G(a)$, $N_G(b)$, $N_G(c)$ and $N_G(d)$ covers $\{x,y,z\}$. W.L.O.G., let $N_G(x) = \{a,b\}$, $N_G(y) = \{b,c\}$, and $N_G(z) = \{c,d\}$. See Fig. 3 for an illustration. If $(a,d) \in E(G)$, then

$\langle a, x, b, y, c, z, d, a\rangle$ is a cycle of length 7, which is a Hamiltonian cycle of G. The argument in Case 1 shows that $\bar{G}$ does not contain F, so Conjecture 1 holds in this case. Now, we discuss the situation when (a, d) does not belong to E(G). The set of edges among $\{a, b, c, d\}$ contains at most $\{(a, b), (b, c), (c, d), (a, c), (b, d)\}$. Removing $\{b, c\}$ results in a graph with at least three components, so G is $\frac{2}{3}$-tough or weaker. It is not possible.

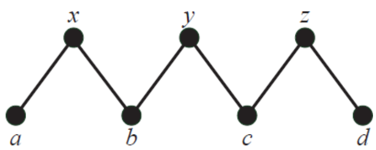

Fig. 3 An illustration for Case 2, (f) in the proof of Theorem 5

g) One of $N_G(a)$, $N_G(b)$, $N_G(c)$ and $N_G(d)$ covers $\{x, y, z\}$. W.L.O.G., let $N_G(a)$ be the one covering $\{x, y, z\}$, and let $N_G(x) = \{a, b\}$, $N_G(y) = \{a, c\}$, and $N_G(z) = \{a, d\}$. It is easy to see that G must be non-Hamiltonian. Moreover, $\bar{G}$ contains the triangle with vertices $\{x, y, z\}$ and the edges $\{(x, c), (x, d), (y, b), (y, d), (z, b), (z, c)\}$. That is, $\bar{G}$ contains F. Since G is 1-tough, it can be observed that E contains $\{(b, c), (c, d), (b, d)\}$ while the edges $(a, b), (a, c), (a, d)$ are optional. See Fig. 4 for an illustration. We note that the graph with $deg_G(a) = 6$ is isomorphic to the graph H in [2]. Thus, Conjecture 1 is true in this case.

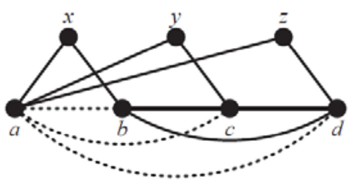

Fig. 4 An illustration for Case 2, (g) in the proof of Theorem 5

From the above derivation, we conclude that for any 1-tough graph G with $|G| = 7$, either G contains a Hamiltonian cycle or G is of the form in Fig. 4, of which the complement contains F.

The following two lemmas are derived in order to obtain the correction for Conjecture 1 for graphs with eight or more vertices. We denote the complement of G by $\bar{G}$. The graph F^* is shown in Fig. 5.

Lemma 2. Let G be a 1-tough graph with $|G| = n \geq 11$. The following three statements are equivalent.

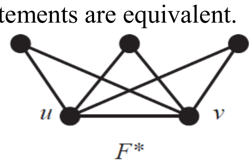

Fig. 5 The graph F^*

i) There exists some nonadjacent pair $\{x, y\}$ in G such that $deg_G(x) + deg_G(y) \leq n - 5$.

ii) There exists an edge (x, y) of $\bar{G}$ such that $deg_{\bar{G}}(x) + deg_{\bar{G}}(y) \geq n + 3$.

iii) $\overline{G}$ contains the graph F^*.

Proof. First of all, we want to show (i) implies (ii). Take the edge (u, v) of $\bar{G}$ such that the nonadjacent vertex pair $\{u, v\}$ in G satisfies $deg_G(u) + deg_G(v) \leq n - 5$. Therefore, $deg_{\bar{G}}(u) + deg_{\bar{G}}(v)$

$$= (n - 1 - deg_G(u)) + (n - 1 - deg_G(v))$$
$$\geq (2n - 2) - (n - 5)$$
$$= n + 3.$$

Secondly, we need to show (ii) implies (iii). Following (ii), there are n – 2 vertices in $V(\bar{G}) - \{u, v\}$. If $N_{\bar{G}}(u) \cap N_{\bar{G}}(v) = \emptyset$, then $deg_{\bar{G}}(u) + deg_{\bar{G}}(v) \leq 1 + 1 + (n - 2) = n$. It violates (ii). Thus, $N_{\bar{G}}(u)$ and $N_{\bar{G}}(v)$ must have at least three common vertices in $V(\bar{G}) - \{u, v\}$. This implies that $\bar{G}$ contains the graph F^*.

Finally, we will show (iii) implies (i). This part will be shown by deducing a contradiction from the opposite assumption. Assume that $\bar{G}$ contains the graph F^*, and $deg_G(x) + deg_G(y) \geq n - 4$ holds for any nonadjacent pair of vertices $\{x, y\}$ of G. With a simple calculation, one can see that $deg_G(x) + \deg(y) \leq n + 2$ holds for any edge (x, y) of $\bar{G}$. As in the previous argument, it means that the endvertices x and y of any edge (x, y) in $\bar{G}$ have at most two common neighbors. Then, $\bar{G}$ cannot contain F^*, which violates (iii). Consequently, there must be some nonadjacent pair of vertices $\{x, y\}$ in G with $deg_G(x) + deg_G(y) \leq n - 5$.

Lemma 3 can be obtained using the similar derivation as in Lemma 2.

Lemma 3. Let G be a 1-tough graph with $|G| = n \in \{8,9,10\}$. The following three statements are equivalent.

i) There exists some nonadjacent pair $\{x, y\}$ in G with $deg_G(x) + deg_G(y) \leq n - 3$.

ii) There exists an edge (x, y) of $\bar{G}$ such that $deg_G(x) + deg_G(y) \geq n + 1$

iii) The complement of G, denoted by $\overline{G}$, contains the graph K_3.

Our correction of Conjecture 1 for graphs with eight or more vertices is presented below.

Theorem 6. Let G be a 1-tough graph with $|G| = n \geq 8$. Then

a) For $n \geq 11$, either $\sigma_2(G) \geq n - 4$ or $\overline{G}$ contains F^*.

b) For $n \in \{8,9,10\}$, either $\sigma_2(G) \geq n - 2$ or $\overline{G}$ contains K_3.

Proof. We will explain (a), where $n \geq 11$, in detail and skip the similar discussion for (b). There are two cases.

Case 1. Suppose that $deg_G(x) + deg_G(y) \leq n - 4$ holds for any nonadjacent pair of vertices $\{x, y\}$ of G. With Theorem 1, G is Hamiltonian. Note that the degree-sum condition is the sufficient condition for G to be Hamiltonian, and the converse is not true.

Case 2. Suppose that there exists some nonadjacent pair of vertices $\{x, y\}$ of G such that $deg_G(x) + deg_G(y) \leq n - 5$. With Lemma 2, it is equivalent to saying that $\bar{G}$ contains F^*.

Combining Case 1 and 2, (a) is verified. When we apply Corollary 1 for (b) concerning the case where $\sigma_2(G) \geq n - 2$, the similar difficulty appears. The fact that the degree-sum condition provides only the sufficient condition for G to be Hamiltonian, not the necessary condition prevents us from a stronger conclusion as in Conjecture 1.

As a result, Theorem 6 corrects Conjecture 1 for $n \geq 8$ and becomes the best that one can have.

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