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Implementation of Quantum Rotation Gates Using Controlled Non-Adiabatic Evolutions

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***Abstract*—**Quantum gates are the basic building blocks in the quantum circuits model. These gates can be implemented using adiabatic or non adiabatic processes. Adiabatic models can be controlled using auxiliary qubits, whereas non adiabatic models can be simplified by using one single-shot implementation. In this paper, the controlled adiabatic evolutions is combined with the single-shot implementation to obtain quantum gates with controlled non adiabatic evolutions. This is an important improvement which can speed the implementation of quantum gates and reduce the errors due to the long run in the adiabatic model. The robustness of our scheme to different types of errors is also investigated.

***Keywords*—**Adiabatic evolutions, non adiabatic evolutions, controlled adiabatic evolutions, quantum rotation gates, dephasing rates, master equation.

I. Introduction

QUANTUM algorithms are powerful algorithms and their use allows one to outperform their classical counterparts. These algorithms can be implemented using either the circuit model [1] or the adiabatic quantum computation model proposed by Farhi et al. [2]. The Quantum circuit model is the most widely used model in quantum information and the computational operations are made in a sequence by using basic quantum gates. However, the advantage of the adiabatic quantum computation model over the circuit model lies in the robustness of the model against some types of errors. Despite this advantage it has been show that the two models are polynomially equivalent [3]. Since, the adiabatic model is based on the adiabatic theorem [4], [5], the extensive run time is a major flaw. This makes the model more vulnerable to the external interaction with the environment. In order to overcome this problem, the non adiabatic quantum computation was proposed [6]–[9]. This model is robust against decoherence and allows a high-speed realization of quantum gates.

Arbitrary quantum gates can be constructed from one- and two-qubit gates. In general, one-qubit gates is a ϕ-rotation about an arbitrary axis. So, the implementation of arbitrary quantum rotation gates are very important. In a recent study by Xu et al. [10] it has been shown how to realize a non adiabatic single quantum gate using a single shot implementation. This has been a great development compared to previous schemes where more than one sequential gates were used to implement a quantum rotation gate [6]–[8]. The proposed scheme uses a three Λ system that is driven by two laser fields and an arbitrary quantum gate can be obtained through the alteration of the detuning, phase and amplitude of the driving laser fields. A square pulses were selected to avoid trouble when varying the frequencies and amplitudes of the lasers. Another interesting approach to realize a single qubit gate was proposed by Italy [11]. It uses a controlled adiabatic evolution which is, so to speak, the circuit model of the adiabatic quantum computation.

The main contribution of this paper is the combination of single shot implementation with controlled adiabatic evolution to attain the controlled non-adiabatic evolution. In Sections II and III, we review the non adiabatic single shot implementation [10] and the controlled adiabatic evolution [11], respectively. In Section IV we propose our model, the controlled non adiabatic evolution and investigate the effect of errors on this model in Section V. Finally, we summarize the results in Section VI.

II. Non Adiabatic Quantum Gates Implementation

In this section we review the single-shot implementation of quantum gates [10]. The model is shown in Fig. 1. A three-level Λ system consists of two computational basis $\{|0\rangle, |1\rangle\}$ and an exited state $|e\rangle$ is driven by two square pulses with the same detuning Δ. The first pulse with the Rabi frequency Ω_0 acts on the transition $|0\rangle \leftrightarrow |e\rangle$ and the second pulse with Rabi frequency Ω_1 acts on the other transition $|1\rangle \leftrightarrow |e\rangle$. The transition $|0\rangle \leftrightarrow |1\rangle$ is a forbidden transition. In order to implement a quantum gate, the detuning and the Rabi frequencies are chosen as follows.

$$\Delta = -2\Omega \sin\gamma, \tag{1}$$

$$\Omega_0 = \Omega \cos\alpha \cos\gamma, \tag{2}$$

$$\Omega_1 = \Omega e^{i\beta} \sin\alpha \cos\gamma, \tag{3}$$

where, Ω is the norm of the vector $[\Delta/2, \Omega_0, \Omega_1]$ and α, β, and γ are arbitrary parameters. Different values of these parameters lead to different quantum rotation gates. The Hamiltonian of the system can be now written as

$$H = -\Delta|e\rangle\langle e| + \sum_{j=0}^{1} \left[\Omega_j |j\rangle\langle e| + \text{H.c.}\right]. \tag{4}$$

This Hamiltonian can be simplified to

$$H = \Omega \sin\gamma I + \Omega \left(\cos\gamma\, \sigma_x + \sin\gamma\, \sigma_z\right), \tag{5}$$

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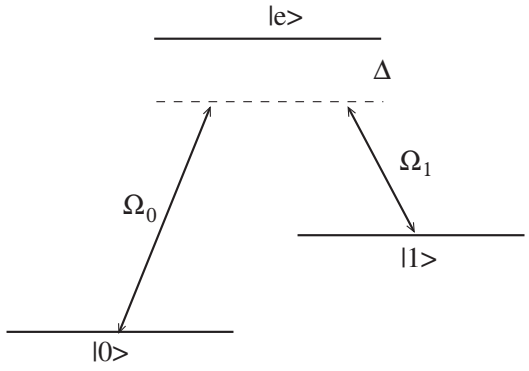


Fig. 1 Three level Λ system used to implement quantum rotation gates with just one-shot implementation

where, I is the identity matrix, the Pauli spins $\sigma_x = |e\rangle\langle b| + |b\rangle\langle e|$ and $\sigma_z = |e\rangle\langle e| - |b\rangle\langle b|$ with $|b\rangle = \cos\alpha|0\rangle + \mathrm{e}^{i\beta}\sin\alpha|1\rangle$. If the duration of the evolution is $\mathcal{T} = \pi/\Omega$, it is easy to obtain the evolution operator at $\mathcal{T}$ (the Hamiltonian is time independent). So, in the basis $\{|e\rangle, |b\rangle, |d\rangle\}$ the evolution operator can be written in a matrix form as

$$U(\mathcal{T}) = \begin{pmatrix} \mathrm{e}^{-i\phi} & 0 & 0 \\ 0 & \mathrm{e}^{-i\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6}$$

where the state $|d\rangle = \sin\alpha|0\rangle - \mathrm{e}^{i\beta}\cos\alpha|0\rangle$ is normal to the state $|b\rangle$. As (6) shows, the state $|d\rangle$ is decoupled from all the other states. Since our computational basis is $\{|0\rangle, |1\rangle\}$, restricting to this logical subspace, the evolution operator is then equivalent to

$$U(\mathcal{T}) \equiv \mathrm{e}^{-i\frac{\phi}{2}(|b\rangle\langle b| - |d\rangle\langle d|)}. \tag{7}$$

This evolution operator is simply a ϕ-rotation gate around the axis determined by the two states $|b\rangle$ and $|d\rangle$.

III. Controlled Adiabatic Evolutions

In this section we review the controlled adiabatic evolutions model proposed in [11]. This model is an important scheme that can be used to rotate any unknown state $|\psi_1\rangle$ by any angle ϕ about any arbitrary axis $|n\rangle$. It consists of two qubits. The first qubit is initially in the unknown state and the second one is an auxiliary qubit. The Hamiltonian is given by the tensor product

$$H(t) = |n\rangle\langle n| \otimes H_0(t) + |n_\perp\rangle\langle n_\perp| \otimes H_\phi(t), \tag{8}$$

where $H_0(t)$ and $H_\phi(t)$ are the adiabatic evolution Hamiltonians. They are acting on their respective subspaces spanned by the states $|n\rangle$ and its orthogonal state $|n_\perp\rangle$. They are time-dependent Hamiltonians and take the form

$$H_\phi(t) = -\cos\theta(t)\sigma_z - \sin\theta(t)(\cos\phi\sigma_x + \sin\phi\sigma_y), \tag{9}$$
$$H_0(t) = -\cos\theta(t)\sigma_z - \sin\theta(t)\sigma_x, \tag{10}$$

where, the time-depend is through the angle $\theta(t)$ which we assume it is a linear function of time, $\theta = \theta_f t/T$. The parameter θ_f is fixed and it is the value of the angle θ at the end of the evolution time T. This evolution time is assumed to be large enough in order to maintain the evolution of the system adiabatic. It is worth to mention that both Hamiltonians have two eigenvalues ± 1.

In the absence of any losses, the evolution of the system is governed by the Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle. \tag{11}$$

To find the solution of (11), the state $|\psi(t)\rangle$ is written as a superposition of its projections on the states $|n\rangle$ and $|n_\perp\rangle$

$$|\psi(t)\rangle = |n\rangle\,|\psi_n(t)\rangle + |n_\perp\rangle\,|\psi_{n\perp}(t)\rangle. \tag{12}$$

This decomposition leads to two decoupled first order differential equations

$$i\frac{\partial}{\partial t}|\psi_n(t)\rangle = H_0|\psi_n(t)\rangle, \tag{13}$$
$$i\frac{\partial}{\partial t}|\psi_{n\perp}(t)\rangle = H_\phi|\psi_{n\perp}(t)\rangle. \tag{14}$$

Let the auxiliary state be in the state $|0\rangle$. The initial state can be written as

$$|\psi(0)\rangle = (\alpha|n\rangle + \beta|n_\perp\rangle)|0\rangle, \tag{15}$$

which can be written as $\psi(0)\rangle = |n\rangle(\alpha|0\rangle) + |n_\perp\rangle(\beta|0\rangle)$, where α and β are arbitrary numbers such that $|\alpha|^2 + |\beta|^2 = 1$. Keeping in mind that the evolution is adiabatic, the system evolves to the the final state

$$\begin{aligned} |\psi(t_f)\rangle = {} & \cos(\theta_f/2)\,(\alpha\,|n\rangle + \beta|n_\perp\rangle)\,|0\rangle \\ & + \sin(\theta_f/2)\left(\alpha\,|n\rangle + \mathrm{e}^{i\phi}\beta|n_\perp\rangle\right)|1\rangle. \end{aligned} \tag{16}$$

When the final angle is chosen to be $\theta_f = \pi$, the first qubit will be a rotation of the initial unknown state by an angle ϕ around the state $|n\rangle$, whereas, the auxiliary state will be with probability one in the state $|1\rangle$.

IV. Controlled Non Adiabatic Evolutions

The adiabatic evolutions require long time run. In order to shorten this time, the non adiabatic evolution was proposed. In this section, we are ready to combine the two scheme, the non adiabatic single-shot implementation with the controlled evolutions. This will be an important improvement of the previous schemes, since it solve the problem of long time run which in turn improve the robustness against decoherence.

The Hamiltonian of the system is similar to that of the controlled adiabatic evolutions [11]. It is given by

$$H(t) = |n\rangle\langle n| \otimes H_0(t) + |n_\perp\rangle\langle n_\perp| \otimes H_\phi(t). \tag{17}$$

However, the Hamiltonians do not depend on time, unlike the controlled adiabatic evolutions. They are chosen almost identical to the single-shot implementation Hamiltonians, (5), such that

$$H_\phi = \Omega(\cos\phi - \sin\phi)\sigma_x + \Omega(\cos\phi + \sin\phi)\sigma_y, \tag{18}$$
$$H_0 = H_{\phi=0} = \Omega\,(\sigma_x + \sigma_y)\,. \tag{19}$$

If the initial state of the first qubit is in unknown state, $|\psi_1\rangle = \alpha|n\rangle + \beta|n_\perp\rangle$, and the auxiliary qubit in the state $|0\rangle$, then, the state at later time t can be obtained from (11). Thus,

$$\begin{aligned}|\psi(t)\rangle &= \cos\left(\sqrt{2}\Omega\, t\right)\left(\alpha|n\rangle + \beta|n_\perp\rangle\right)|0\rangle \\ &\quad + \mathrm{e}^{-i\pi/4}\sin\left(\sqrt{2}\Omega\, t\right)\left(\alpha|n\rangle + \mathrm{e}^{i\phi}\beta|n_\perp\rangle\right)|1\rangle. \end{aligned} \tag{20}$$

If the evolution time is chosen to be $T = \frac{\pi}{\sqrt{8}\,\Omega}$, then the system ends up in the state

$$|\psi(T)\rangle = \left(\alpha|n\rangle + \beta\mathrm{e}^{i\phi}|n_\perp\rangle\right)|1\rangle, \tag{21}$$

where we have omitted the global phase. This means that the first qubit evolves to a rotated state by an angle ϕ around the axis determined by the state $|n\rangle$. This rotated state is simply the rotation of the unknown state $|\psi_1\rangle$. In addition to that, the auxiliary qubit ends up in the state $|1\rangle$. Since the Hamiltonian (17) acts differently on the two subspaces spanned by $|n\rangle$ and its orthogonal state $|n_\perp\rangle$, the adiabatic evolution of the auxiliary state evolves also differently and in parallel on these subspaces [11]. This is of cause due to the adiabatic theorem [4], [5] which states that if the Hamiltonian evolves slowly with respect to the characteristic time of the system, the state of the system is kept close to an instantaneous eigenvalue at later time. To show this parallel evolutions we choose two quantum gates, the NOT gate and square NOT gate. The NOT gate is simply the Pauli-X gate which is given in matrix form by

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{22}$$

and it is a rotation of $\phi = \pi$ around the x-axis, i.e,. $|n\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$. The square root of NOT gate denoted by $\sqrt{\text{NOT}}$ gate is represented by the matrix

$$\sqrt{\sigma_x} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}, \tag{23}$$

and it is a rotation of $\phi = \pi/2$ around the x-axis. Fig. 2a shows the parallel evolutions of the auxiliary state initially in $|0\rangle$ for the NOT gate. At the end of the evolution $T = \pi/\sqrt{8}$, the auxiliary state ends up in $|1\rangle$. It is clear that the auxiliary qubit follows two different parallel paths on the Bloch sphere. The angle between the two longitudinal paths is π. Similarly, Fig. 2b illustrates the evolution of the auxiliary qubit for the $\sqrt{\text{NOT}}$ gate following two different longitudinal paths. The angle between these paths is $\pi/2$.

V. Analysis of Controlled Non Adiabatic Evolutions Model

Our Model described above is an ideal model. There is always interaction with the environment. So, our model is subject to different types of decoherence which cause an imperfect implementation of quantum rotation gates. In this section we focus on pulse area error and decoherence. The pulse area error is due to the imperfect control of the pulses and the decoherence is due to either the decay of the excited state or dephasing.

A. Pulse Area Error

The pulse area error occurs in case there is an inaccurate control in Ω which is the only relevant parameter. In this case, the fidelity at the end of the evolution time, $t = T$ can be easily obtained and it is given by

$$F = \sqrt{\frac{1}{2}\left(1 + \cos\left(\frac{\pi\xi}{\Omega}\right)\right)}, \tag{24}$$

where ξ is the error in the value of Ω. The fidelity is the same for all rotation gates and does not depend on the initial state of the first qubit. For small values $\xi \ll 1$, the fidelity is a quadratic function of ξ and it is given by

$$F \approx 1 - \frac{1}{8}\left(\frac{\pi\xi}{\Omega}\right)^2. \tag{25}$$

So, we have $\lim_{\Omega\to\infty} F = 1$. Thus, Increasing the Rabi frequency Ω leads to increasing the fidelity and also shorten the time run.

B. Dephasing

The effect of the decoherence can be studied by replacing the Schrödinger equation (11) by the the Lindblad master equation [12], [13]

$$\begin{aligned}\frac{d\rho}{dt} &= -i\left[H, \rho\right] \\ &\quad + \frac{1}{2}\sum_i \left(2\, C_i \rho C_i^\dagger - C_i^\dagger C_i \rho - \rho C_i^\dagger C_i\right), \end{aligned} \tag{26}$$

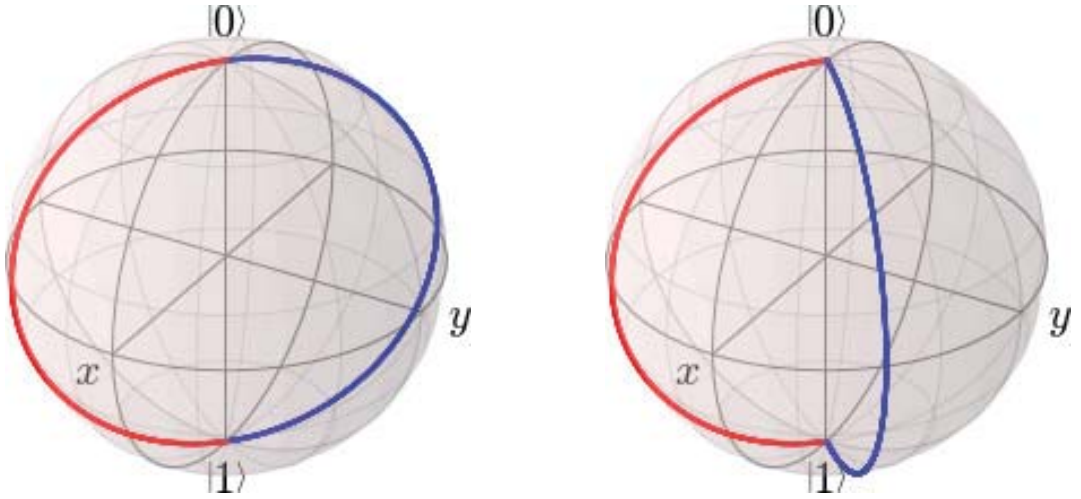
where ρ is the density operator, H is the Hamiltonian operator given by (17), and C_i are the Lindblad operators associated with different types of decoherence. We will focus on two important operators, $C = \sqrt{2\kappa}|1\rangle\langle 0|$ and $C = \sqrt{2\kappa}\sigma_z$. For simplicity we assume that both qubits have the same type of decoherence with the same rate κ. To compute the fidelity we can use either the master equation (26) or the quantum trajectory approach [14]–[17]. The quantum trajectory approach unravels the master equation so that it can show the evolution of an individual trajectory. This is very important if we are interested in the relative phase between the two paths followed by the auxiliary qubit.

The numerical computations of the fidelities are obtained from the master equation (26) using 5000 initial random states uniformly distributed on the Bloch sphere

$$|\psi(0)\rangle = \left(\sqrt{u}|0\rangle + \mathrm{e}^{i2\pi v}\sqrt{1-u^2}|1\rangle\right)|0\rangle, \tag{27}$$

where, u and v are two random numbers uniformly distributed over the unit interval $[0, 1]$.

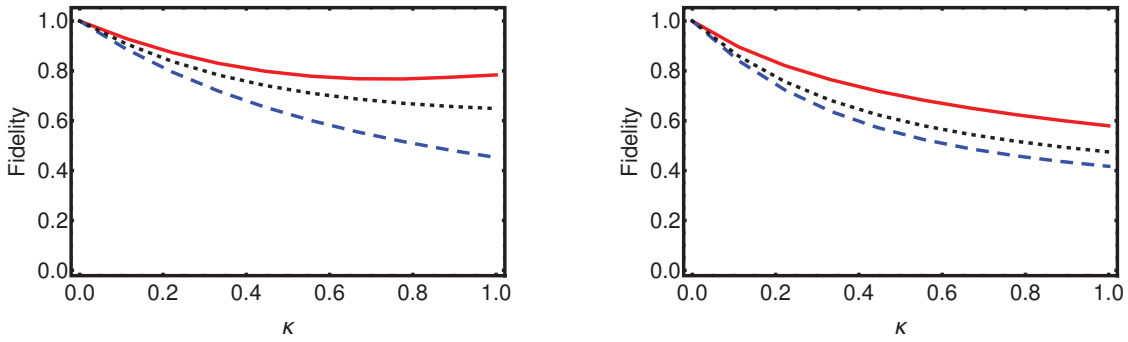
Figs. 3 and 4 show the maximum, the minimum, and the average fidelities as a function of the rate κ for the NOT gate where the common Lindblad operator $C = \sqrt{2\kappa}|1\rangle\langle 0|$ and $C = \sqrt{2\kappa}\sigma_z$. In case when $C = \sqrt{2\kappa}\sigma_z$, there is no big deviation between the fidelities for different initial states. However, there is a large deviation when $C = \sqrt{2\kappa}|1\rangle\langle 0|$. This indicates that the fidelity depends on the initial state of the unknown state. We have also plot the fidelity as a function of Ω. Fig. 5 shows that for large values of Ω the fidelity approaches 1. So, this means that increasing the value of the Rabi frequency compensate the errors due to dephasing and area pulse error.



(a) For the NOT gate. The rotation angle is $\phi = \pi$

(b) For the $\sqrt{\text{NOT}}$ gate. The rotation angle is $\phi = \pi/2$.

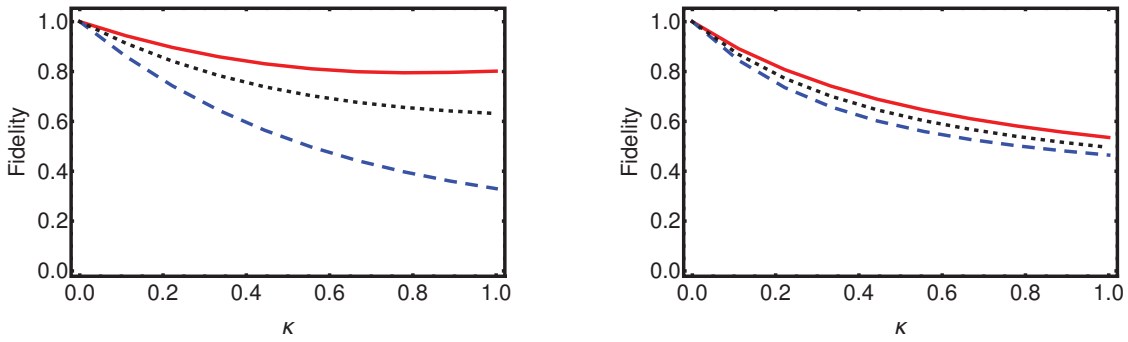
Fig. 2 The evolution of the auxiliary qubit follows two different longitudinal paths on the Bloch sphere. The parameters are: $\Omega = 1$, $|n\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and the evolution time $T = \pi/\sqrt{8}$



(a) The Linddblad operator $C_i = \sqrt{2\kappa}|1\rangle\langle 0|$

(b) The Linddblad operator $C_i = \sqrt{2\kappa}\sigma_z$

Fig. 3 The fidelity for the NOT gate as function of κ. The maximum (solid line), the minimum (dashed line) and the average (doted line) fidelities. The value of the Rabi frequency is $\Omega = 1$



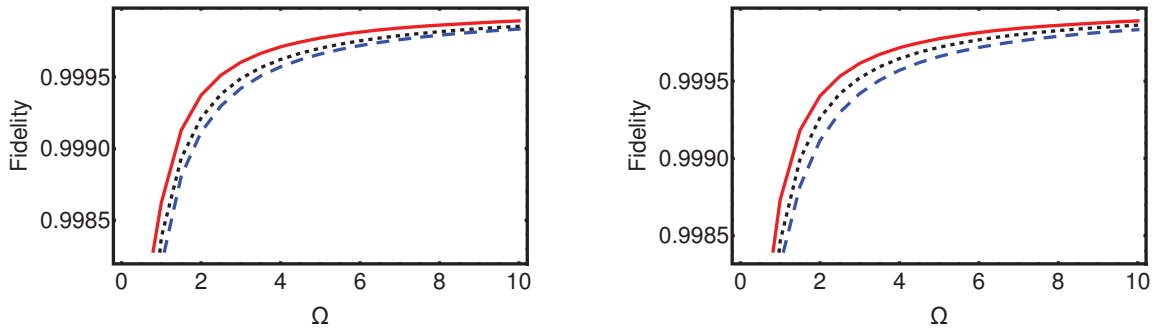
(a) The Linddblad operator $C_i = \sqrt{2\kappa}|1\rangle\langle 0|$

(b) The Linddblad operator $C_i = \sqrt{2\kappa}\sigma_z$

Fig. 4 The fidelity for the $\sqrt{\text{NOT}}$ gate as function of κ. The maximum (solid line), the minimum (dashed line) and the average (doted line) fidelities. The value of of the Rabi frequency is $\Omega = 1$.

VI. Conclusion

In the field of quantum computing, non-adiabatic and adiabatic schemes can be used to implement any quantum gate. Rotation gates are very important gates and can be implemented using both schemes. Non-adiabatic schemes established an increasing attention for its robustness against certain errors. Nevertheless, all previous studies demonstrated that with two or more sequentially implemented gates, a



(a) For the NOT Gate

(b) For the $\sqrt{\text{NOT}}$ gate

Fig. 5 Fidelity for the NOT and $\sqrt{\text{NOT}}$ gates as a function of the Rabi frequency Ω. The Lindblad operators are $C_i = \sqrt{2\kappa}\sigma_z$ and $\kappa = 0.001$. The maximum (solid line), the minimum (dashed line) and the average (doted line) fidelities

general one-qubit gate can be obtained and constructed. In recent studies, it has been demonstrated that with non adiabatic evolution a gate can be realized using single shot implementation, and on another study it demonstrated an implementation of a quantum gate using controlled adiabatic evolution. In this paper, we have combined the two approaches and obtain what is called controlled non adiabatic evolutions. Our approach is important improvement over the previous schemes. Our focus is on the NOT and $\sqrt{\text{NOT}}$ gates but it can be used for any rotation gate. We have also discussed the robustness of our model against the pulse area error and the dephasing. We Have calculated the maximum, minimum, and average fidelities for 5000 initial states uniformly distributed on the Block sphere. The run time which is given by $pi/\sqrt{8}\Omega$ and can be chosen as small as desired by increasing the value of the Rabi frequency. This will shorten the time run as well as make the fidelity goes to unity.

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