World Academy of Science, Engineering and Technology
International Journal of Electronics and Communication Engineering
Vol:11, No:12, 2017

Identification of LTI Autonomous All Pole System Using Eigenvector Algorithm

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Abstract—This paper presents a method for identification of a linear time invariant (LTI) autonomous all pole system using singular value decomposition. The novelty of this paper is two fold: First, MUSIC algorithm for estimating complex frequencies from real measurements is proposed. Secondly, using the proposed algorithm, we can identify the coefficients of differential equation that determines the LTI system by switching off our input signal. For this purpose, we need only to switch off the input, apply our complex MUSIC algorithm and determine the coefficients as symmetric polynomials in the complex frequencies. This method can be applied to unstable system and has higher resolution as compared to time series solution when, noisy data are used. The classical performance bound, Cramer Rao bound (CRB), has been used as a basis for performance comparison of the proposed method for multiple poles estimation in noisy exponential signal.

Keywords—MUSIC algorithm, Cramer Rao bound, frequency estimation.

I. Introduction

POLE estimation of noisy exponential signal appears in many areas such as radar signal processing, sonar signal processing, communication, speech signal processing etc. Frequency estimation has also applications in power quality assessment. Different parametric and nonparametric methods have been used for pole and frequency estimation [1]-[3]. A comparison of different nonparametric methods such as spectral filtering, which is a block processing nonparametric approach, Fast Fourier transform (FFT), interpolated FFT, short time Fourier transform, chirp Z transform, wavelet transform, Hilbert transform, least squares sine fitting has been done in [4]. The nonparametric methods involve forming a weighted periodogram or the time series data and have very poor resolution owing to the time-frequency uncertainty principle applicable to finite record data length, however, their computational complexity is very small. Least squares method (LS), linear prediction (LP) and the maximum likelihood (ML) methods have been used for frequency estimation for different applications [5]-[7]. We proposed MUSIC algorithm for complex frequency estimation. The advantage of using the MUSIC algorithm over the ML estimation is that the latter cannot be implemented in closed form i.e. search algorithms are required for estimation, while, no such search algorithm is required for the MUSIC algorithm [8]. To implement MUSIC, $O(N^3)$ multiplications must be performed for determining the eigenvectors of the $(N \times N)$ correlation matrix. In the time series algorithm, again a set of linear equations must be solved for obtaining the autoregressive (AR) coefficients and this is $O(p^3)$, if, $p \times p$ is the size of the correlation matrix or equivalently p is the length of AR polynomial. However, after obtaining the AR coefficients, the complex frequencies must be determined by rooting the AR polynomial and this involves a Newton Raphson iteration [9]. Further, AR models do not remove noise and hence, their resolution is poor as compared with the MUSIC method, which removes noise as the minimum eigenvector of the correlation matrix. Thus, the MUSIC method has higher resolution than the time series method, but, also has higher complexity $(N > p \Rightarrow O(N^3) > O(p^3))$. The MUSIC method has lower resolution and accuracy than the ML method, but lesser complexity [10]-[11]. Reference [12] proposed a method for estimating the frequency of a complex sinusoid in complex white Gaussian noise. The method attains the Cramer Rao bound down to lower signal to noise ratio values. [13] proposed an improved eigen decomposition based algorithm which shows that the frequency can be alternatively estimated from a set of estimates of $e^{j\omega}$. It has better performance because more estimates of each frequency are

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available to average out the numerical error. This paper presents a method for identification of an all pole system defined by a linear differential equation with constant coefficients. Single and two sinusoids frequency estimation has been proposed by [12], [13], but the method presented in this paper can be used for multiple frequency estimation.

II. Proposed Method

With $x(t)$ as the input process and $y(t)$ as the output, the system is defined by,

$$\frac{d^p y(t)}{dt^p}+a_1\frac{d^{p-1}y(t)}{dt^{p-1}}+....+a_{p-1}\frac{dy(t)}{dt}+a_p y(t)=x(t) \tag{1}$$

Identification of this system from input data amounts to eliminating the coefficients a_1, a_2,...,a_p. One way is to take discrete measurements of the input process, approximate the differential equation by a difference equation and estimate a_1, a_2, ...,a_n by a time domain least squares method. If the sampling rate is $1/T$ and measurements $y[n] = y(nT)$, $x[n] = x(nT)$ are taken, then $dy(t)/dt$ is replaced by Δ/T, where Δ is the finite difference operator, i.e. $\Delta y[n] = y[n] - y[n-1]$. The time domain least squares method amounts to minimizing $\sum(T^{-p}\Delta^p y[n] + a_1 T^{-p+1}\Delta^{p-1}y[n] + ... + a_{p-1}T^{-1}\Delta y[n] + a_p y[n] - x[n])^2$ with respect to $a_1, a_2, ..., a_p$. This is rather inaccurate due to discretization, depends on measuring both input - output data and is also computationally expensive. In this paper, we propose an algorithm for estimating a_1, a_2,....,a_p using high resolution eigen space method and using only measurements on the output data. To do so, we note that if $x(t) = 0$, then the output at time t is given by $y(t) = \sum_{k=1}^{p} A_k e^{s_k}t$, where, $s_1, s_2, ..., s_p$ are the roots of the polynomial $A(s) = s^p + a_1 s_{p-1} + ... + a_{p-1}s + a_p$. Assuming that s_1, s_2,....,s_p are all distinct, then a_1, a_2,...a_p are symmetric functions of the poles s_1, s_2,...,s_p related by $\prod_{k=1}^{p}(s-s_k) = s^p + a_1 s_{p-1} + ...a_{p-1}s + a_p$. Thus, for example, $a_1 = -\sum_{k=1}^{p} s_k$, $a_2 = \sum_{1\le k<m\le p} s_k s_m$, $a_{p-1} = (-1)^{p-1}\sum_{1\le i_1<i_2<...<i_{p-1}\le p} s_{i_1}...s_{i_{p-1}}$

and $a_p = (-1)^p \prod_{i=1}^{p} s_i$. The A_k's are completely determined by the initial conditions, $y^{(k)}(0)$, where, $0 \le k \le p-1$;

$$y^{(k)}(0) = \sum_{r=1}^{p} s_r^k A_r for 0 \le k \le p-1 \tag{2}$$

or,

$$\begin{pmatrix} y(0) \\ y^{'}(0) \\ y^{''}(0) \\ . \\ . \\ . \\ y^{(p)}(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1.. & 1 \\ s_1 & s_2 & s_3 & s_4.. & s_p \\ s_1^2 & s_2^2 & s_3^2 & s_4^2.. & s_p^2 \\ . & & & & \\ . & & & & \\ . & & & & \\ s_1^{p-1} & s_2^{p-1} & s_3^{p-1} & s_4^{p-1}.. & s_p^{p-1} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ . \\ . \\ . \\ A_p \end{pmatrix} \tag{3}$$

The Van-Der-Monde matrix is inverted to solve $(A_1, A_2, ..., A_p)^T$. Now, suppose we give several initial conditions, say, $(y_r(0), y_r^{'}(0), ..., y_r^{p-1}(0))_{r=1}^{M}$. Then corresponding to the r^{th} set of initial conditions , let $A_m^{(r)}$ be the amplitude of $e^{s_m t}$ in $y_r(t)$, the output is given by

$$y_r(t) = \sum_{m=1}^{p} A_m^{(r)} e^{s_m t} for, 1 \le r \le M \tag{4}$$

We collect N samples of the outputs for each set of initial conditions at the rate of $1/T$.

$$\mathbf{y_r} = \; [y_r(0), y_r(T), y_r(2T), ..., y_r((N-1)T)]^T \tag{5}$$

Arranging $\mathbf{y_r}$ as column of an $N \times M$ matrix Y.

$$\mathbf{Y} = [\mathbf{y_1}, \mathbf{y_2}, ..., \mathbf{y_M}] = \sum_{\mathbf{m=1}}^{\mathbf{p}} \mathbf{e}(\mathbf{s_m})[\mathbf{A_1^{(1)}}, \mathbf{A_2^{(2)}}, ..., \mathbf{A_p^{(M)}}] \tag{6}$$

or equivalently,

$$\mathbf{Y} = \mathbf{E}(\mathbf{s})\mathbf{A} \tag{7}$$

where,

$$\mathbf{E(s)} = [\mathbf{e}(s_1), ..., \mathbf{e}(s_p)] \in \mathbb{C}^{N\times p}$$

$$\mathbf{e}(s_k) = [e^{s_k t_1}, e^{s_k t_2}, ..., e^{s_k t_N}]^T$$

and

$$\mathbf{A} = [\mathbf{A_1}^T, \mathbf{A_2}^T, ..., \mathbf{A}_p{}^T]$$

and

$$\mathbf{A}_k{}^T = [A_{k_1}, A_{k_2}, ..., A_{k_M}]$$

$N, M \gg p$ taken, so that it is possible to assume that $rank(\mathbf{E(s)}) = p = rank(\mathbf{A})$. Here $(\mathbf{A})$ is an unknown matrix, $\mathbf{s}$ is an unknown parameter vector, $\mathbf{Y}$ is known NXM matrix with rank $(Y) = p$.

III. The MUSIC Estimator

Estimation of poles using MUSIC algorithm is based on the eigen decomposition of the autocorrelation matrix into two subspaces, a signal subspace and noise subspace. Using the following signal model

$$\mathbf{X} = \mathbf{Y} + \mathbf{W} \tag{8}$$

where, $\mathbf{W}$ is white Gaussian noise. We form the autocorrelation matrix as

$$\boldsymbol{R} = \mathbb{E}(\boldsymbol{X}^\dagger \boldsymbol{X}) = \mathbb{E}[(\boldsymbol{E}(\boldsymbol{s})\boldsymbol{A}\boldsymbol{A}^\dagger \boldsymbol{E}^\dagger(\boldsymbol{s})) + \sigma_w^2 \boldsymbol{I}] \tag{9}$$

Then $\mathbf{R}$ is an $M \times M$ positive semi definite matrix of rank p. $\mathbf{AA}^\dagger$ is a $p \times p$ positive definite matrix. σ_W^2 is the noise variance. $X^\dagger$ denotes conjugate transpose of X. The eigen structure of $\mathbf{R}$ is of the form

$$\boldsymbol{R} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\dagger + \sigma_w^2 \boldsymbol{I} \tag{10}$$

where, $D = diag[\lambda_1, \lambda_2, ..., \lambda_p, \lambda_{min}, \lambda_{min}, ..., \lambda_{min}] \in \mathbb{R}^{M\times M}$, and $\lambda_1 > \lambda_2 > ..... > \lambda_p > \lambda_{min}$ are the eigen values of R. It has p eigen values greater than λ_{min} and $(M-p)$ eigen values are λ_{min} . Also $\lambda_1 > \lambda_2 > ..... > \lambda_p > \lambda_{min}$. U is a unitary matrix. Its column is denoted by

$$\boldsymbol{U} = [\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_M}] \tag{11}$$

and

$$U^{\dagger}U = I \tag{12}$$

i.e.

$$u_{\alpha}^{\dagger}u_{\beta} = \delta_{\alpha\beta} \tag{13}$$

Thus,

$$\boldsymbol{R} = \sum_{\alpha=1}^{p} \lambda_{\alpha} u_{\alpha} u_{\alpha}^{\dagger} \tag{14}$$

$\{\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_p}\}$ are the signal eigen vectors while $\{\mathbf{u_{p+1}}, ..., \mathbf{u_M}\}$ are the noise eigen vectors. Thus,

$$span(\mathbf{u_1}, ..., \mathbf{u_p}) = span(\mathbf{u_{p+1}}, ..., \mathbf{u_M})^{\perp} \tag{15}$$

$$span(\mathbf{E(s)}) = span(\mathbf{e}(s_1), ..., \mathbf{e}_(s_p)) \tag{16}$$

It follows that the function

$$P_{MUSIC} = \frac{1}{\sum_{\alpha=p+1}^{M} \left|\boldsymbol{u}_{\alpha}^{\dagger} e(s)\right|^2} = \frac{1}{\boldsymbol{e}^{\dagger}(s)\boldsymbol{e}(s) - \sum_{\alpha=1}^{p} \left|\boldsymbol{u}_{\alpha}^{\dagger}\boldsymbol{e}(s)\right|^2} \tag{17}$$

peaks precisely, when $s \in \{s_1, ..., s_p\}$.

IV. Cramer Rao Bound

The CRB [14] appear on main diagonal of the inverse of the Fisher information matrix (FIM) J. The $(i,j)^{th}$ element of FIM is given by

$$J_{i,j} = -\mathbb{E}\{[\frac{\partial lnp(\mathbf{x} \mid \boldsymbol{\theta})}{\partial \theta_i}]^* [\frac{\partial lnp(\mathbf{x} \mid \boldsymbol{\theta})}{\partial \theta_j}]\} \tag{18}$$

where, $p(\mathbf{x} \mid \boldsymbol{\theta})$ denotes the conditional probability density function, $\mathbf{x}$ is the data vector and $\boldsymbol{\theta}$ is the parameter vector. The Cramer Rao bound gives a lower bound on the variance of estimator assuming zero bias. For an unbiased estimator, the total mean square error (MSE) depends only on the variance. Therefore, the lower bound on the variance is directly related to a lower bound of the total mean square error. But, in the case of bias estimators, MSE is the sum of variance and the squared norm of the bias. If $\hat{\theta}$ is an estimate of θ, then the Cramer Rao inequality for unbiased estimator states that $var(\hat{\theta}) > J^{-1}$. var denotes variance. For biased estimator, $var(\hat{\theta}) = (1 + B'(\theta)J^{-1})$, where

$$B' = \partial B / \partial \theta$$

If $(\hat{\theta}))$ is an estimate of θ, then the mean square error (MSE) for biased estimator [15] states that

$$\mathbb{E}[(\hat{\theta}) - \theta][(\hat{\theta}) - \theta]^T] \geq (I + B'(\theta))J(\theta)^{-1}(I + B'(\theta))^T \tag{19}$$

where,

$$B(\theta) = \mathbb{E}[(\hat{\theta}) - \theta]$$

So, for biased estimators, we have,

$$\begin{aligned} var(\hat{\theta}) &= \mathbb{E}[(\hat{\theta} - \mathbb{E}(\hat{\theta}))(\hat{\theta} - \mathbb{E}(\hat{\theta}))^T] \\ &= \mathbb{E}[(\hat{\theta} - \theta)(\hat{\theta} - \theta)^T] - B(\theta)B(\theta)^T \\ &\geq (I + B'(\theta))J(\theta)^{-1}(I + B'(\theta))^T - B(\theta)B(\theta)^T \end{aligned} \tag{20}$$

V. Simulations and Results

The proposed method has been applied to two different examples. First example consists of two real poles $s_1 = 1$ and $s_2 = -2$. Second example consists of two complex poles. $s_{1c} = 1 + i$ and $s_{2c} = 1 - i$. N=10 time samples has been used for simulation. In measurements, the performance of an unbiased estimator after bias correction is of interest, rather than the performance of the original biased estimator as bias often occur in practice. So, the performance of MUSIC estimator has been compared for unbiased case and unbiased case after bias correction. The CRB has been calculated for first and second example in Appendix I and Appendix II respectively. The SNR for either sinusoid is defined as $10log_{10}\frac{A_1^2}{\sigma_w^2}$. MSE between the estimated frequency and actual frequency has been plotted. For each SNR, the MSE of estimation has been obtained from 30 realizations. In each case, the CRB is shown by red color.

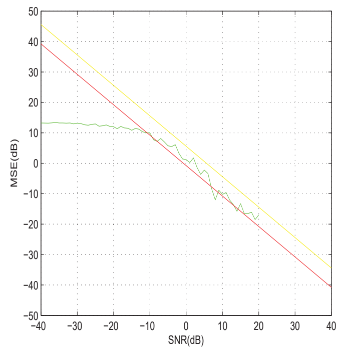

Fig. 1 Comparison of MSE of unbiased estimator and unbiased estimator after bias correction with CRB for s_1. Red color shows CRB. Green and yellow color shows MSE for unbiased and biased estimators

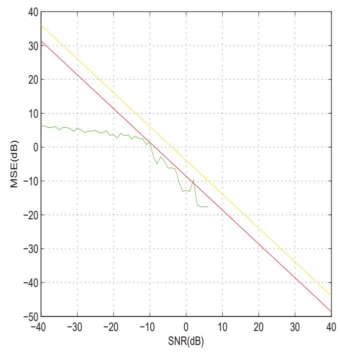

Fig. 2 Comparison of MSE of unbiased estimator and unbiased estimator after bias correction with CRB for s_2. Red color shows CRB. Green and yellow color shows MSE for unbiased and biased estimators

VI. Conclusions

MUSIC estimation is suboptimal compared to Maximum Likelihood (ML) operator, which is optimal. ML method creates highly nonlinear equations, which can be solved using approximation schemes like gradient search method. ML method can not be implemented in the closed form for parameter estimation. The

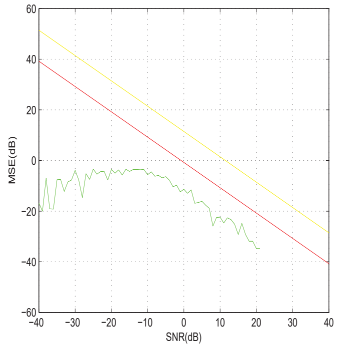

Fig. 3 Comparison of MSE of unbiased estimator and unbiased estimator after bias correction with CRB for s_{1c}. Red color shows CRB. Green and yellow color shows MSE for unbiased and biased estimators

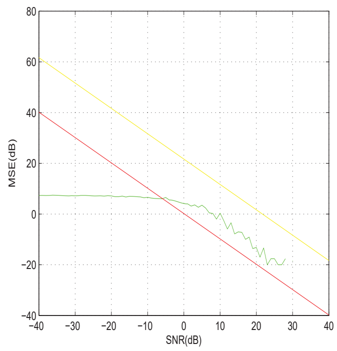

Fig. 4 Comparison of MSE of unbiased estimator and unbiased estimator after bias correction with CRB for s_{2c}. Red color shows CRB. Green and yellow color shows MSE for unbiased and biased estimators

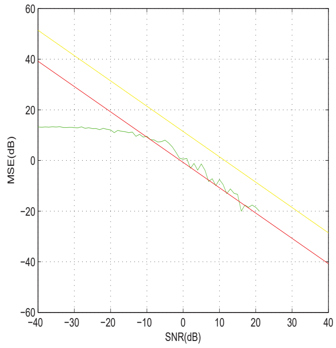

Fig. 5 Comparison of MSE of unbiased estimator and unbiased estimator after bias correction with CRB for s_1. Red color shows CRB. Green and yellow color shows MSE for unbiased and biased estimators

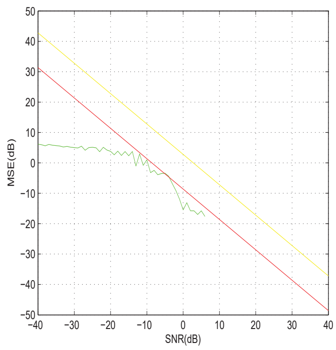

Fig. 6 Comparison of MSE of unbiased estimator and unbiased estimator after bias correction with CRB for s_2. Red color shows CRB. Green and yellow color shows MSE for unbiased and biased estimators

proposed method can be used for multiple complex frequency estimation from real measurements only. Further, the method requires only the output data for estimation. Also the CRB of two real and two complex frequency estimation have been derived from N time samples. Due to importance of the performance of an unbiased estimator, after bias correction in measurements as bias often occur in practice, we compared both cases with CRB.

VII. Appendix A1

The signal consisting of two exponentials in white Gaussian noise with zero mean and σ^2 variance.

$$X(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + w \tag{21}$$

where A_1 and A_2 are signal amplitudes and s_1 and s_2 are poles. Discretization of this equation using $t = n\Delta$ gives,

$$(\mathbf{X}(n\Delta))_{n=0}^{N-1} = \mathbf{X} = A_1\mathbf{e}(s_1) + A_2\mathbf{e}(s_2) + \mathbf{w} \tag{22}$$

where Δ is the step size. w is the complex white Gaussian noise then the probability density function of $\mathbf{x}$ is

$$p(\mathbf{x} \mid s_1, s_2) = \exp(-\frac{1}{2\sigma^2} \mid\mid \mathbf{x} - A_1\mathbf{e}(s_1) - A_2\mathbf{e}(s_2) \mid\mid^2) \tag{23}$$

$$\mathbf{e}(s) = \begin{bmatrix} 1 \\ e^{s\Delta} \\ e^{2s\Delta} \\ . \\ . \\ . \\ e^{(N-1)s\Delta} \end{bmatrix} \tag{24}$$

Or,

$$-logp = \frac{1}{2\sigma^2} \mid\mid \mathbf{x} - A_1\mathbf{e}(s_1) - A_2\mathbf{e}(s_2) \mid\mid^2$$

$$= \frac{1}{\sigma^2}(\mathbf{x} - A_1\mathbf{e}(s_1) - A_2\mathbf{e}(s_2))^*(\mathbf{x} - A_1\mathbf{e}(s_1) - A_2\mathbf{e}(s_2)) \tag{25}$$

so that the Fisher Information matrix is

$$J_1(s_1, s_2) = -\mathbb{E}\begin{pmatrix} (\frac{\partial^2 lnp(\mathbf{X}|s_1,s_2)}{\partial s_1 \partial s_1}) & (\frac{\partial^2 lnp(\mathbf{X}|s_1,s_2)}{\partial s_1 \partial s_2}) \\ (\frac{\partial^2 lnp(\mathbf{X}|s_1,s_2)}{\partial s_2 \partial s_1}) & (\frac{\partial^2 lnp(\mathbf{X}|s_1,s_2)}{\partial s_2 \partial s_2}) \end{pmatrix} \tag{26}$$

where,

$$J_1(1,1) = \frac{1}{\sigma^2}[2A_1^2 \mid \mathbf{e}'(s_1) \mid^2] \tag{27}$$

where $\mathbf{e}'(s_1) = \frac{d\mathbf{e}(s_1)}{dt}$

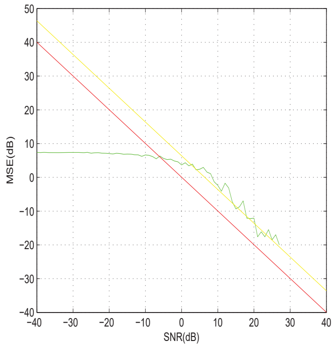

Fig. 7 Comparison of MSE of unbiased estimator and unbiased estimator after bias correction with CRB for s_{1c}. Red color shows CRB. Green and yellow color shows MSE for unbiased and biased estimators

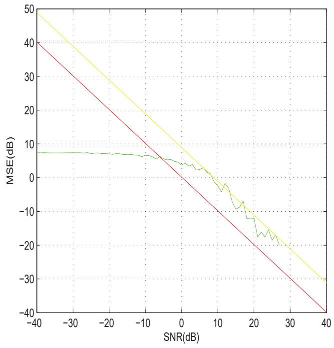

Fig. 8 Comparison of MSE of unbiased estimator and unbiased estimator after bias correction with CRB for s_{2c}. Red color shows CRB. Green and yellow color shows MSE for unbiased and biased estimators

$$J_1(1,2) = \frac{1}{\sigma^2}[A_1 A_2 \mathbf{e}'^T(s_1)\mathbf{e}'(s_2) + A_1 A_2 \mathbf{e}'^T(s_2)\mathbf{e}'(s_1)] \quad (28)$$

where $\mathbf{e}'^T(s_1)$ represents transpose of $\mathbf{e}'(s_1)$

$$J_1(2,1) = \frac{1}{\sigma^2}[A_2 \mathbf{e}'^T(s_1)\mathbf{e}'(s_2) + A_1 A_2 \mathbf{e}'^T(s_2)\mathbf{e}'(s_1)] \quad (29)$$

$$J_1(2,2) = \frac{1}{\sigma^2}[2A_2^2 \mid \mathbf{e}'(s_2) \mid^2] \quad (30)$$

$$\mathbf{e}'(s) = \begin{bmatrix} 0 \\ \Delta e^{s\Delta} \\ 2\Delta e^{2s\Delta} \\ . \\ . \\ . \\ (N-1)\Delta e^{(N-1)s\Delta} \end{bmatrix} \quad (31)$$

VIII. Appendix A2

For complex poles, we used

$$s_i = s_{R_i} + j s_{I_i} i = 1,2 \quad (32)$$

$$\mathbf{X} = \mathbf{E(s)}A + \mathbf{w} \quad (33)$$

where $\mathbf{w}$ is white Gaussian noise. The probability density function of $\mathbf{X}$ is

$$p(\mathbf{X} \mid \theta) = \exp(\frac{1}{2\sigma^2} \parallel \mathbf{X} - \mathbf{E}(s)A \parallel)^2 \quad (34)$$

where $\theta = (s_{R_1}, s_{I_1}, s_{R_2}, s_{I_2})$

$$\ln p(\mathbf{X} \mid \theta) = \frac{1}{2\sigma^2}(\mathbf{X} - \mathbf{E}(s)A)^*(\mathbf{X} - \mathbf{E}(s)A) \quad (35)$$

The Fisher Information Matrix for this is given

$$J_2 = -\mathbb{E}\begin{pmatrix} (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{R_1}\partial s_{R_1}}) & (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{R_1}\partial s_{I_1}}) & (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{R_1}\partial s_{R_2}}) & (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{R_1}\partial s_{I_2}}) \\ (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{I_1}\partial s_{R_1}}) & (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{I_1}\partial s_{I_1}}) & (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{I_1}\partial s_{R_2}}) & (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{I_1}\partial s_{I_2}}) \\ (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{R_2}\partial s_{R_1}}) & (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{R_2}\partial s_{I_1}}) & (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{R_2}\partial s_{R_2}}) & (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{R_2}\partial s_{I_2}}) \\ (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{I_2}\partial s_{R_1}}) & (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{I_2}\partial s_{I_1}}) & (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{I_2}\partial s_{R_2}}) & (\frac{\partial^2 lnp(\mathbf{X}|\theta)}{\partial s_{I_2}\partial s_{I_2}}) \end{pmatrix} \quad (36)$$

where,

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{R_1}\partial s_{R_1}} = \frac{1}{\sigma^2}[4A_1\overline{A_1}\mathbf{e}(s_1)^* D^2\mathbf{e}(s_1) + \overline{A_1}A_2\mathbf{e}(s_1)^* D^2\mathbf{e}(s_2) + A_1\overline{A}_2\mathbf{e}(s_2)^* D^2\mathbf{e}(s_1)$$

$$-2Re\{A_1\overline{A}_2\mathbf{e}(s_2)^* D^2\mathbf{e}(s_1) + A_1\overline{A}_1\mathbf{e}(s_1)^* D^2\mathbf{e}(s_1)\}] \quad (37)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{R_1}\partial s_{I_1}} = 2Im\frac{1}{\sigma^2}[\{A_1\overline{A_2}\mathbf{e}(s_1)^* D^2\mathbf{e}(s_2)\}$$

$$-Re\{jA_1\overline{A_1}\mathbf{e}(s_1)^* D^2\mathbf{e}(s_1) + jA_1\overline{A_2}\mathbf{e}(s_2)^* D^2\mathbf{e}(s_1)\}] \quad (38)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{R_1}\partial s_{R_2}} = \frac{1}{\sigma^2}[\overline{A}_1 A_2\mathbf{e}(s_1)^* D^2\mathbf{e}(s_2) + A_1\overline{A}_2\mathbf{e}(s_1)^* D^2\mathbf{e}(s_2)] \quad (39)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{R_1}\partial s_{I_2}} = -2Im\frac{1}{\sigma^2}[\{\overline{A}_1 A_2\mathbf{e}(s_1)^* D^2\mathbf{e}(s_1)\}] \quad (40)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{I_1}\partial s_{R_1}} = \frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{R_1}\partial s_{I_1}} \quad (41)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{I_1}\partial s_{I_1}} = \frac{1}{\sigma^2}[-\overline{A}_1 A_2\mathbf{e}(s_1)^* D^2\mathbf{e}(s_2) - A_1\overline{A}_2\mathbf{e}(s_2)^* D^2\mathbf{e}(s_1)$$

$$+2Re\{A_1\overline{A}_1\mathbf{e}(s_1)^* D^2\mathbf{e}(s_1) + A_1\overline{A}_2\mathbf{e}(s_2)^* D^2\mathbf{e}(s_1)\}] \quad (42)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{I_1}\partial s_{R_2}} = -2Im\frac{1}{\sigma^2}[\{j\overline{A}_1 A_2\mathbf{e}(s_2)^* D^2 e(s_1)\}] \quad (43)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{I_1}\partial s_{I_2}} = \frac{1}{\sigma^2}[\overline{A}_1 A_2\mathbf{e}(s_1)^* D^2\mathbf{e}(s_2) + A_1\overline{A}_2\mathbf{e}(s_2)^* D^2\mathbf{e}(s_1)$$

$$+2Re\{\overline{A}_1 A_2\mathbf{e}(s_1)^* D^2\mathbf{e}(s_2) + A_2\overline{A}_2\mathbf{e}(s_2)^* D^2\mathbf{e}(s_2)\}] \quad (44)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{R_2}\partial s_{R_1}} = \frac{\partial^2 p(\mathbf{X})}{\partial s_{R_2}\partial s_{R_1}} \quad (45)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{R_2}\partial s_{I_1}} = \frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{I_1}\partial s_{R_2}} \quad (46)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{R_2} \partial s_{R_2}} = \frac{1}{\sigma^2}[\overline{A}_1 A_2 \mathbf{e}(s_1)^* D^2 \mathbf{e}(s_2) + A_1 \overline{A}_2 \mathbf{e}(s_2)^* D^2 \mathbf{e}(s_1)$$
$$+4A_2 \overline{A}_2 \mathbf{e}(s_2)^* D^2 \mathbf{e}(s_2)$$
$$-2Re\{\overline{A}_1 A_2 \mathbf{e}(s_1)^* D^2 \mathbf{e}(s_2) + A_2 \overline{A}_2 \mathbf{e}(s_2)^* D^2 \mathbf{e}(s_2)\}] \quad (47)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{R_2} \partial s_{I_2}} = 2Im \frac{1}{\sigma^2}[\{\overline{A}_1 A_2 \mathbf{e}(s_1)^* D^2 \mathbf{e}(s_2)\}] \quad (48)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{I_2} \partial s_{R_1}} = \frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{R_1} \partial s_{I_2}} \quad (49)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{I_2} \partial s_{I_1}} = \frac{\partial^2 p(\mathbf{X})}{\partial s_{I_1} \partial s_{I_2}} \quad (50)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{I_2} \partial s_{R_2}} = \frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{R_2} \partial s_{I_2}} \quad (51)$$

$$\frac{\partial^2 p(\mathbf{X} \mid \theta)}{\partial s_{I_2} \partial s_{I_2}} = -\frac{1}{\sigma^2}[\overline{A}_1 A_2 \mathbf{e}(s_1)^* D^2 \mathbf{e}(s_2) - A_1 \overline{A}_2 \mathbf{e}(s_2)^* D^2 \mathbf{e}(s_1)$$
$$+2Re\{\overline{A}_1 A_2 \mathbf{e}(s_1)^* D^2 \mathbf{e}(s_2) + A_2 \overline{A}_2 \mathbf{e}(s_2)^* D^2 \mathbf{e}(s_2)\}] \quad (52)$$

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