World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:12, No:4, 2018

Total Chromatic Number of Δ-Claw-Free 3-Degenerated Graphs

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Abstract**—**The total chromatic number $\chi''(G)$ of a graph G is the minimum number of colors needed to color the elements (vertices and edges) of G such that no incident or adjacent pair of elements receive the same color Let G be a graph with maximum degree $\Delta(G)$. Considering a total coloring of G and focusing on a vertex with maximum degree. A vertex with maximum degree needs a color and all $\Delta(G)$ edges incident to this vertex need more $\Delta(G)+1$ distinct colors. To color all vertices and all edges of G, it requires at least $\Delta(G)+1$ colors. That is, $\chi''(G)$ is at least $\Delta(G)+1$. However, no one can find a graph G with the total chromatic number which is greater than $\Delta(G)+2$. The Total Coloring Conjecture states that for every graph G, $\chi''(G)$ is at most $\Delta(G)+2$.

In this paper, we prove that the Total Coloring Conjectur for a Δ-claw-free 3-degenerated graph. That is, we prove that the total chromatic number of every Δ-claw-free 3-degenerated graph is at most $\Delta(G)+2$.

Keywords**—**Total colorings, the total chromatic number, 3-degenerated.

I. Introduction

A m*-coloring* of a graph G is a coloring $f : V(G) \to \{1, 2, \ldots, m\}$. A m-coloring is *proper* if adjacent vertices have different colors. A graph is m*-colorable* if it has a proper m-coloring. The *chromatic number* $\chi(G)$ is the least positive integer m such that G is m-colorable.

A m*-edge coloring* of a graph G is a coloring $f : E(G) \to \{1, 2, \ldots, m\}$. A m-edge coloring is *proper* if incident edges have different colors. A graph is m*-edge-colorable* if it has a proper m-edge coloring. The *edge chromatic number* $\chi'(G)$ of a graph G is the least positive integer m such that G is m-edge-colorable.

A m*-total coloring* of a graph G is a coloring $f : V(G) \cup E(G) \to \{1, 2, \ldots, m\}$. A m-total coloring is *proper* if incident edges have different colors, adjacent vertices have different colors, and edges and its endpoints have different colors. A graph is m*-total colorable* if it has a proper m-total coloring. The *total chromatic number* $\chi''(G)$ of a graph G is the least positive integer m such that G is m-total colorable.

However, no one can find a graph G with $\chi''(G) > \Delta(G)+2$. *The Total Coloring Conjecture*, introduced independently by Behzad [1] and Vizing [2], states that for every graph G, $\chi''(G) \leq \Delta(G)+2$. In 2003, Zhou, Matsuo and Nishizeki [3] found the total chromatic number of a series parallel graph which is a 2-degenerated graph. Furthermore, the Total Coloring Conjecture has been proved for graphs of sufficiently small maximum degree. It was proved for $\Delta(G) = 3$ by Rosenfeld [4] and indepently by Vijayaditya [5], and an algorithmic proof was presented by Yap [6]. For $\Delta(G) = 4$ Kostochka [7] gave a proof of $\chi''(G) \leq 6$. The case $\Delta(G) = 5$ was settled in the doctoral thesis of Kostochka [8], [9], who proved that $\chi''(G) \leq \Delta(G)+2$ is valid for all graphs G with $\Delta(G) \leq 5$.

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Proposition 1. *Let G be a nontrivial graph. We obtain $\chi''(G) \geq 3$.*

Proof: Since G is a nontrivial graph, there is an edge uv where $u, v \in V(G)$. We need 3 colors to label vertices u, v and edge uv. Thus $\chi''(G) \geq 3$. ■

The following statments are the chromatic number, the edge chromatic number and the total chromatic number of some well known graphs such as cycle and complete graphs. A *cycle* is a graph with a single cycle through all vertices. A cycle with n vertices is denoted by C_n. A *complete graph* is a graph whose vertices are pairwise adjacent. The complete graph with n vertices is denoted by K_n.

Remark 1. $\chi(C_n) = \chi'(C_n) = \begin{cases} 2 & \textit{if } n \textit{ is even,} \\ 3 & \textit{if } n \textit{ is odd.} \end{cases}$

Proof: Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. If n is even, we color all vertices by color 1 and 2 alternatively to obtain $\chi(C_n) = 2$. Similarly, $\chi'(C_n) = 2$ when n is even. If n is odd, we color a vertex of C_n by color 1 and color remaining vertices by color 2 and color 3 alternatively to obtain $\chi(C_n) = 3$. Similarly, $\chi'(C_n) = 3$ when n is odd. ■

Proposition 2. [10] $\chi''(C_n) = \begin{cases} 3 & \textit{if } \ n \equiv 0 \pmod 3, \\ 4 & \text{otherwise.} \end{cases}$

Theorem 1. [11], [12] *For every graph G, $\chi(G) \leq \Delta(G)+1$. The equality holds if and only if G is a complete graph or an odd cycle.*

Remark 2. $\chi''(C_n) \geq \chi'(C_n) = \chi(C_n)$.

Proof: By Remark 1, we obtain $\chi(C_n) = \chi'(C_n)$. By Theorem 1, $\chi(C_n) = \chi'(C_n) \leq 3$. By Proposition 2, $\chi(C_n) = \chi'(C_n) \leq 3 \leq \chi''(C_n)$. ■

Proposition 3. $\chi(C_n) = \chi'(C_n) = \chi''(C_n)$ *if and only if* $n \equiv 3 \pmod 6$.

Proof: Sufficiency. Assume that $n \equiv 3 \pmod 6$. Since C_n is an odd cycle, we get $\chi(C_n) = 3$ and $\chi'(C_n) = 3$. By Proposition 2, we get $\chi''(C_n) = 3$. Therefore, $\chi(C_n) = \chi'(C_n) = \chi''(C_n)$.
Necessity. We will prove by contrapositive. Assume that $n \not\equiv 3 \pmod 6$. By the division algorithm, $n = 6k, 6k+1, 6k+$

$2, 6k+4$ or $6k+5$ for some integer k.
Case 1. $n = 6k, 6k+2$ or $6k+4$.
Since C_n is an even cycle, we get $\chi(C_n) = 2$. However, $\chi''(C_n) \geq \Delta(C_n)+1 = 3$. Then $\chi(C_n) \neq \chi''(C_n)$.
Case 2. $n = 6k+1$ or $n = 6k+5$.
Since n is not divisible by 3, by Proposition 2, we get $\chi''(C_n) = 4$. By Theorem 1, $\chi(C_n) \leq \Delta(C_n)+1 = 3$ and $\chi''(C_n) = 4$. Then $\chi(C_n) \neq \chi''(C_n)$. Therefore, $\chi(C_n) = \chi'(C_n) = \chi''(C_n)$ if and only if $n \equiv 3 \pmod 6$. ■

It is easy to find a coloring of a complete graph K_n. That is, $\chi(K_n) = n$ However, it is quite complicated to find an edge coloring or a total coloring of a complete graph K_n. An edge coloring of K_n was found by Fiorini and Wilson [13] and a total coloring of K_n was found by Bezhad, Chartrand and Cooper [14].

Proposition 4. [13] $\chi'(K_n) = \begin{cases} n & \textit{if } n \textit{ is odd,} \\ n-1 & \textit{if } n \textit{ is even.} \end{cases}$

Proposition 5. [14] $\chi''(K_n) = \begin{cases} n & \textit{if } n \textit{ is odd,} \\ n+1 & \textit{if } n \textit{ is even.} \end{cases}$

Let G be a graph. It requires more colors to color all vertices and edges of G than to color only vertices of G. Hence, $\chi''(G) \geq \chi(G)$. Similarly, $\chi''(G) \geq \chi'(G)$. However, it is not true that $\chi'(G) \geq \chi(G)$ or $\chi'(G) \leq \chi(G)$. For example, $\chi'(K_4) = 3$ but $\chi(K_4) = 4$.

Proposition 6. *If n is odd then $\chi(K_n) = \chi'(K_n) = \chi''(K_n)$. Otherwise, $\chi(K_n) = \chi'(K_n)+1 = \chi''(K_n)-1$.*

Proof: Case 1. n is odd. By Proposition 4 and Proposition 5, we get $\chi(K_n) = \chi'(K_n) = \chi''(K_n) = n$.
Case 2. n is even
By Proposition 5, we get $\chi''(K_n) = n+1$. However, $\chi(K_n) = n$. Thus $\chi(K_n) = \chi''(K_n)-1$. By Proposition 4, we get $\chi'(K_n) = n-1$. Thus $\chi(K_n) = \chi''(K_n)+1$. ■

Theorem 2. *Let G be a graph. If G is not a complete graph of even degree, then $\chi''(G) \geq \chi'(G) \geq \chi(G)$. Otherwise, $\chi(G) = \chi'(G)-1 = \chi''(G)+1$.*

Proof: Case 1. G is neither a complete graph nor an odd cycle. By Theorem 1, $\chi(G) \leq \Delta(G)$. Since $\Delta(G) \leq \chi'(G)$ and $\chi'(G) \leq \chi''(G)$, we get $\chi''(G) \geq \chi'(G) \geq \chi(G)$.
Case 2. G is an odd cycle. By Remark 2, $\chi''(G) \geq \chi'(G) \geq \chi(G)$.
Case 3. G is a complete graph. If n is odd then $\chi(K_n) = \chi'(K_n) = \chi''(K_n)$ and if n is even then $\chi(K_n) = \chi'(K_n)+1 = \chi''(K_n)-1$ by Proposition 6. ■

The following theorem gives necessary and sufficient conditions for the equality of the chromatic number, the edge-chromatic number and the total chromatic number.

Theorem 3. *Let G be a graph with n vertices. $\chi(G) = \chi'(G) = \chi''(G)$ if and only if G is C_n where $n \equiv 3 \pmod 6$ or K_n where n is odd.*

Proof: Sufficiency. $\chi(G) = \chi'(G) = \chi''(G)$ by Proposition 3 and Proposition 6.
Necessity. Assume that $\chi(G) = \chi'(G) = \chi''(G)$. By Theorem 1 and Remark **??**, we get $\chi(G) \leq \Delta(G)+1 \leq \chi''(G)$. Then $\chi(G) = \Delta(G)+1 = \chi''(G)$. Thus $\chi(G) > \Delta(G)$. From Theorem 1, G is an odd cycle or a complete graph. By Proposition 3 and Proposition 6, G is a cycle of length $n \equiv 3 \pmod 6$ or a complete graph of order n when n is odd. ■

In Fig. 1, we can remove all vertices by this order $v_7, v_6, v_5, v_4, v_3, v_2, v_1$ which satisfy the definition of 3-degenerated graph.

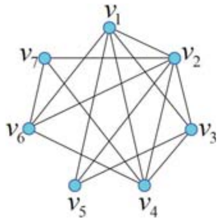

Fig. 1 A 3-degenerated graph

A *m-claw* in a graph G is the bipartite $K_{1,3}$ whose all leaves are vertices with degree m in G. A graph G is *m-claw-free* if G has no m-claw as an induced subgraph. A Δ-claw in a graph G is the bipartite $K_{1,3}$ whose all leaves are vertices with maximum degree in G. A graph G is Δ-claw-free if G has no Δ-claw as an induced subgraph.

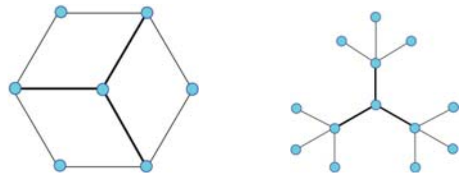

Fig. 2 graphs which have exactly 1 Δ-claw

Although each graph in Fig. 2 has 4 claws, it has only 1 Δ-claw. In this paper, $N(v)$ is denoted the set of all vertices adjacent to a vertex v.

II. Main Result

Our main result is that every Δ-claw-free 3-degenerated graph satisfies the Total Coloring Conjecture. Let G be a Δ-claw-free 3-degenerated graph. We will prove that if $m \geq \Delta(G)+2$ and G is $(m-2)$-claw-free, then $\chi''(G) \leq m$ by induction on the number of vertices. Since G is a 3-degenerated graph, it can be succesive removal vertices with degree at most 3. Let v the first removal vertex. The proof is divided into four cases; the first case is $d(v) = 1$, the second case is $d(v) = 2$, the third case and the fourth case are $d(v) = 3$ with different conditions.

Lemma 1. *Let G be a graph and contain a vertex v with degree 1. If $\chi''(G-v) \leq m$ where m is an integer such that $m \geq \Delta(G)+2$ then $\chi''(G) \leq m$.*

Proof:
Let $m \geq \Delta(G-v)+2$ be an integer. Assume that $\chi''(G-v) \leq m$. Then there is a proper total coloring $f : V(G-v) \cup E(G-v) \to [m]$. Since $d_{G-v}(u)+1 \leq m-1$, there exists a remaining color in $[m]$, say r, which is not used to color u and edges incident to u in $G-v$. Since $m \geq \Delta(G)+2 \geq$

$d(v)+2=3$. Thus we can pick a color s which differs from $f(u)$ and r.

Let $f' : V(G) \cup E(G) \to [m]$ be a total coloring defined by

$$f'(x) = \begin{cases} f(x) & \text{if } \ x \in V(G-v) \cup E(G-v), \\ r & \text{if } \ x = uv, \\ s & \text{if } \ x = v. \end{cases}$$

The properties of the proper total coloring f, color r and color s yield that f' is a proper total coloring from $V(G) \cup E(G)$ to $[m]$. Therefore $\chi''(G) \leq m$. ■

Lemma 2. *Let v be a vertex with degree 2 of a graph G and $m \geq \Delta(G)+2$. If $\chi''(G-v) \leq m$ then $\chi''(G) \leq m$.*

Proof: Let u_1 and u_2 be the vertices which are adjacent to v. Let $m \geq \Delta(G)+2$. Assume that $\chi''(G-v) \leq m$. If $\Delta(G) \leq 2$, each component of G is a path or a cycle. Then $\chi''(G) \leq \Delta(G)+2 \leq m$. Assume that $\Delta(G) \geq 3$.

It suffices to show that there is a proper total coloring from $V(G) \cup E(G)$ to $[m]$.

Since $\chi''(G-v) \leq m$, there is a proper total coloring $f : V(G-v) \cup E(G-v) \to [m]$. Since $d_{G-v}(u_1)+1 \leq d_G(u_1) \leq \Delta(G) \leq m-2$, we use at most $m-2$ colors to color u_1 and edges incident to u_1 in $G-v$. Then there are 2 remaining colors for coloring u_1v. Let one be r_1. Similarly, there are 2 remaining colors for coloring u_2v. Pick the one which differs from r_1, say r_2. Since $\Delta(G) \geq 3$, we get $m \geq 5$. Let s be a color which differs from $f(u_1), f(u_2), r_1$ and r_2. Let $f' : V(G) \cup E(G) \to [m]$ be a total coloring defined by

$$f'(x) = \begin{cases} f(x) & \text{if } \ x \in V(G-v) \cup E(G-v), \\ r_1 & \text{if } \ x = u_1v, \\ r_2 & \text{if } \ x = u_2v, \\ s & \text{if } \ x = v. \end{cases}$$

By Properties of f, color r_1, color r_2 and color s. Then f' is a proper total coloring from $V(G) \cup E(G)$ to $[m]$. Hence $\chi''(G) \leq m$. ■

Lemma 3. *Let v be a vertex with degree 3 of a graph G and $m \geq \Delta(G)+2$.*
If $\exists u \in N(v), d_G(u) \leq m-3$ and $\chi''(G-v) \leq m$, then $\chi''(G) \leq m$.

Proof: Let v be a vertex with degree 3 of a graph G and $m \geq \Delta(G)+2$. Assume that $\exists u \in N(v), d_G(u) \leq m-3$ and $\chi''(G-v) \leq m$. As mention in first page, for any graph G such that $\Delta(G) \leq 5$, we know that $\chi''(G) \leq \Delta(G)+2 \leq m$. Suppose that $\Delta(G) \geq 6$. Let u_1, u_2 and u_3 be the vertices which are adjacent to v. Without loss of generality, assume that $d_G(u_3) \leq m-3$. Since $\chi''(G-v) \leq m$, there is a proper total coloring $f : V(G-v) \cup E(G-v) \to [m]$.

Since $\chi''(G-v) \leq m$, there is a proper total coloring $f : V(G-v) \cup E(G-v) \to [m]$. Since $d_{G-v}(u_1)+1 \leq d_G(u_1) \leq \Delta(G) \leq m-2$, we use at most $m-2$ colors to color u_1 and edges incident to u_1 in $G-v$. Then there are 2 remaining colors for coloring u_1v. Let one be r_1. Similarly there are 2 remaining colors for coloring u_2v. Pick the one which differs from r_1, say r_2. Since $d_{G-v}(u_3)+1 \leq d_G(u_3) \leq m-3$, there are 3 remaining colors for coloring u_3v. Pick the one which differs from r_1 and r_2, say r_3. Since $\Delta(G) \geq 6$, we get $m \geq 8$. Let s be a color which differs from $f(u_1), f(u_2), f(u_3), r_1, r_2$ and r_3. Let $f' : V(G) \cup E(G) \to [m]$ be a total coloring defined by

$$f'(x) = \begin{cases} f(x) & \text{if } \ x \in V(G-v) \cup E(G-v), \\ r_1 & \text{if } \ x = u_1v, \\ r_2 & \text{if } \ x = u_2v, \\ r_3 & \text{if } \ x = u_3v, \\ s & \text{if } \ x = v. \end{cases}$$

Then f' is a proper total coloring from $V(G) \cup E(G)$ to $[m]$. Hence $\chi''(G) \leq m$. ■

Theorem 4. [12] *For sets $A_1, A_2, \ldots, A_n$, $\exists a_i \in A_i$ such that $a_i \neq a_j$ for $i \neq j$ if and only if $|\bigcup_{i \in S} A_i| \geq |S|$ for every $S \subseteq [n]$.*

Remark 3. *Let A_1, A_2, A_3 be sets containing at least 2 elements. If $A_1 \cup A_2 \cup A_3$ has at least 3 elements, then there are $a_1 \in A_1, a_2 \in A_2, a_3 \in A_3$ such that a_1, a_2, a_3 are different.*

Proof: Let A_1, A_2, A_3 be sets containing at least 2 elements. Assume that $A_1 \cup A_2 \cup A_3$ has at least 3 elements. To use Theorem 4, we consider following sets

- $|A_1|, |A_2|, |A_3| \geq 1$,
- $|A_1 \cup A_2|, |A_2 \cup A_3|, |A_1 \cup A_3|$,
- $|A_1 \cup A_2 \cup A_3| \geq 3$.

Thus there are $a_1 \in A_1, a_2 \in A_2, a_3 \in A_3$ such that a_1, a_2, a_3 are different. ■

Lemma 4. *Let v be a vertex with degree 3 of a graph G and m be integer such that $m \geq \Delta(G)+2$. If $N(v)$ is not an independent set and $\chi''(G-v) \leq m$ then $\chi''(G) \leq m$.*

Proof: Let v be a vertex with degree 3 of a graph G and $m \geq \Delta(G)+2$. Assume that $N(v)$ is not an independent set and $\chi''(G-v) \leq m$. As mention in first page, for any graph G such that $\Delta(G) \leq 5$, we know that $\chi''(G) \leq \Delta(G)+2 \leq m$. Suppose that $\Delta(G) \geq 6$. Let u_1, u_2 and u_3 be the vertices which are adjacent to v. Without loss of generality, assume that u_1 and u_2 are adjacent. Since $\chi''(G-v) \leq m$, there is a proper total coloring $f : V(G-v) \cup E(G-v) \to [m]$. Since $d_{G-v}(u_1)+1 \leq d_G(u_1) \leq \Delta(G) \leq m-2$, we use at most $m-2$ colors to color u_1 and edges incident to u_1 in $G-v$. Then there are 2 remaining colors for coloring vu_1, say r_1, r_2. Similarly, there are 2 remaining colors for coloring vu_2, say s_1, s_2 and there are 2 remaining colors for coloring vu_3, say t_1, t_2. Let $R = \{r_1, r_2\}, S = \{s_1, s_2\}, T = \{t_1, t_2\}$.
*Case*1. $|R \cup S \cup T| \geq 3$.
By Remark 3, there is $r \in R, s \in S, t \in T$ such that r, s, t are different

Since $\Delta(G) \geq 6$, we get $m \geq 8$. Let c be a color which differs from $f(u_1), f(u_2), f(u_3), r, s, t$. Let $f' : V(G) \cup$

$E(G) \to [m]$ be a total coloring defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(G-v) \cup E(G-v), \\ r & \text{if } x = u_1v, \\ s & \text{if } x = u_2v, \\ t & \text{if } x = u_3v, \\ c & \text{if } x = v. \end{cases}$$

Then f' is a proper total coloring from $V(G) \cup E(G)$ to $[m]$. Hence $\chi''(G) \leq m$.
*Case*2. $|R \cup S \cup T| = 2$.
Thus $R = S = T$. Without loss of generality, let $r_1 = s_1 = t_1$ and $r_2 = s_2 = t_2$. Let $g : V(G-v) \cup E(G-v) \to [m]$ be a total coloring of a graph $G - v$ defined by

$$g(x) = \begin{cases} r_1 & \text{if } x = u_1u_2, \\ g(x) & \text{otherwise.} \end{cases}$$

Then g is a proper total coloring from $V(G-v) \cup E(G-v)$ to $[m]$. Moreover, remaining color sets for vu_1, vu_2 and vu_3 are $\{f(u_1u_2), r_2\}$, $\{f(u_1u_2), r_2\}$ and $\{r_1, r_2\}$, respectively. Since $f(u_1u_2) \neq r_1, r_2$, we get g is in Case 1. Similar to Case 1, we can use g to define a proper total coloring from $V(G) \cup E(G)$ to $[m]$. ■

The main result is obtained by combining Lemma 1, Lemma 2, Lemma 3 and Lemma 4.

Theorem 5. *Every Δ-claw-free 3-degenerated graph satisfies the Total Coloring Conjecture.*

Proof: First, we will prove that for a 3-degenerated graph G with n vertices, if $m \geq \Delta(G) + 2$ and G is $(m-2)$-claw-free, then $\chi''(G) \leq m$.

Let $P(n)$ be the statement that for a 3-degenerated graph G with n vertices, if $m \geq \Delta(G)+2$ and G is $(m-2)$-claw-free, then $\chi''(G) \leq m$.

It is easy to see that $P(1)$ holds. Assume that $P(1), P(2), \ldots, P(k-1)$ hold. Let G be a 3-degenerated graph with k vertices. Then G has a vertex with degree at most 3, say v. Assume that $m \geq \Delta(G) + 2$ and G is $(m-2)$-claw-free. Then $G - v$ is also 3-degenerated and $(m-2)$-claw-free. Thus $\chi''(G-v) \leq m$.
*Case*1. $d_G(v) = 1$. By Lemma 1, we get $\chi''(G) \leq m$.
*Case*2. $d_G(v) = 2$. By Lemma 2, we get $\chi''(G) \leq m$.
*Case*3. $d_G(v) = 3$.
Since G is $(m-2)$-claw-free, $\exists u \in N(v), d_G(v) \neq m-2$ or $N(v)$ is not an independent set.
(3.1) $\exists u \in N(v), d_G(v) \neq m-2$. Since $m \geq \Delta(G)+2$, we get $d_G(u) \leq m-3$. By Lemma 3, we get $\chi''(G) \leq m$.
(3.2) $N(v)$ is not an independent set. By Lemma 4, we get $\chi''(G) \leq m$. Hence $P(k)$ hold.
By mathematic induction, $P(n)$ holds for any natural number n.

Let G be Δ-claw-free 3-degenerated graph To prove the Total Coloring Conjecture, we focus only when $m = \Delta(G) + 2$. Thus $m - 2 = \Delta(G)$; hence, G is $(m-2)$-claw-free. By the statement, $\chi''(G) \leq m = \Delta(G) + 2$. That is, G satisfies the Total Coloring Conjecture. ■

REFERENCES

[1] M. Behzad, The total chromatic number of a graph *Combinatorial Mathematics and its Applications, Proceedings of the Conference Oxford Academic Press N. Y.* 1-9, 1971.
[2] V. G. Vizing, On evaluation of chromatic number of a p-graph (in Russian) *Discrete Analysis, Collection of works of Sobolev Institute of Mathematics SB RAS* 3 3-24, 1964.
[3] X. Zhou, Y. Matsuo, T. Nishizeki, List total colorings of series-parallel Graphs, Computing and Combinatorics,*Lecture Notes in Comput. Sci.* 2697, Springer Berlin, 172-181, 2003.
[4] M. Rosenfeld, On the total coloring of certain graphs. *Israel J. Math.* 9 396-402, 1971.
[5] N. Vijayaditya, On total chromatic number of a graph, *J. London Math. Soc.* 3 405-408, 1971.
[6] H. P. Yap, Total colourings of graphs. *Bull. London Math. Soc.* **21** 159-163, 1989.
[7] A. V. Kostochka, The total colorings of a multigraph with maximal degree 4. *Discrete Math.* 17, 161-163, 1977.
[8] A. V. Kostochka, Upper bounds of chromatic functions of graphs (in Russian). Doctoral Thesis, Novosibirsk, 1978.
[9] A. V. Kostochka, Exact upper bound for the total chromatic number of a graph (in Russian). In: *Proc. 24th Int. Wiss. Koll.,Tech. Hochsch. Ilmenau,1979* 33-36, 1979.
[10] H. P. Yap, *Total coloring of graphs*, Lecture Note in Mathematics Vol. 1623, Springer Berlin, 1996.
[11] R. L. Brooks, On coloring the nodes of a network, *Proc. Cambridge Phil. Soc.* 37 194-197, 1941.
[12] D. B. West, *Introduction to Graph Theory*, Prentice Hall, New Jersey, 2001.
[13] S. Fiorini, R. J. Wilson, *Edge Coloring of Graphs*, Pitman London, 1977.
[14] M. Bezhad, G. Chartrand, J. K. Cooper, The colors numbers of complete graphs, *J. London Math. Soc.* 42 225-228, 1967.