

# Strict Stability of Fuzzy Differential Equations by Lyapunov Functions

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**Abstract**—In this study, we have investigated the strict stability of fuzzy differential systems and we compare the classical notion of strict stability criteria of ordinary differential equations and the notion of strict stability of fuzzy differential systems. In addition that, we present definitions of stability and strict stability of fuzzy differential equations and also we have some theorems and comparison results. Strict Stability is a different stability definition and this stability type can give us an information about the rate of decay of the solutions. Lyapunov's second method is a standard technique used in the study of the qualitative behavior of fuzzy differential systems along with a comparison result that allows the prediction of behavior of a fuzzy differential system when the behavior of the null solution of a fuzzy comparison system is known. This method is a usefull for investigating strict stability of fuzzy systems. First of all, we present definitions and necessary background material. Secondly, we discuss and compare the differences between the classical notion of stability and the recent notion of strict stability. And then, we have a comparison result in which the stability properties of the null solution of the comparison system imply the corresponding stability properties of the fuzzy differential system. Consequently, we give the strict stability results and a comparison theorem. We have used Lyapunov second method and we have proved a comparison result with scalar differential equations.

**Keywords**—Fuzzy systems, fuzzy differential equations, fuzzy stability, strict stability.

## I. INTRODUCTION

**S**TABILITY of differential equations is very important topic for math, engineering and related areas [1],[2], [6]-[15], [17]-[21], [24]-[40], [46]-[48]. Strict Stability is a different stability definition and this stability can give us some information about the rate of decay of the solutions [22]. Strict stability of differential equations, strict stability in impulsive functional differential equations and strict stability of differential equations with initial time difference studied and new results obtained [22], [38]-[45], [49].

In this paper, we studied the definitions and comparison theorems of strict stability of fuzzy differential equations.

## II. PRELIMINARIES

First of all, we must define the fundamental properties of fuzzy sets, fuzzy derivative and fuzzy calculus for the fuzzy differential equations and the stability of fuzzy differential equations. For fuzzy sets, we take as the base space  $R^n$ . The fuzzy set  $u \in E^n$  is a function  $u : R^n \rightarrow [0, 1]$ . Where  $E^n$  is the space of all fuzzy subsets of  $R^n$  and  $E^n = \{u : R^n \rightarrow [0, 1] \mid u \text{ satisfies (i) to (iv)}\}$  below

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(i)  $u$  is normal if there exists an  $x_0 \in R^n$  such that  $u(x_0) = 1$ ;

(ii)  $u$  is fuzzy convex if for  $x, y \in R^n$  and  $0 \leq \lambda \leq 1$  such that

$$u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)];$$

(iii)  $u$  is upper semi-continuous;

(iv)  $[u]^0 = [x \in R^n : u(x) > 0]$  is compact subset of  $R^n$ .

The function  $u(x)$  for  $x \in R^n$  taking values in  $[0, 1]$ . Now we state the following membership grade of the  $u(x)$ ,  $u(x) = 0$  meaning to nonmembership,  $0 < u(x) < 1$  to partial membership and  $u(x) = 1$  to full membership.

Let  $K_c^n$  be the family of all nonempty compact, convex subsets of  $R^n$ .

For  $0 < \alpha \leq 1$ , we denote the  $\alpha$ -level sets as  $[u]^\alpha = [x \in R^n : u(x) \geq \alpha]$ . Then from (i) to (iv), it follows that  $[u]^\alpha \in K_c^n$  for  $0 \leq \alpha \leq 1$ .

Let  $d_H(A, B)$  be the Hausdorff distance between the sets  $A, B \in K_c^n$ . Then we define

$$d[u, v] = \sup_{0 \leq \alpha \leq 1} d_H([u]^\alpha, [v]^\alpha)$$

which defines a metric in  $E^n$  and  $(E^n, d)$  is a complete metric space. We have the following properties of  $d[u, v]$ :

$$(i) \quad d[u + w, v + w] = d[u, v],$$

$$(ii) \quad d[u, v] = d[v, u],$$

$$(iii) \quad d[\lambda u, \lambda v] = |\lambda| d[u, v],$$

$$(iv) \quad d[u, v] \leq d[u, w] + d[w, v],$$

for all  $u, v, w \in E^n$  and  $\lambda \in R$ .

For  $x, y \in E^n$  if there exists a  $z \in E^n$  such that  $x = y + z$ , then  $z$  is called the  $H$ -difference of  $x$  and  $y$  and is denoted by  $x - y$ .

**Definition 1:** Let  $F : (a, b) \rightarrow E^n$ ,  $t_0 \in (a, b)$  and  $F$  is differentiable at  $t_0$ . If there exists  $F'(t_0) \in E^n$  such that

(i) For all  $h > 0$  sufficiently small there exist  $F(t_0 + h) - F(t_0)$ ,  $F(t_0) - F(t_0 - h)$  and the limits  $(\text{in}(E^n, d))$

$$\lim_{h \rightarrow 0^+} (1/h)(F(t_0 + h) - F(t_0)) = \lim_{h \rightarrow 0^+} (1/h)(F(t_0) - F(t_0 - h)) = F'(t_0) \quad (1)$$

under the similar conditions, and other definition is,

(ii) For all  $h < 0$  sufficiently small there exist  $F(t_0 + h) - F(t_0)$ ,  $F(t_0) - F(t_0 - h)$  and the limits  $(\text{in}(E^n, d))$

$$\lim_{h \rightarrow 0^-} (1/h)(F(t_0 + h) - F(t_0)) = \lim_{h \rightarrow 0^-} (1/h)(F(t_0) - F(t_0 - h)) = F'(t_0) \quad (2)$$

For details please see [3]-[5].

### III. FUNDAMENTAL DEFINITIONS AND THEOREMS OF STRICT STABILITY CRITERIA OF FUZZY DIFFERENTIAL SYSTEMS

Consider the initial value problem for the fuzzy differential equations

$$u' = f(t, u), u(t_0) = u_0 \text{ for } t \geq t_0 \geq 0 \quad (3)$$

$$u' = f(t, u), u(\tau_0) = v_0 \text{ for } t \geq \tau_0 \geq 0 \quad (4)$$

where  $f \in C[R_+ \times S\rho, E^n]$ ,  $S\rho = [u \in E^n : d[u, \tilde{0}] < \rho < \infty]$ ,  $f(t, \tilde{0}) \equiv \tilde{0}$  for  $t \geq 0$ .

Before we can establish our comparison theorem and Lyapunov functions with strict stability criteria of fuzzy differential equations we need to introduce the following definitions.

**Definition 2:** The trivial solution of the system (3) is said to be strictly stable, if given  $\epsilon_1 > 0$  and  $t_0 \in R_+$ , there exist a  $\delta_1 = \delta_1(\epsilon_1, t_0) > 0$  such that  $d[u_0, \tilde{0}] < \delta_1$  implies  $d[u(t), \tilde{0}] < \epsilon_1$ ,  $t \geq t_0$ , and for every  $0 < \delta_2 \leq \delta_1$ , there exists an  $0 < \epsilon_2 < \delta_2$  such that

$$\delta_2 < d[u_0, \tilde{0}] \text{ implies } \epsilon_2 < d[u(t), \tilde{0}] \text{ for } t \geq t_0$$

If  $\delta_1, \delta_2$  and  $\epsilon_2$  in Definition 2 are independent of  $t_0$ , then the trivial solution of the system (3) is said to be strictly uniformly stable for  $t \geq t_0$ .

**Definition 3:** The trivial solution of the system (3) is said to be strictly attractive, if given  $\alpha_1 > 0$ ,  $\epsilon_1 > 0$  and  $t_0 \in R_+$ , for every  $\alpha_2 \leq \alpha_1$ , there exists  $\epsilon_2 < \epsilon_1$  and  $T_1 = T_1(\epsilon_1, t_0)$  and  $T_2 = T_2(\epsilon_2, t_0)$  such that

$$\alpha_2 < d[u_0, \tilde{0}] < \alpha_1$$

implies that

$$\epsilon_2 < d[u(t), \tilde{0}] < \epsilon_1 \text{ for } T_1 + t_0 \leq t \leq T_2 + t_0.$$

If  $T_1, T_2$  in Definition 3 are independent of  $t_0$ , then the trivial solution of the system (3) is said to be uniformly strictly attractive.

**Definition 4:** The trivial solution of the system (3) is said to be strictly asymptotically stable if Definition 3 holds and the trivial solution is stable.

The trivial solution of the system (3) is said to be strictly uniformly asymptotically stable if trivial solution of the system (3) is uniformly strictly attractive and the trivial solution of the system (3) is strictly uniformly stable.

**Definition 5:** A function  $\phi(r)$  is said to belong to the class  $\mathcal{K}$  if  $\phi \in C[(0, \rho), R_+]$ ,  $\phi(0) = 0$ , and  $\phi(r)$  is strictly monotone increasing in  $r$ . It is said to belong to class  $\mathcal{K}_\infty$  if  $\rho = \infty$  and  $\phi(r) \rightarrow \infty$  as  $r \rightarrow \infty$ .

**Definition 6:** For a real-valued function  $V(t, u) \in C[R_+ \times E^n, R_+]$  we define the Dini derivatives as follows

$$D^+V(t, u) \equiv \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(t+h, u+hf(t, u)) - V(t, u)]$$

$$D^-V(t, u) \equiv \liminf_{h \rightarrow 0^-} \frac{1}{h} [V(t+h, u+hf(t, u)) - V(t, u)]$$

for  $(t, x) \in R_+ \times E^n$ . In our earlier some studies and in the study of some others [39], [41]-[43], the differences between

the classical notion of stability and initial time difference stability did not allow the use of the behavior of the null solution in our initial time difference stability analysis. The main result presented in this section resolves those difficulties with a new approach that allows the use of the stability of the null solution of the comparison system to predict the stability properties of  $v(t, \tau_0, v_0)$  the solution of (3) with respect to  $\tilde{u}(t) = u(t - \eta, t_0, u_0)$  where  $u(t, t_0, u_0)$  is any solution of the system (3).

**Theorem 1:** Assume that  $f, F \in C[J \times E^n, E^n]$  and  
 (i)  $\lim_{h \rightarrow 0^+} \sup \frac{1}{h} [d[v - \tilde{u} + h(F(t, v) - \tilde{f}(t, \tilde{u})), \tilde{0}] - d[v - \tilde{u}, \tilde{0}]] \leq G(t, d[v - \tilde{u}, \tilde{0}])$   
 for  $t \in J$  and  $\tilde{u}, v \in E^n$  where  $G \in C[J \times R_+, R]$ ;  
 (ii)  $r(t) = r(t, \tau_0, w_{10})$  is the maximal solution of the scalar differential equation exists on  $J$

$$u' = G(t, u), u(\tau_0) = u_0 \geq 0 \text{ for } t \geq \tau_0 \text{ in } J. \quad (5)$$

Then, if  $u(t)$  and  $v(t)$  are any solutions of (3) and (4) through  $(t_0, u_0)$  and  $(\tau_0, v_0)$  respectively on  $J$ , then we have

$$d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), \tilde{0}] \leq r(t, \tau_0, w_{10}), t \in J \quad (6)$$

provided that  $d[v_0 - u_0, \tilde{0}] \leq w_{10}$ . For details [16], [23]

**Corollary 1:** Assume that  $f, F \in C[J \times E^n, E^n]$  and either  
 (i)

$$d[F(t, v) - \tilde{f}(t, \tilde{u}), \tilde{0}] \leq G(t, d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), v_0 - u_0])$$

or

(ii)  $\lim_{h \rightarrow 0^+} \sup \frac{1}{h} [d[v - \tilde{u} + h(F(t, v) - \tilde{f}(t, \tilde{u})), \tilde{0}] - d[v - \tilde{u}, \tilde{0}]] \leq G(t, d[v - \tilde{u}, \tilde{0}])$  where  $G \in C[J \times R_+, R]$ .

Then, if  $d[v_0 - u_0, \tilde{0}] \leq w_{10}$ , we have

$$d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), \tilde{0}] \leq r(t, \tau_0, w_{10}) \text{ for } t \in J \quad (7)$$

where  $r(t) = r(t, \tau_0, w_{10})$  is the maximal solution of the Fuzzy scalar differential equation

$$w_1' = G(t, w_1), w_1(\tau_0) = w_{10} \geq 0 \text{ for } t \geq \tau_0, t \in J \quad (8)$$

For details [16], [23]

### IV. STRICT STABILITY CRITERIA OF THE FUZZY DIFFERENTIAL SYSTEMS AND A COMPARISON RESULT IN STRICT STABILITY OF FUZZY DIFFERENTIAL EQUATIONS

Now, we are in a position to prove main result for the strict stability criteria of the Fuzzy differential systems.

Before we prove the general result in terms of the comparison principle. Let us consider the uncoupled comparison differential systems:

$$\begin{cases} (i) w_1' = g_1(t, w_1), w_1(\tau_0) = w_{10} \geq 0 \\ (ii) w_2' = g_2(t, w_2), w_2(\tau_0) = w_{20} \geq 0 \end{cases} \text{ where } g_1, g_2 \in C[R_+^2, \mathbb{R}] \quad (9)$$

The comparison system (9) is said to be strictly stable:

If given any  $\epsilon_1 > 0$  and  $t \geq \tau_0, \tau_0 \in \mathbb{R}_+$ , there exist a  $\delta_1 > 0$  such that

$$w_{10} \leq \delta_1 \text{ implies } w_1(t) < \epsilon_1 \text{ for } t \geq \tau_0$$

and for every  $\delta_2 < \delta_1$  there exists an  $\epsilon_2 > 0, 0 < \epsilon_2 < \delta_2$  such that

$$w_{20} \geq \delta_2 \text{ implies } w_2(t) > \epsilon_2 \text{ for } t \geq \tau_0.$$

Here,  $w_1(t)$  and  $w_2(t)$  are any solutions of (i) in (9) and (ii) in (9); respectively.

The comparison system (9) is said to be strictly attractive:

If given any  $\alpha_1 > 0, \gamma_1 > 0, \epsilon_1 > 0$  and  $\tau_0 \in \mathbb{R}_+$ , for every  $\alpha_2 < \alpha_1$ , there exist  $\epsilon_2 < \epsilon_1, T_1 = T_1(\epsilon_1, \tau_0) > 0$  and  $T_2 = T_2(\epsilon_1, \tau_0) > 0$  such that

$$w_1(t, \tau_0, w_{10}) < \epsilon_1 \text{ for } T_1 + \tau_0 \leq t \leq T_2 + \tau_0 \text{ whenever } w_{10} \leq \alpha_1$$

and

$$w_2(t, \tau_0, w_{20}) > \epsilon_2 \text{ for } T_2 + \tau_0 \geq t \geq T_1 + \tau_0 \text{ whenever } w_{20} \geq \alpha_2.$$

If  $T_1$  and  $T_2$  are independent of  $\tau_0$ , then the comparison system (9) is initial time difference uniformly strictly attractive for  $t \geq \tau_0$ .

Following main result based on this definition that result is formulated in terms of comparison principle.

In this section, we have a comparison theorem in strict stability of fuzzy differential systems via scalar differential equation and we give the proof of this theorem.

**Theorem 2** Assume that

(S<sub>1</sub>) Let  $V_\mu \in C[\mathbb{R}_+ \times S_\rho, \mathbb{R}_+]$ ,  $|V(t, u) - V(t, v)| \leq Ld[u, v], L > 0$  and for  $(t, u) \in \mathbb{R}_+ \times S_\rho$ , where  $S_\rho = [u \in E^n : d[u(t), \hat{0}] < \rho]$ , for each  $\mu, 0 < \mu < \rho$ , and  $V_\mu$  is locally Lipschitzian in  $z$  and for  $(t, z) \in \mathbb{R}_+ \times S_\rho$  and  $d[z, \hat{0}] \geq \mu$ , such that

$$b_1(d[z, \hat{0}]) \leq V_\mu(t, z) \leq a_1(d[z, \hat{0}]), a_1, b_1 \in \mathcal{K}$$

$$D^+ V_\mu(t, v - \tilde{u})$$

$$= \limsup_{h \rightarrow 0^+} \frac{1}{h} [V_\mu(t+h, v - \tilde{u} + h(F(t, v) - \tilde{f}(t, \tilde{u}))) - V_\mu(t, v - \tilde{u})]$$

$$\leq g_1(t, V_\mu(t, z));$$

(10)

(S<sub>2</sub>) for each  $\theta, 0 < \theta < \rho, V_\theta \in C[\mathbb{R}_+ \times S_\rho, \mathbb{R}_+]$  and  $V_\theta$  is locally Lipschitzian in  $z$  and for  $(t, z) \in \mathbb{R}_+ \times S_\rho$  and  $d[z, \hat{0}] \leq \theta$ ,

$$b_2(d[z, \hat{0}]) \leq V_\theta(t, z) \leq a_2(d[z, \hat{0}]), a_2, b_2 \in \mathcal{K},$$

$$D^+ V_\theta(t, v - \tilde{u})$$

$$= \limsup_{h \rightarrow 0^+} \frac{1}{h} [V_\theta(t+h, v - \tilde{u} + h(F(t, v) - \tilde{f}(t, \tilde{u}))) - V_\theta(t, v - \tilde{u})]$$

$$\geq g_2(t, V_\theta(t, z));$$

(11)

where  $g_2(t, w) \leq g_1(t, w), g_1, g_2 \in C[\mathbb{R}_+^2, \mathbb{R}], g_1(t, 0) = g_2(t, 0) = 0$  and  $z(t) = v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0)$  for  $t \geq \tau_0, v(t, \tau_0, v_0)$  of the system (4) through  $(\tau_0, v_0)$  and

$u(t - \eta, t_0, u_0)$ , where  $u(t, t_0, u_0)$  is any solution of the system (3) for  $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$  and  $\eta = \tau_0 - t_0$ .

We have the strict stability properties of the comparison systems give us the corresponding strict stability concept of the fuzzy differential equation solution  $v(t, \tau_0, v_0)$  of the system (4) through  $(\tau_0, v_0)$  with respect to the solution  $u(t - \eta, t_0, u_0)$  of the system (3) with initial time difference where  $u(t, t_0, u_0)$  is any solution of the system (3) for  $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$ .

**Proof of Theorem 2:** We will only prove the case of strict uniformly asymptotically stability. Suppose that the comparison differential systems in (9) is strictly uniformly asymptotically stable, then for any given  $\epsilon_1, 0 < \epsilon_1 < \delta$ , there exist a  $\delta^* > 0$  such that  $w_{10} \leq \delta^*$  means that  $w_1(t, \tau_0, w_{10}) < b_1(\epsilon_1)$  for  $t \geq \tau_0$ .

For this  $\epsilon_1$  we choose  $\delta_1$  and  $\delta_0$ , such that  $a_1(\delta_1^*) \leq \delta^*$  and  $\delta_1^* < \epsilon_1$  where  $\delta_1^* = \max\{\delta_1, \delta_0\}$ , then we claim that

$$d[v_0 - u_0, \hat{0}] < \delta_1, d[\tau_0 - t_0, \hat{0}] < \delta_0 \text{ means that}$$

$$d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), \hat{0}] < \epsilon_1 \text{ for } t \geq \tau_0.$$

If it is not true, then there exist  $t_1$  and  $t_2, t_2 > t_1 > \tau_0$  and a solution  $q(t)$  of

$$q' = \tilde{f}(t, q), q(\tau_0) = v_0 - u_0 \text{ for } t \geq \tau_0$$

$$\text{with } d[\tau_0 - t_0, \hat{0}] < \delta_{11} \text{ and } d[v_0 - u_0, \hat{0}] < \delta_1$$

$$d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), \hat{0}] < \delta_1^*,$$

$$d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), \hat{0}] = \epsilon_1 \text{ and}$$

$$\delta_1^* \leq d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), \hat{0}] < \epsilon_1 \text{ for } [t_1, t_2].$$

If we choose  $\mu = \delta_1^*$  and using the theory of differential inequalities, we get

$$b_1(\epsilon_1) = b_1(d[v(t_2, \tau_0, v_0) - u(t_2 - \eta, t_0, u_0), \hat{0}])$$

$$\leq V_\mu(t_2, v(t_2, \tau_0, v_0) - u(t_2 - \eta, t_0, u_0))$$

$$\leq r(t_2, t_1, V_\mu(t_1, v(t_1, \tau_0, v_0) - u(t_1 - \eta, t_0, u_0)))$$

$$\leq r(t_2, t_1, a_1(\delta_1^*))$$

$$\leq r(t_2, t_1, \delta^*)$$

$$< b_1(\epsilon_1).$$

and then we have a contradiction here that this inequality is impossible  $b_1(\epsilon_1) < b_1(\epsilon_1)$ , and this inequality that we obtain is a contradiction. Here the function  $r(t, \tau_0, w_{10})$  is the maximal solution of the comparison system. We have that the strict stability of the fuzzy differential equation is true and we have uniformly stability with initial time difference. Then, we have proved that the fuzzy differential equation is strictly uniformly attractive with initial time difference.

For any given  $\delta_2, \epsilon_2 > 0, \delta_2 < \delta^*$ , we choose  $\bar{\delta}_2$  and  $\bar{\epsilon}_2$  such that  $a_1(\delta_2) < \bar{\delta}_2$  and  $b_1(\epsilon_2) \geq \bar{\epsilon}_2$ . For these  $\bar{\delta}_2$  and  $\bar{\epsilon}_2$ , since (9) is strictly uniformly attractive, for any  $\bar{\delta}_3 < \bar{\delta}_2$  there exist  $\bar{\epsilon}_3$  and  $T_1$  and  $T_2$  and we assume that the inequality  $T_2 < T_1$  such that  $\bar{\delta}_3 < w_{10} = w_{20} < \bar{\delta}_2$  implies

$$\begin{aligned} r(t, \tau_0, w_{10}) &\leq r(t, \tau_0, \bar{\delta}_2) < \bar{\epsilon}_2 \\ \rho(t, \tau_0, w_{20}) &\geq \rho(t, \tau_0, \bar{\delta}_3) > \bar{\epsilon}_2 \end{aligned}$$

where  $r(t, \tau_0, w_{10})$  and  $\rho(t, \tau_0, w_{20})$  is the maximal solution and minimal solution of the comparison system (9) part (i) and the comparison system (9) part (ii); respectively.

Now, for any  $\delta_3$ , let  $b_2(\delta_3) \geq \bar{\delta}_3$ . We choose  $\epsilon_3$  such that  $a_2(\epsilon_3) < \bar{\epsilon}_3$ . Then by using comparison system (9), (i) and  $(A_1)$ , we have

$$\begin{aligned} &b_1(d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), \tilde{0}]) \\ &\leq V_\mu(t, v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0)) \\ &\leq r(t, \tau_0, V_\mu(\tau_0, v_0 - u_0)) \\ &\leq r(t, \tau_0, a_1(d[v_0 - u_0, \tilde{0}])) \\ &\leq r(t, \tau_0, \bar{\delta}_2) \\ &< \bar{\epsilon}_2 \leq b_1(\epsilon_2) \end{aligned}$$

So that we obtain that we have the inequality

$$b_1(d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), \tilde{0}]) < b_1(\epsilon_2)$$

if we have  $b_1^{-1}$  and it exists which implies that  $d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), \tilde{0}] < \epsilon_2$  for  $t \in [\tau_0 + T_2, \tau_0 + T_1]$ .

We have used the comparison principle and (ii) and  $(S_2)$  and then we have

$$\begin{aligned} &a_2(d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), \tilde{0}]) \\ &\geq V_\theta(t, v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0)) \\ &\geq \rho(t, \tau_0, V_\theta(\tau_0, v_0 - u_0)) \\ &\geq \rho(t, \tau_0, b_2(\delta_3)) \\ &\geq \rho(t, \tau_0, \bar{\delta}_3) \\ &> \bar{\epsilon}_3 \geq a_2(\epsilon_3) \end{aligned}$$

Then we have an important inequality that we obtain the inequality

$$a_2(d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), \tilde{0}]) > a_2(\epsilon_3)$$

if we have that  $a_2^{-1}$  exists which implies that for  $d[v(t, \tau_0, v_0) - u(t - \eta, t_0, u_0), \tilde{0}] > \epsilon_3$  for  $t \in [\tau_0 + T_2, \tau_0 + T_1]$ , and we have  $v(t, \tau_0, v_0)$  solution of (4) through  $(\tau_0, v_0)$  is strictly uniformly attractive with respect to the

solution  $u(t - \eta, t_0, u_0)$  is solution of the system (3) for  $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$ . So that we have completed the proof of the comparison theorem of fuzzy differential equations with strict stability of fuzzy differential equations with Lyapunov functions.

This completes the proof of the Theorem 2.

## V. CONCLUSION

In this study, we present some results for the study of stability of fuzzy differential equations and the strict stability of fuzzy systems. First of all, we present the fundamental concept of fuzzy calculus and differential equations in fuzzy calculus. Firstly, we consider and define the strict stability of fuzzy differential equations by Lyapunov functions. Secondly, we develop a new comparison principle for the strict stability of fuzzy differential systems, then we prove the strict stability criteria with Lyapunov functions. It may provide a greater prospect of solving problems which exhibit strict stability of fuzzy differential equations by Lyapunov functions.

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