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# Concept of a Pseudo-Lower Bound Solution for Reinforced Concrete Slabs

M. De Filippo, J. S. Kuang



*Abstract*—In construction industry, reinforced concrete (RC) slabs represent fundamental elements of buildings and bridges. Different methods are available for analysing the structural behaviour of slabs. In the early ages of last century, the yield-line method has been proposed to attempt to solve such problem. Simple geometry problems could easily be solved by using traditional hand analyses which include plasticity theories. Nowadays, advanced finite element (FE) analyses have mainly found their way into applications of many engineering fields due to the wide range of geometries to which they can be applied. In such cases, the application of an elastic or a plastic constitutive model would completely change the approach of the analysis itself. Elastic methods are popular due to their easy applicability to automated computations. However, elastic analyses are limited since they do not consider any aspect of the material behaviour beyond its yield limit, which turns to be an essential aspect of RC structural performance. Furthermore, their applicability to non-linear analysis for modeling plastic behaviour gives very reliable results. *Per contra*, this type of analysis is computationally quite expensive, i.e. not well suited for solving daily engineering problems. In the past years, many researchers have worked on filling this gap between easy-to-implement elastic methods and computationally complex plastic analyses. This paper aims at proposing a numerical procedure, through which a pseudo-lower bound solution, not violating the yield criterion, is achieved. The advantages of moment distribution are taken into account, hence the increase in strength provided by plastic behaviour is considered. The lower bound solution is improved by detecting over-yielded moments, which are used to artificially rule the moment distribution among the rest of the non-yielded elements. The proposed technique obeys Nielsen's yield criterion. The outcome of this analysis provides a simple, yet accurate, and non-time-consuming tool of predicting the lower-bound solution of the collapse load of RC slabs. By using this method, structural engineers can find the fracture patterns and ultimate load bearing capacity. The collapse triggering mechanism is found by detecting yield-lines. An application to the simple case of a square clamped slab is shown, and a good match was found with the exact values of collapse load.

*Keywords*—Computational mechanics, lower bound method, reinforced concrete slabs, yield-line.

## I. Introduction

The assessment of load bearing capacity of RC slabs and plates has extensively been treated through yield-line method. Ingerslev [6] was the first to coin the term 'yield-line' which describes an amount of subsequent points along which yielding is occurring. Derivation of such yield lines is of particular interest since it leads to deriving the collapse mechanism of the slab. Such field of research has attracted a consistent interest throughout the years. Johansen [8] has subsequently further developed and confirmed the theory. Its method represents a traditional hand solution of the upper bound problem, aiming to obtain an approximation of the ultimate load carrying capacity, along with its critical yield-line pattern. Numerous researchers have then attempted to further improve such method by implementing new features such as full automation, optimization procedures, mesh-less approaches etc. The latest updates on the upper bound solution are given by Gilbert [5], who has implemented a discontinuous layout optimization technique into a mesh-less geometry.

Results obtained through upper bound solution, from its definition, will over-estimate the ultimate load capacity. Moreover, there is plenty of uncertainty that the most critical collapse mechanism has been found. Validation of the obtained results through other means is then imperative. A lower bound solution instead gives an under-estimation of the critical load. However, using such approach engineers can enjoy the luxury of dealing with conservative results, which ensures an over-estimation of the load-carrying capacity would never occur. Given such an advantage, however the lower bound approach has historically attracted less interest, probably due to its complexity. Jackson and Middleton [7] have set themselves this task, and have proposed an optimization-based technique to find the most critical collapse mechanism, and its related transverse load.

In this paper, the concept of an alternative lower bound method is presented. The main objective is to propose an easy-to-compute FE based technique simulating the real structural behavior of an RC slab under monotonic increasing transverse load.

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## II. Problem Statement and Assumptions

### A. Problem Statement

In mathematics, given a function $f$ of domain $\mathcal{D}$, and codomain $\mathcal{C}$, the exact solution, or the so called 0 of the polynomial, is given by the element $y \in \mathcal{C}$ for which it is verified $y = f(x)$, for each $x \in \mathcal{D}$. Considering now to be dealing with a function $f$ so complex that finding its 0-polynomial turns to be particularly difficult, then it would instead be preferable to find an approximate solution. An element $y \in \mathcal{C}$ is an upper bound of $f$ if $y > f(x)$, for each $x \in \mathcal{D}$. Similarly, an element $y \in \mathcal{C}$ is a lower bound of $f$ if $y < f(x)$, for each $x \in \mathcal{D}$ [9].

In mechanics of materials, the above definition can be applied through analogy of respectively $x$ with stresses, and of $y = f(x)$ with the yield condition. If all stresses are contained within the yield-surface, a lower bound solution is achieved. *Per contra*, if the yield-criterion is anywhere violated an upper bound solution is obtained. The aim is to find the load-carrying capacity, and collapse mechanism of the slab, getting as close as possible to the exact solution, provided by the 0 of the polynomial. Such concepts can be closely related to Elastic and Plastic Analyses. Nowadays, the Finite Element Method (FEM) has found endless applications for equilibrium related problems within the field of civil engineering. The solution of elastic equations of equilibrium is a task commonly solved through FEM. However, the analysis is linear-elastic, meaning that stresses and strains linearly increase with displacements, and a failure threshold is never reached. On the other hand, a more reliable solution can be achieved through plastic analyses. In this case, equilibrium is satisfied everywhere in the physical domain taking into account the non-linearity of materials' constitutive models. Such challenge is normally faced through iterative procedures such as Newton-Raphson Method, Modified Newton-Raphson Method, Arc-Length Method, etc. This latter technique provides a more reliable estimation of the structural behavior since the phenomenon of yielding is taken into account, but with a dramatical increase in computational cost.

This paper proposes a modified elastic-based FE analysis of the plate bending problem including the yielding effects of both materials constituting RC: concrete and steel. The results aim at representing a feasible solution for the previously-mentioned lower bound problem of RC slabs.

### B. *Assumptions*

Given the evident complexity of the problem, in order to obtain a reliable approximate solution it is inevitably required to reduce its complexity with a few assumptions. They are hereby mentioned as:

- The problem is restricted to static plate bending due to transverse loads.
- The slab is assumed to behave as a Kirchhoff plate, thus strains and displacements are small, and flexure is assumed to be independent of the effects of shear or any vertical stress. [3]
- It is assumed that in-plane, stresses (or the so-called Membrane Forces) do not have any significant effect in increasing the maximum carrying capacity of the slab. Given that elastic deflections are not significant, such assumption holds.
- The FEM is run only for the elastic problem, so plastic deformations of the slab are not taken into account. Yielding is instead simulated through a modification of the state of stresses.
- There is large experimental evidence that collapse of the slab due to monotonically increasing transverse load is solely caused due to bending. For such reason, an eventual collapse analysis due to shear should be separately threated. The maximum load carrying capacity is hence solely evaluated in function of bending moments.
- The slab is assumed to be ductile enough to not fail in a brittle manner. A non-ductile failure, due to highly stiff structural behavior, may cause a significant change in terms of load carrying capacity.

## III. Methodology

### A. *Yield-criterion*

Given the previous definition of lower bound, in order to check whether a stress point reached yielding or not, it is imperative to define a yield-criterion. For the present case, Nielsen's yield criterion is the most appropriate choice, hence yielding condition is solely dependent on state of bending moments. Large evidence was shown on the applicability of such criterion to several cases of slabs [11]. A lower bound solution is then achieved if (1) is fulfilled.

$$-m_R^-(\phi) \leq m(\phi) \leq m_R^+(\phi) \tag{1}$$

where $\phi$ is Variable angle of potential yield-line. $m(\phi)$ is Triad of Bending Moments at $\phi$. $m_R^-(\phi)$, $m_R^+(\phi)$ is Triad of Resisting Moments at $\phi$.

Such condition is broke down in the set of (2). [1]

$$\begin{cases} (M_{xy} - M_{R,xy}^+)^2 - (M_x - M_{R,x}^+)(M_y - M_{R,y}^+) \leq 0 \\ (M_{xy} + M_{R,xy}^-)^2 - (M_x + M_{R,x}^-)(M_y + M_{R,y}^-) \leq 0 \\ +M_x - M_{R,x}^+ \leq 0 \\ -M_x - M_{R,x}^- \leq 0 \\ +M_y - M_{R,y}^+ \leq 0 \\ -M_y - M_{R,y}^- \leq 0 \end{cases} \tag{2}$$

where $M_x$, $M_y$, $M_{xy}$ is Bending Moments. $M_{R,x}^+$, $M_{R,y}^+$, $M_{R,xy}^+$ is Resisting Sagging Moments. $M_{R,x}^-$, $M_{R,y}^-$, $M_{R,xy}^-$ is Resisting Hogging Moments.

The fulfillment of such conditions determines whether a moment triad, ($M_x$, $M_y$, $M_{xy}$), is within the surface or not. The yield criterion is dependent on the specifications of the cross-section and material. Its derivation is achieved through the solution of the set of (2). For simplicity, a linearized bi-conical version of the criterion, has been derived, as shown in Fig. 1. Through such an approach, computation time has been significantly decreased, and a more conservative criterion is obtained, since this latter is contained within the original quadratic one.

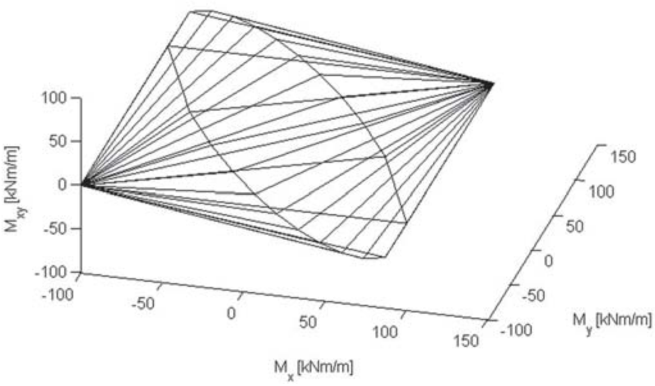


Fig. 1 Linearized Bi-Conical Nielsen's Yield Criterion

### B. *Plate Bending Problem*

The problem is based on a regular linear-elastic FE analysis of a bending plate. The plate is constituted by square and equidimensional QUAD4 elements, with 3 degrees of freedom per node: transverse displacement, $w$, (in out of plane direction $z$), and two rotations, $\theta_x$ and $\theta_y$, with reference to in-plane directions $x$ and $y$. A representation of the slab mesh, in the x-y plane, for a representative mesh of 8 elements in both x- and y-direction, is shown in Fig. 2. Elements with index $e$, are physically detectable through indices of columns and rows on the mesh, respectively referred with indices $i$ and $j$.

The plate bending problem is tackled through Mindlin-Reissner theory, so that also the effects of transverse shear deformations are taken into account in terms of displacements. Deformations are obtained through the classic FE formulation given in:

$$[K]\{U\} = \{F\} \tag{3}$$

where $K$ is Stiffness Matrix; $U$ is Deformations Vector; $F$ is Force Vector.

Bending moments are then computed as displayed in (4). [4]

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \cdot \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \tag{4}$$

where $D$ is Constitutive Matrix and $\nu$ is Poisson's ratio.

### C. Proposed Algorithm

The formulations leading to the creation of yield-criterion, and solution of the elastic equilibrium problem have been given in the previous two sections. They constitute the basis of the pseudo-lower bound method that is going to be introduced. A fully automated algorithm was built in MATLAB [10], and a flow chart for a sample of RC clamped square slab under distributed load is given in Fig. 3. The upcoming discussion is based on such algorithm representation.

The algorithm starts with an input of cross-section geometry, material properties, boundary conditions, and load. The method provides as final output the maximum load bearing capacity of the slab, and a map of yielded and non-yielded elements. As shown in Fig. 3, the program is structured with three loops: Loop 1 increasing load until failure does not occur, and Loop 2 and 3 checking whether yielding is occurring or not. Loop 1 starts at the beginning of the algorithm for an initial value of load. From the input, the maximum resisting sagging and hogging moments are automatically calculated according to the Hong Kong Code of Practice [2]. A yield-criterion is then obtained accordingly by plugging the derived values into (1). The yield surface is represented in a ($M_x$, $M_y$, $M_{xy}$) space.

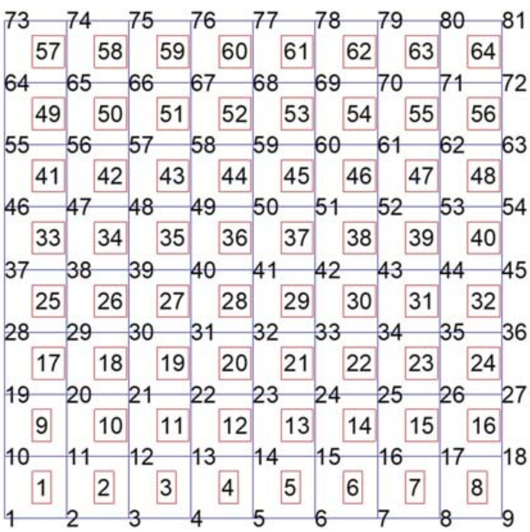


Fig. 2 A representation the slab domain meshed with QUAD4 elements.

Given the physical domain as the whole RC slab, a mesh of Melosh QUAD4 elements [4] is created, and a FE analysis is run. Equations (3) and (4) lead to obtaining the values of $M_x$, $M_y$, and $M_{xy}$ for each single element. Then, through the solution of an eigenvalue problem the principal moments and directions are derived. At this point, given that the performed analysis was purely linear-elastic, the yield-surface may have been violated somewhere (Loop 2). A check has to be performed to establish whether yielding occurred or not. If in the moments space, each point is contained within the surface, hence satisfying all the conditions given in the set of equations (2) (green dots on yield surfaces in Fig. 3), then the slab is behaving elastically, and yielding did not occur anywhere. Collapse cannot have occurred yet, so the whole program can restart with an increased value of load in Loop 1. Otherwise, the program continues running at the same intensity of load.

Triad of moments falling out of the yield surface are called 'Over-yielded moments' (red dots on yield surfaces in Fig. 3), since they violate the laws of plasticity by which a state of stress can, at maximum, be on the surface. Given $m^e_{o-y}$ as over-yielded bending moment triads of the element $e$, falling out of the yield surface, then

$$m^e_{o-y} = \{M^e_x,\ M^e_y,\ M^e_{xy}\}^T \mid f > 0 \tag{5}$$

where $e$ is Element located at $i$-th column, and $j$-th row and $f > 0$ is is the condition of violation of yield surface $f$.

Such condition is not physically allowable, because $f > 0$ is not plastically possible, hence the over-yielded moments are elastically dragged back to the surface (blue dots on yield surfaces in Fig. 3), and the triad of yielded moments $m^e_y$ is obtained.

$$m^e_y = \{M^e_x,\ M^e_y,\ M^e_{xy}\}^T \mid f = 0 \tag{6}$$

where $f = 0$ is the yielding condition.

At this position, the points in the bending moments space are not violating any plasticity condition, but are not allowed to swing on the surface. The idea is to compensate this moment reduction through a distribution of 'residual moments' among non-yielded neighbors of yielded elements. Normally, in a structural element, when a single part reaches its ultimate capacity, and load keeps increasing, other parts of the structure tend to absorb the residual part. With a monotonically increasing load, such operation keeps repeating. The previously-mentioned idea tries to simulate such structural behaviour. In order to perform this operation, at first yielded and non-yielded elements have to be distinguished in the physical space ($x$,$y$), so that their location can be visualized. In Fig. 3, they are respectively represented as blue- and green-filled elements on the bi-dimensional mesh. Subsequently, a 'Neighborhood Relationship' is established, with the aim of determining the left-, right-, bottom-, and top-neighbours of each single element. Non-yielded neighbours of yielded elements are detected, and represented as red-filled elements. Each yielded core (group of yielded neighbours of yielded elements) is divided into x- and y-stripes, and the residual moments corresponding to each x- and y-stripe are, respectively, split among x- and y-non-yielded neighbors. No variation of moments is instead applied to moment triads contained in the yield surface, hence non-yielded non-neighbours of yielded elements.

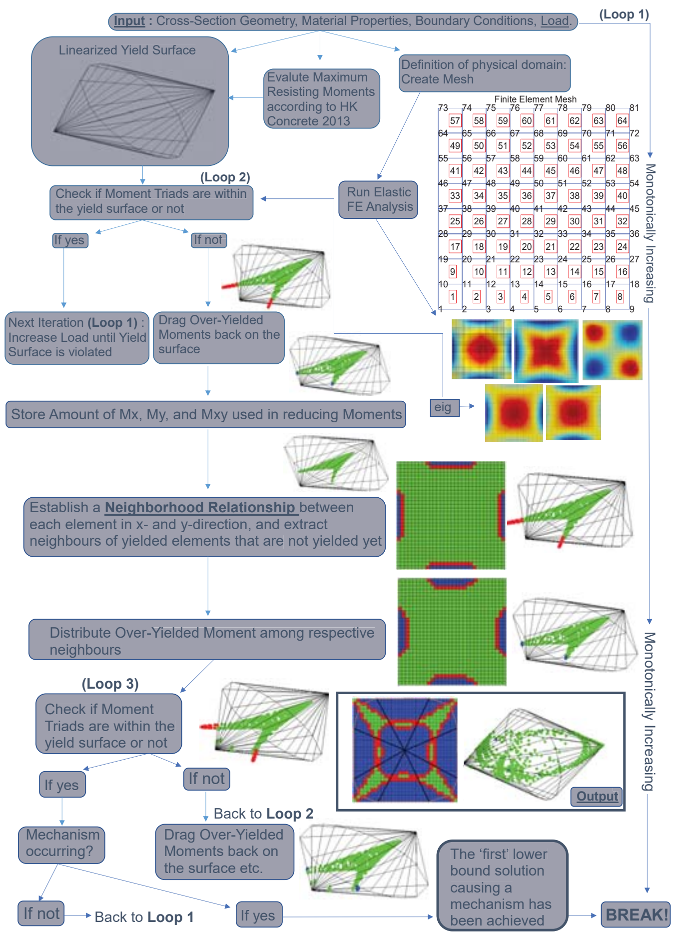


Fig. 3 Flow chart of Pseudo-Lower Bound Algorithm

Defined the residual moment triads at the element $e$, $\Delta m^e$, as the difference of the two moment vectors given in (5) and (6), then it follows that

$$\Delta m^e = m^e_{o-y} - m^e_y \tag{7}$$

The residual moments in the x-direction associated to the $j$-th x-stripe, $\Delta M^j_x$, are equal to the sum of the $\Delta M_x$ moments associated with each element contained in the x-stripe, and similarly the residual moments in the y-direction associated to the $i$-th y-stripe, $\Delta M^i_y$ are equal to the sum of the $\Delta M_y$ moments associated with each element contained the y-stripe. Hence

$$\Delta M^j_x = \sum_j^n \Delta M^{i,j}_x \tag{8}$$

$$\Delta M^i_y = \sum_i^m \Delta M^{i,j}_y \tag{9}$$

where $n$ is Amount of rows of elements in the mesh; $m$ is Amount of columns of elements in the mesh; $\Delta M^{i,j}_x$ is Residual moment in x-direction associated with the element at location $(i, j)$; $\Delta M^{i,j}_y$ is Residual moment in y-direction associated with the element at location $(i, j)$.

Logically, for $j$-th x-stripe, and $i$-th y-strip along which no yielded elements are detected, $\Delta M^j_x$ and $\Delta M^i_y$ are null. When they are not null, instead, yielded elements are detected. Then, $\Delta M^j_x$ and $\Delta M^i_y$ are, respectively, equally distributed among the x-neighbours of the $j$-th x-stripes, and the y-neighbours of the $i$-th y-stripes.

The above mentioned mathematical artifice is a representation of the phenomenon of stress redistribution occurring when a single part reaches its ultimate capacity, load keeps increasing, and other parts of the structure tend to 'absorb' the residual part of stresses. This is exactly what happens when an over-yielded moments is artificially dragged back to the yield surface, and its $\Delta M^e_x$ and $\Delta M^e_y$ components are, respectively, distributed among the x- and y-neighbours of the x- and y-stripe in which it is contained. Such concept does not apply to $\Delta M^e_{xy}$ moments instead. They are equally distributed among all the non-yielded elements of yielded elements, independently of their neighbourhood relationship. The whole moment configuration has been artificially altered, meaning that yielding may have occurred again somewhere in the slab. Then, such condition has to be checked again.

Loop 3 is reached. If yielding occurred, then the program goes back to Loop 2, and starts dragging over-yielded moments back to the surface, and then re-distributing moments again. Otherwise, if all the bending moments are contained in the surface, a pseudo-lower bound solution has been reached.

The last step is constituted by a check of whether a mechanism causing collapse of the slab is occurring or not. If that is not the case, the program goes back to Loop 1, and restart the whole analysis increasing load intensity. If instead a mechanism is occurring, a lower bound solution and its relative collapse mechanism are given as output, and the program breaks.

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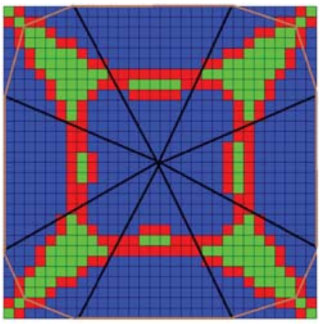

Fig. 4 Collapse Mechanism of a square clamped RC slab obtained with the Pseudo-lower bound solution

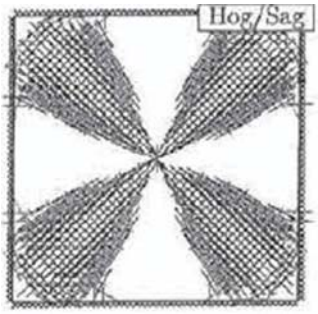


Fig. 5 Collapse Mechanism of a square clamped RC slab obtained with Jackson's lower bound solution [7]

### *D. Mechanism Detection Criterion and Preliminary Results*

In order to detect, at the last step of the algorithm, whether a mechanism is occurring or not, an occurrence criterion has to be established. The obtained collapse mechanism for the studied case is shown in Fig. 4. Yielded elements have obviously already reached their maximum capacity in terms of strength. Potential yield-lines may form along yielded neighbours then. In this way, a domain is for the existence of yield-lines has been created. Fig. 4 also gives an idea about which are the next elements to yield, namely the non-yielded neighbours of yielded elements. A further increase in load, or eventual redistribution of moments, will cause them to yield.

Yield-lines spread out from the centre of load (in this case coinciding with the centre of the slab), once they touch the boundaries, and yield occurs also along the lines joining such intersections of yield-lines and boundaries, namely axes of rotation, a collapse mechanism occurs. The problem now is reduced to minimizing the amount of energy necessary to cause collapse. Minimization of yield-lines and axes of rotations are computed through analogies with two well-known combinatorial algorithms, respectively Shortest Path Problem (SPP) and Travelling Salesman Problem (TSP). In Fig. 4, the black lines represent the yield-lines, and the orange lines represent the axes of rotation. If a solution is found, satisfying the above mentioned conditions, collapse occurs.

As shown by Fig. 5, the derived mechanism shows good agreement with the yield-line indicators obtained by Jackson, which were validated with experimental results [7].

## IV. Conclusions & Future Developments

The concept of a lower bound method for the assessment of RC slabs has been presented in this paper. Investigations of various researchers on the topic have been studied, and their progresses have tried to be improved. The main idea of the proposed method is to reproduce a simulation of the overall historical process, in terms of stresses, from steady unloaded condition until collapse. The method has been purposely called *'pseudo'*-lower bound to underline the non-naturally mechanical root of the analysis. The aim was to build a simple algorithm based on a linear method, which may run in a shorter time, and give results that take into account the effects of plasticity. The elastic solution is altered through the described iterative moment distribution. This concept is significantly related to the real structural behaviour, hence gaining valid reliability. As it was mentioned in the previous section, the obtained results well fit the exact solutions, and the method has high potential of improvement. With the future implementation of a wider library of element types, boundary conditions, and load pattern, the program could also be adapted to more complex geometrical configurations. Furthermore, with reference to non-linear FE analyses, computation costs are significantly reduced.

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