Stochastic Repair and Replacement with a Single Repair Channel

Mohammed A. Hajeeh

Abstract—This paper examines the behavior of a system, which upon failure is either replaced with certain probability p or imperfectly repaired with probability q. The system is analyzed using Kolmogorov's forward equations method; the analytical expression for the steady state availability is derived as an indicator of the system's performance. It is found that the analysis becomes more complex as the number of imperfect repairs increases. It is also observed that the availability increases as the number of states and replacement probability increases. Using such an approach in more complex configurations and in dynamic systems is cumbersome; therefore, it is advisable to resort to simulation or heuristics. In this paper, an example is provided for demonstration.

Keywords—Repairable models, imperfect, availability, exponential distribution.

I. Introduction

OPERATING systems, such as mechanical systems, consist of components; these components may fail in different modes such as fatigue, leak, rupture, creep, wear, corrosion, and deformation. Failed components may either be replaced or repaired; repairable systems when failed usually receive repair or maintenance actions that restore their functions. These actions affect the system behavior and alter the makeup of the system. Repair brings the system either to as good as new "perfect repair" or to a status prior to failure "minimal repair", or to an inferior state called 'imperfect repair". The aim of the different maintenance actions is to enhance systems' performance; reliability is one of the performance measures used to insure system effectiveness and produce quality characteristic products required by consumers. Another important performance criterion is availability, which encompasses both reliability and maintainability; it is the probability that a system is operational and executes its required functions at a given point of time

Systems' performance has been addressed extensively in the literature; for example, Abdel-Hameed [1] examined an age-dependent minimal repair model where the failed system is either perfectly repaired with probability $p(t)$ or minimally repaired with probability $1-p(t)$. Brown and Proschan [2] developed optimal replacement policies for a system, which upon failure either undergoes perfect repair with probability p or is minimally repaired with probability $(1-p)$. Meanwhile, Beichelt [3] considered the total repair cost limit replacement policy, where the system is replaced as soon as its total repair

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cost reaches a specific level. Moustafa [4] studied Markov models for the transient analysis of the reliability of k-out-of-N: G systems with and without repair subject to M failure modes. Zhao [5] studied the failure pattern of repairable components when a failed component is either perfectly or imperfectly repaired. In this work, the lifetime of a component is assumed to have a general distribution. Several asymptotic quantities were derived such as the mean number of failures in a specific component position, the fraction of time the system is down due to failure in a particular component position, and the availability of the system.

Dimitrov et al. [6] examined an age-dependent repair model with imperfect repair, obtaining the warranty costs for the products under a nonhomogeneous Poisson process scenario. The behavior of multiple repairable systems was inspected by Pan and Rigdon [7] using Bayesian methods for models that are between as "bad as old" and "good-as-new". Monte Carlo methods were used to approximate the properties of the posterior distributions. Pandey et al. [8] developed a mathematical model to assist decision makers in selecting proper maintenance scenarios under imperfect repair. Examples were used to validate the applicability of the proposed method. Results indicated that the introducing of imperfect repair better facilitates the allocation of maintenance resources. Meanwhile, El-Damcese and Shama [9] studied the performance of a 2-state repairable system with two types of failures. Laplace transform techniques were utilized to develop expressions for several performance measures including availability, reliability, and mean time to failure under exponentially distributed times between failures and repair times. Neguyen et al. [10] studied repairable systems under imperfect repair where the time between the failures of a new system follows the Weibull distribution. Furthermore, an analytical approach for the distribution of the inter-failure times was obtained in addition to producing under steady conditions, an optimal preventive maintenance policy under a static, a dynamic, or a failure limit policy.

II. System Descriptions

The purpose of this paper is to study the availability of a system, which upon failure is either replaced with certain probability or imperfectly repaired. The failure and repair rate times are assumed to follow exponential distribution. The system is analyzed using Kolmogorov's forward equations method. Other assumptions include availability, and the travel time to repair station is negligible. The analysis is based on the availability of one repair person.

At each failure, the system is either imperfectly repaired

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with probability q or is perfectly repaired with probability p. After each perfect repair, the system becomes as new, while following the imperfect repair, the system's performance decreases as time progresses. Thus, the failure rate increases after each repair ($\alpha_{i+1} \geq \alpha_i$ for $i = 1, 2, n$); similarly, the rate of repair declines after each repair; $\varphi_{i+1} \leq \varphi_i$ for $i = 1, 2,.., n$. The system is replaced after undergoing a specified number of repairs (n).

A pictorial presentation of the state transitions is shown in Fig. 1. The rectangular shapes in the figure represent the operational state, while the oval shapes are the failed states.

The process starts at state 1, where the system is new; after failure (α_1), at state 1, the system moves to state 2 for repair. At state 2, the system is either perfectly repaired with probability $p\varphi_{n+1}$ to state 1 or imperfectly repaired with probability $q\varphi_1$ (q + p =1) to state 3. Similarly, after failure with a failure rate of α_1, the system transitions to state 4. At state 4, the system is either perfectly repaired to state 1 or imperfectly repaired to state 2 with probability $q\varphi_2$. Finally, after the n+1 failure, the system is replaced and moves to state 1, and the process is regenerated.
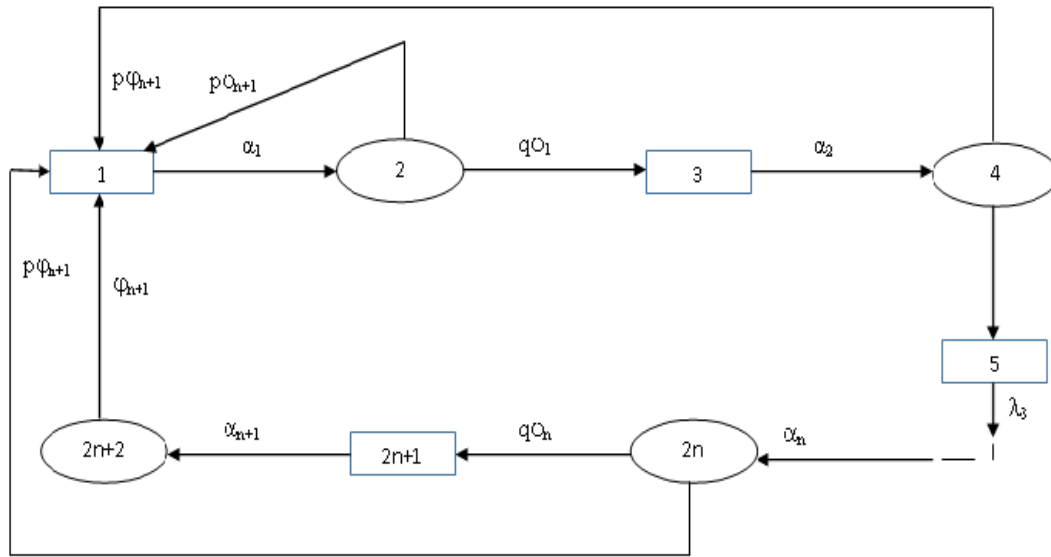

Fig. 1 A pictorial presentation of the transition process of the system

A. Availability Analysis

The transition probability P(t) for the systems is as:

$$\tilde{p}_1(t) = -\alpha_1 p_1(t) + p\varphi_{n+1} p_2(t) + p\varphi_{n+1} p_4(t) + ... + p\varphi_{n+1} p_{2N}(t)$$
$$\tilde{p}_2(t) = \alpha_1 p_1(t) - (p\varphi_{n+1} + q\varphi_1) p_2(t)$$
$$\vdots$$
$$\tilde{p}_{2n}(t) = \alpha_n p_{2n-1}(t) - (p\varphi_{n+1} + q\varphi_n) p_{2n}(t) \qquad (1)$$
$$\tilde{p}_{2n+1}(t) = \varphi_n p_{2n}(t) - \alpha_{n+1} p_{2n+1}(t)$$
$$\tilde{p}_{2n+2}(t) = \alpha_{n+1} p_{2n+1}(t) - \varphi_{n+1} p_{2n+2}(t)$$

The steady state system with exponential Markov with continuous time is expressed as:

$$Q\pi = 0$$

where π is the state transition probability matrix and Q is the state transition rate matrix Q is:

$$Q = \begin{bmatrix}
-\alpha_1 & p\mu_{n+1} & 0 & p\mu_{n+1} & 0 & .... & p\mu_{n+1} & 0 & \mu_{n+1} \\
\alpha_1 & -(p\mu_{n+1}+q\mu_1) & 0 & 0 & 0 & .... & 0 & 0 & 0 \\
0 & q\mu_1 & -\alpha_2 & 0 & 0 & .... & 0 & 0 & 0 \\
0 & 0 & \alpha_2 & -(p\mu_{n+1}+q\mu_2) & 0 & .... & 0 & 0 & 0 \\
0 & 0 & 0 & q\mu_2 & -\alpha_3 & .... & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & -(p\mu_{n+1}+q\mu_n) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q\mu_n & -\alpha_{n+1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{n+1} & -\mu_{n+1}
\end{bmatrix}$$

Solving the system of equations $Q\pi = 0$, the steady state transition probability is:

$$\pi_{2k} = \frac{\alpha_j}{(q\varphi_j + p\varphi_{n+1})} \pi_{2k-1}; \text{for } 1 \leq k \leq n \qquad (2)$$

$$\pi_{2k+1} = \frac{q\varphi_j}{\alpha_{J+1}} \pi_{2k} \qquad \text{for } 1 \leq k \leq n \qquad (3)$$

$$\pi_{2n+2} = \frac{\alpha_{n+1}}{\varphi_{n+1}} \pi_{2n+1} \qquad (4)$$

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$$\sum_{i=1}^{n} \pi_k = 1 \tag{5}$$

Equations can be written in terms of π_1 as follows:

$$\pi_{2k} = q^{k-1} \frac{\alpha_1}{(p\varphi_{n+1} + q\varphi_1)} \prod_{i=1}^{k-1} \frac{\varphi_i}{(p\varphi_{n+1} + q\varphi_{i+1})} \pi_1 \tag{6}$$
$$\text{for } 1 \leq k \leq n+1$$

$$\pi_{2k+1} = q^k \frac{\varphi_k}{\alpha_{k+1}} \frac{\alpha_1}{(p\varphi_{n+1} + q\varphi_1)} \prod_{i=1}^{k-1} \frac{\varphi_i}{(p\varphi_{n+1} + q\varphi_{i+1})} \pi_1 \tag{7}$$
$$\text{for } 1 \leq k \leq n$$

where $\prod_{i=1}^{0} \dfrac{\varphi_i}{(p\varphi_{n+1} + q\varphi_{i+1})} = 1$.

Utilizing (5)-(7), the expression for the probability of being in the initial state 1(π_1) is as:

$$a = \frac{\alpha_1}{q\varphi_1 + p\varphi_{n+1}}$$

$$b(k) = \prod_{i=1}^{k-1} \frac{\varphi_i}{q\varphi_{i+1} + p\varphi_{n+1}}$$

$$\pi_1 = \frac{1}{1 + a\left[\sum_{k=1}^{n+1} q^k b(k) + \sum_{k=1}^{n+1} q^k \frac{\varphi_k}{\alpha_{k+1}} b(k)\right]} \tag{8}$$

$$A = \frac{\sum_{k=1}^{n+1} q^k \frac{\varphi_k}{\alpha_{k+1}} b(k)}{1 + a\left[\sum_{k=1}^{n+1} q^k b(k) + \sum_{k=1}^{n+1} q^k \frac{\varphi_k}{\alpha_{k+1}} b(k)\right]} \tag{9}$$

$$A = \frac{\sum_{k=1}^{n+1} q^k \frac{\varphi_k}{\alpha_{k+1}} b(k)}{1 + a\left[\sum_{k=1}^{n+1} q^k b(k) + \sum_{k=1}^{n+1} q^k \frac{\varphi_k}{\alpha_{k+1}} b(k)\right]} \tag{10}$$

III. EXAMPLES

To illustrate the process, an example is presented for a system that undergoes imperfect repairs before complete failure and regeneration; the transition states are presented in Fig. 2.

The state transition matrix for this problem is presented as:

$$Q = \begin{bmatrix} -\alpha_1 & p\varphi_1 & 0 & p\varphi_2 & 0 & \varphi \\ \alpha_1 & -(p\varphi + q\varphi_1) & 0 & 0 & 0 & 0 \\ 0 & q\varphi_1 & -\alpha_2 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & -(p\varphi + q\varphi_2) & 0 & 0 \\ 0 & 0 & 0 & q\varphi_2 & -\alpha_3 & 0 \\ 0 & 0 & 0 & 0 & \alpha_3 & -\varphi \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \\ \pi_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{11}$$
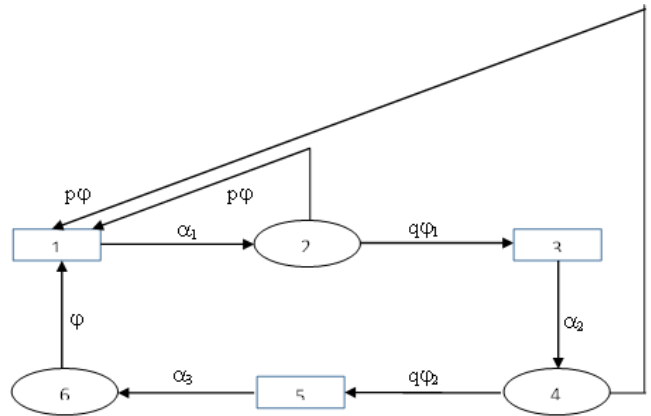

Fig. 2 A pictorial presentation of a system with two imperfect repairs before replacement

From the above matrix, the system's transition probabilities are presented and the expression for the steady state probability of being at state 1 is derived as follows:

$$\pi_1 = \frac{\alpha_2\alpha_3\varphi(p\varphi + q\varphi_1)(p\varphi + q\varphi_2)}{\begin{bmatrix} \alpha_2\alpha_3\varphi(p\varphi + q\varphi_1)(p\varphi + q\varphi_2) + \alpha_1\alpha_2\alpha_3\varphi(p\varphi + q\varphi_2) \\ + \alpha_1\alpha_3 q\varphi\varphi_1(p\varphi + q\varphi_2) \\ + \alpha_1\alpha_2\alpha_3\varphi q\varphi_1 + \alpha_1\alpha_2\varphi q\varphi_1 q\varphi_2 + \alpha_1\alpha_2\alpha_3 q\varphi_1 q\varphi_2 \end{bmatrix}} \tag{12}$$

Summing the probabilities of the operational states, we obtain the steady state availability of the system as follows:

$$A = \frac{\varphi\begin{bmatrix} \alpha_2\alpha_3(p\varphi + q\varphi_1)(p\varphi + q\varphi_2) + \alpha_1\alpha_3 q\varphi_1(p\varphi + q\varphi_2) \\ + \alpha_1\alpha_2 q\varphi_1 q\varphi_2 \end{bmatrix}}{\varphi\begin{bmatrix} \alpha_2\alpha_3(p\varphi + q\varphi_1)(p\varphi + q\varphi_2) + \alpha_1\alpha_3 q\varphi_1(p\varphi + q\varphi_2) \\ + \alpha_1\alpha_2 q\varphi_1 q\varphi_2 \end{bmatrix} + \alpha_1\alpha_2\alpha_3\varphi(p\varphi + q\varphi_2) + \alpha_1\alpha_2\alpha_3\varphi q\varphi_1 + \alpha_1\alpha_2\alpha_3 q\varphi_1 q\varphi_2}$$

Dividing the numerator and the denominator by $\alpha_1\alpha_2\alpha_3\varphi(p\varphi + q\varphi_1)(p\varphi + q\varphi_2)$ and rearranging and simplifying, the following expression is derived:

$$A = \frac{\left[\frac{1}{\alpha_1} + \frac{q\varphi_1}{(p\varphi + q\varphi_1)}\left[\frac{1}{\alpha_2} + \frac{q\varphi_2}{\alpha_3(p\varphi + q\varphi_2)}\right]\right]}{\left[\frac{1}{\alpha_1} + \frac{q\varphi_1}{(p\varphi + q\varphi_1)}\left\{\frac{1}{\alpha_2} + \frac{q\varphi_2}{\alpha_3(p\varphi + q\varphi_2)}\right\} + \frac{1}{(p\varphi + q\varphi_1)}\left\{1 + \frac{q\varphi_1}{(p\varphi + q\varphi_2)}\left[1 + \frac{q\varphi_2}{\varphi}\right]\right\}\right]} \tag{13}$$

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The system availability for various values of repair and failure rates is exhibited in Figs. 3 and 4. In Fig. 4, for a constant repair, the availability increases as the failure rates increase.
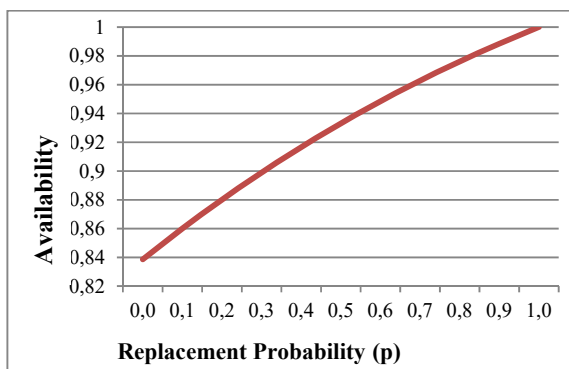

Fig. 3 Availability vs. replacement probability for the system ($\alpha_1 = 0.05$, $\alpha_2 = 0.30$, $\alpha_3 = 0.70$; $\varphi_1 = 2.0$, $\varphi_2 = 1.75$, $\varphi_3 = 1.5$; $\varphi = 3.0$)
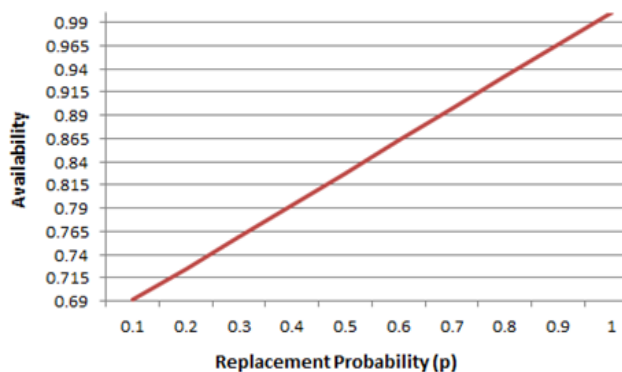

Fig. 4 Availability vs. replacement probability for the system ($\alpha_1 = 0.6$ $\alpha_2 = 0.90$, $\alpha_3 = 1.5$; $\varphi_1 = 2.0$, $\varphi_2 = 1.75$, $\varphi_3 = 1.5$; $\varphi = 3.0$)

IV. Conclusions and Future Research

Most the maintenance problems that are investigated in literature are based on the assumption of perfect maintenance; this work is an attempt to study imperfect repair phenomena, which often occurs in real-world situations. In this regard, expressions for several performance measures are derived expressions. These expressions are generic and apply not only to imperfect repair, but also to perfect repair. The expressions become more complicated as the number of states increases. Furthermore, the system may become rather cumbersome, if non-exponential distributions are used. Complexity is also attributed to the increase in the number of repair channels, and by adding the time element. Simulation may be utilized to address large scale, dynamic maintenance problems.

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