Free Vibration of Axially Functionally Graded Simply Supported Beams Using Differential Transformation Method

A. Selmi

Abstract—Free vibration analysis of homogenous and axially functionally graded simply supported beams within the context of Euler-Bernoulli beam theory is presented in this paper. The material properties of the beams are assumed to obey the linear law distribution. The effective elastic modulus of the composite was predicted by using the rule of mixture. Here, the complexities which appear in solving differential equation of transverse vibration of composite beams which limit the analytical solution to some special cases are overcome using a relatively new approach called the Differential Transformation Method. This technique is applied for solving differential equation of transverse vibration of axially functionally graded beams. Natural frequencies and corresponding normalized mode shapes are calculated for different Young's modulus ratios. MATLAB code is designed to solve the transformed differential equation of the beam. Comparison of the present results with the exact solutions proves the effectiveness, the accuracy, the simplicity, and computational stability of the differential transformation method. The effect of the Young's modulus ratio on the normalized natural frequencies and mode shapes is found to be very important.

Keywords—Differential transformation method, functionally graded material, mode shape, natural frequency.

I. INTRODUCTION

WITH the increasing demands in modern technologies, it seems that the conventional homogeneous materials no longer can efficiently meet the needs of industries; thus, new advanced materials with special mechanical characteristics should be fabricated.

Functionally graded material (FGM) is achieved by controlling the volume fractions of the material constituents during manufacturing. FGM has exceptional properties compared to traditional homogeneous materials due to the continuous transition of material properties. These kinds of materials are increasingly being used in wide applications in modern industries including aerospace, mechanical, electronics, optics, chemical, biomedical, nuclear, and civil engineering.

Design of structures based on FGM to resist dynamic forces, such as wind and earthquakes, requires knowledge of their vibration properties particularly their natural frequencies and their mode shapes.

Few analytical solutions are found for arbitrary gradient

A. Selmi is with the National Engineering School of Tunis, Civil Engineering Laboratory B.P. 37, Le belvédère 1002, Tunis, Tunisia (e-mail: selmi_fr2016@yahoo.com).

change due to the difficulty of mathematical treatment of the problem [1]. For example, Aydogdu used the semi-inverse method to treat a large class of problems involving graded beams of special forms and obtained explicit fundamental frequency [2]. However, the semi-inverse method cannot apply for graded beams of any axial non-homogeneity. In addition, with the aid of special functions, Huang and Li solved some free vibration and buckling problems of axially graded beams [3]. Nevertheless, the assumption of non-homogeneity or non-uniformity still has special requirements and is not arbitrary. Consequently, the application of numerical techniques seems to be inevitable. Various numerical methods are developed in order to calculate the frequencies and mode shapes of beams. Different variational techniques such as Rayleigh Ritz and Galerkin methods had been applied in the past. Due to advancement in computational techniques and availability of software, FE method is quite a less cumbersome than the conventional methods [4].

To the best knowledge of the authors, no research effort has been devoted to find the solution of vibrational behavior of a linearly axially FG beam by employing differential transformation method (DTM). Motivated by these considerations, the need for investigation of vibrational behavior of the linearly axially FG beams using DTM is necessary.

In this study, the free vibration of axially functionally graded simply supported Euler-Bernoulli beams is considered. The rule of mixture is adopted to determine the mechanical properties of the composite.

II. BASIC IDEA OF DTM

The DTM is a numerical method that uses Taylor series for the solution of differential equations. This method has wide applications in structural dynamics, fluid flow, and heat transfer problems and nonlinear oscillators problems [5]-[7]. DTM is directly used to solve governing equations and gives the solutions for whole domain. In this method, incorporation of boundary conditions is easily performed. DTM does not pose any restrictions on both the type of material gradation and the variation of the cross-section profile; hence, it could cover most of the engineering problems dealing with the mechanical behavior of non-uniform and non-homogenous structures.

The differential transformation of a sufficiently differentiable function $u(x)$ is defined as

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$$U(k) = \frac{1}{k!}\left(\frac{d^k u(x)}{dx^k}\right)_{x_0} \qquad (1)$$

where k is a positive integer.

In this paper, x_0 is set to zero. The inverse differential transform is known as a presentation of $u(x)$ by power series:

$$u(x) = \sum_{k=0}^{\infty} x^k U(k) \qquad (2)$$

III. FORMULATION OF THE PROBLEM

Let us consider a rectangular beam of uniform cross-section A, length L, and made of FGM. The volume fraction of the reinforcement is assumed to vary continuously along the length direction according to linear-law form:

$$v_x = v_L + v_R \frac{X}{L} \qquad (3)$$

where v_L and v_R are the volume fraction of the reinforcement at the left and right side of the beam.

According to the Euler Bernoulli beam theory, the governing differential equation for free transverse vibration of axially FG beams is given as:

$$\rho(X)A\frac{\partial^2 w(X,t)}{\partial t^2} + \frac{\partial^2}{\partial X^2}\left[E(X)I\frac{\partial w(X,t)}{\partial X^2}\right] = 0 \qquad (4)$$

where $w(X, t)$ represents transverse displacement of the beam, $E(X)$ is the modulus of elasticity, I is the moment of inertia and $\rho(X)$ is the density of the beam material. X is the distance from the left end of the beam.

For simply supported beam, the corresponding boundary conditions can be stated as:

$$w(0,t) = 0; \frac{\partial w(0,t)}{\partial X} = 0; w(L,t) = 0; \frac{\partial w(L,t)}{\partial X} = 0 \qquad (5)$$

Assuming the transverse displacement of the beam as follows, where ω is the circular natural frequency:

$$w(X,t) = Y(X)\exp(i\omega t) \qquad (6)$$

Substituting the expression of the transverse displacement into the governing differential equation, (4) takes the form:

$$-\rho(X)AY(X)\omega^2 + \frac{d^2}{dX^2}\left[E(X)I\frac{d^2 Y(X)}{dX^2}\right] = 0 \qquad (7)$$

which can be written as

$$\frac{d^2 Y(X)}{dX^4} + 2I\frac{E(X)}{dX}\frac{d^3 Y(X)}{dX^2} + \frac{d^2 E(X)}{dX^2}\frac{d^2 Y(X)}{dX^2} = \rho(X)A\omega^2 Y(X) \qquad (8)$$

The substitution of the expression of the transverse displacement into the boundary conditions equations, (5) takes the form.

$$Y(0) = 0; \frac{\partial Y(0)}{\partial X} = 0; Y(L) = 0; \frac{\partial Y(L)}{\partial X} = 0$$

IV. DIMENSIONLESS FORM OF THE GOVERNING DIFFERENTIAL EQUATION AND BOUNDARY CONDITIONS

Defining the non-dimensional parameters:

$$x = \frac{X}{L}, \quad u(x) = \frac{Y(X)}{L}, \quad S(X) = \frac{E(X)}{E_m}, \quad p(x) = \frac{\rho(X)}{\rho_m}$$

where E_m and ρ_m are the matrix Young's modulus and matrix density, respectively.

The equation of motion can be rewritten in terms of dimensionless variables as:

$$p(x)\Omega^2 u(x) = S(x)\frac{d^4 u(x)}{dx^4} + 2I\frac{S(x)}{dx}\frac{d^3 u(x)}{dx^3} + \frac{d^2 S(x)}{dx^2}\frac{d^2 u(x)}{dx^2} \qquad (9)$$

where $\Omega^2 = \omega^2\frac{\rho_m AL^4}{E_m I}$ and the dimensionless boundary conditions can be expressed as:

$$u(0) = 0, \quad \frac{d^2 u}{dx^2}(0) = 0, \quad u(1) = 0, \quad \frac{d^2 u}{dx^2}(1) = 0 \qquad (10)$$

Applying the principle of DTM to the non-dimensional governing equation, the following recurrence relations are obtained:

For homogenous reinforcement distribution:

$$U_{k+4} = \Omega^2 \frac{U_k}{(k+1)(k+2)(k+3)(k+4)} \qquad (11)$$

For linear reinforcement distribution:

$$U_{k+4} = \Omega^2 \frac{eU_k + mU_{k-1}}{(k+1)(k+2)(k+3)(k+4)} - \frac{f}{g}\frac{k+2}{k+4}U_{k+3} \qquad (12)$$

where:

$$f = \frac{(E_r - E_m)v_r}{E_m} = (r-1)v_r$$

$$g = \frac{(E_r - E_m)v_r}{E_m} + \frac{E_m}{E_m} = (r-1)v_r + 1$$

$$m = \frac{(\rho_r - \rho_m)v_r}{\rho_m} = (r_\rho - 1)v_r$$

$$e = \frac{(\rho_r - \rho_m)v_r}{\rho_m} + \frac{\rho_m}{\rho_m} = (r_\rho - 1)v_r + 1$$

The application of the DTM to the non-dimensional boundary conditions (10) yields

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$$U_0 = 0, \quad U_2 = 0, \quad \sum_{k=0}^{\infty} U_k, \quad \sum_{k=0}^{\infty} k(k+1)U_k \qquad (13)$$

U_1 and U_3 are the set for unknown constants: $U_1 = c$ and $U_3 = d$. Equation (13) gives nonlinear equation in terms of Ω and linear in terms of c and d. Putting the boundary conditions (13) in matrix form, we get,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (14)$$

where A and C are the coefficients of c, B and D are the coefficients of d. Since c and d are different from zero, the determinant of matrix must be equal to zero. Hence,

$$A.D - B.C = 0 .$$

Depending upon the number of terms k taken, we get a higher degree polynomial in Ω. The solution of the polynomial equation gives the dimensionless frequency Ω.

V. NUMERICAL RESULTS

In order to check the convergence of the DTM, the case of a homogenous beam, made of a matrix reinforced with 20%

randomly distributed inclusions, is considered. For this case, the natural frequencies can be exactly calculated.

Evaluated results of first three non-dimensional natural frequencies and the exact ones are tabulated in Table I.

TABLE I
FIRST THREE NON-DIMENSIONAL FREQUENCIES OF HOMOGENOUS EULER
BEAMS FOR DIFFERENT NUMBER OF TERMS K

k	Ω_1	Ω_2	Ω_3
19	9.8696044699	39.4169139196	-
20	9.8696044699	39.4169139196	-
25	9.8696044011	39.4784501712	87.8912222720
30	9.8696044011	39.4784176959	88.8107229014
35	9.8696044011	39.4784176044	88.8264496362

From Table I, one can find that numerical results have rapid convergence. It can be seen that the first non-dimensional frequency converges first (k=25) then the second one (k=35) and finally the third non-dimensional frequency which requires relatively high number of terms (k=45).

One can conclude that the estimated first three normalized frequency using DTM are identical to the exact results, which indicates that the present approach is very efficient.

It is observed that increasing the number of terms, k, improves the accuracy of results and leads to convergent solutions at k=45. Hence, k=45 is used in the following numerical calculations.

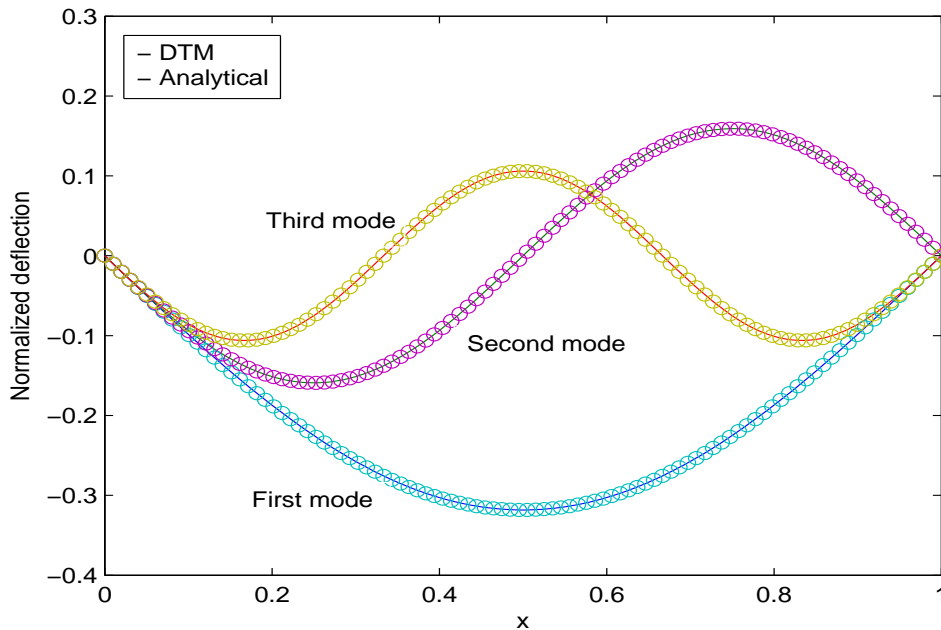Fig. 1 First three mode shapes of homogenous beam

TABLE II
FIRST THREE NON-DIMENSIONAL FREQUENCIES OF FG EULER BEAMS FOR
DIFFERENT VALUES OF R

r	Ω_1	Ω_2	Ω_3
1	9.8696044011	39.4784176044	88.8264396098
2	10.8103016470	43.2396217674	97.2880894790
5	13.2300426379	52.9063156317	119.0244088607
10	16.4851145184	65.9041393487	148.2077909727

Fig. 1 shows the first three mode shapes of homogenous Euler beams. From Fig. 1, it can be seen that the first three mode shapes drawn using DTM are very close to the mode shapes drawn using exact method.

Table II gives results of the first three non-dimensional frequencies of simply supported FG Euler beams with different values of r. The reinforcement volume fraction is

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considered to vary linearly along the longitudinal direction.

The reinforcement content varies from 15% on the left to 25% on the right beam surface. It can be demonstrated that the first three frequencies of the simply non-uniform beam increase with increasing the Young's modulus ratio.

Figs. 2-4 show, respectively, the first, the second, and the third mode shapes of simply supported FG Euler beams with different Young's modulus ratios.
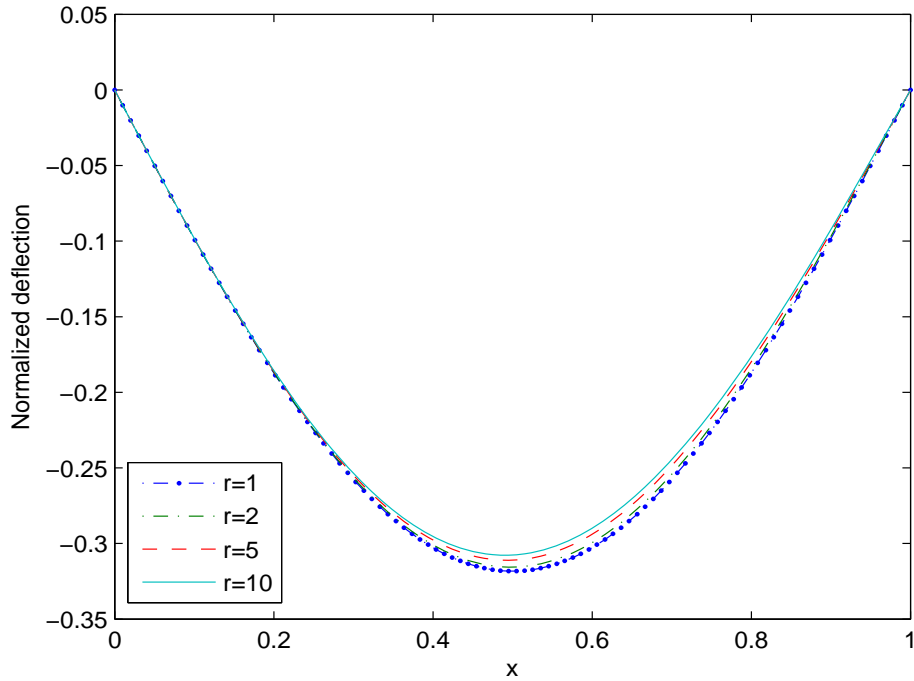

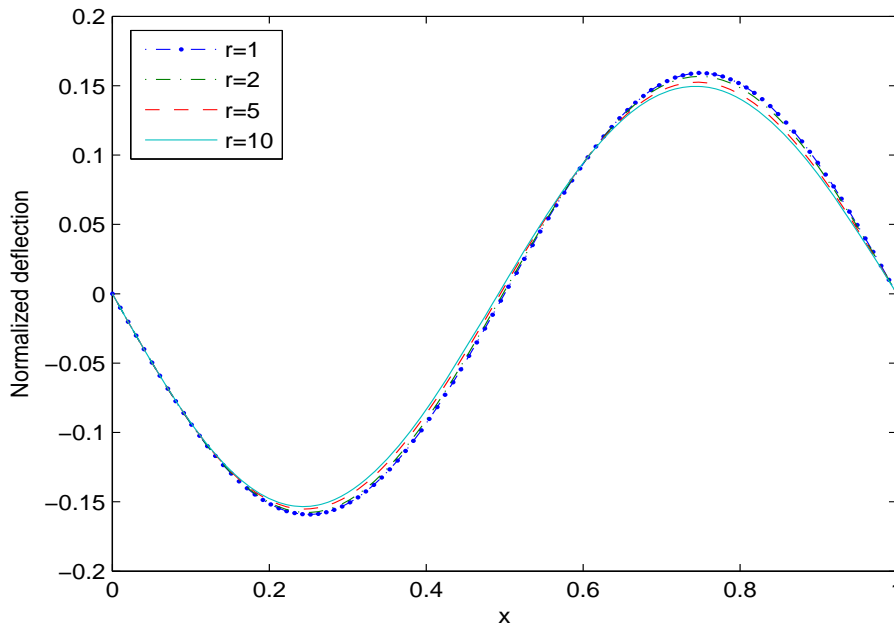Fig. 2 First mode shapes for four different values of r

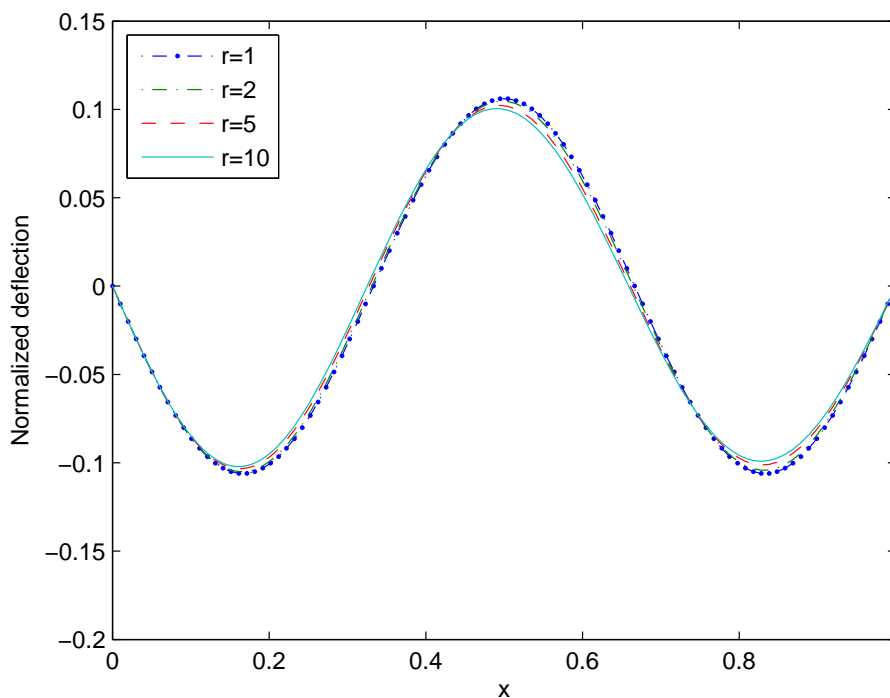Fig. 3 Second mode shapes for four different values of r

Fig. 4 Third mode shapes for four different values of r

From Figs. 2-4, it can be seen that the value of r has a significant effect on the mode shapes and the deflection values. Deflection of FG beams decreases as the reinforcement Young's modulus to the matrix Young's modulus ratio increases (r values increases).

VI. CONCLUSION

The DTM approach has been presented to solve free vibration of Euler Bernoulli axially linear FG beams. The effect of the reinforcement Young's modulus to the matrix Young's modulus ratio on the fundamental frequencies and mode shapes of free vibration of FG beam is investigated.

Based on the numerical results, it is found that the frequency results and mode shapes of FG beams are affected considerably by the changes in the ratio, r.

The frequency of non-homogenous beam decreases with decreasing of the ratio r. Also, it is observed that the deflection of the non-homogenous Euler beam is decreased as the ratio r increased.

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