Generalized Fuzzy Subalgebras and Fuzzy Ideals of BCI-Algebras with Operators

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***Abstract*—The aim of this paper is to introduce the concepts of generalized fuzzy subalgebras, generalized fuzzy ideals and generalized fuzzy quotient algebras of BCI-algebras with operators, and to investigate their basic properties.

***Keywords*—BCI-algebras with operators, generalized fuzzy subalgebras, generalized fuzzy ideals, generalized fuzzy quotient algebras.

I. Introduction

The fuzzy set is a generalization of the classical set. After the introduction of fuzzy sets, there have been a number of generalizations of this fundamental concept, especially, in the branches of mathematics. Imai and Iseki [1], [2] introduced the concept of BCK/BCI-algebras, which are generalizations of BCK-algebras. In 1980, Ming et al. [13] introduced the neighbourhood structure of a fuzzy point.

In 1991, Xi [3] applied the fuzzy sets to BCK-algebras; fuzzy BCK/BCI-algebras have been widely researched. Meng et al. [4] introduced the concept of fuzzy implicative ideals of BCK-algebras in 1997. Liu and Meng [6], [7] introduced the notions of fuzzy positive implicative ideals and fuzzy implicative ideals of BCI-algebras. Zheng [5] defined operators in BCK-algebras and raised the concept of BCI-algebras with operators and gave some isomorphism theorems of it. In 2002, Liu [8] introduced the concept of the fuzzy quotient algebras of BCI-algebras. In 2004, Jun [9] introduced the (α,β)-fuzzy ideals of BCK/BCI-algebras and established the characterizations of $(\in,\in\vee q)$-fuzzy ideals. In 2006, Liao et al. [11] introduced the $(\in,\in\vee q_{(\lambda,\mu)})$-fuzzy normal subgroup. In 2009, Jun et al. [12] introduced the concept of $(\in,\in\vee q)$-ideals of BCI-algebras. In 2011, Liu and Sun [10] introduced the concept of generalized fuzzy ideals of BCI-algebra and investigate some basic properties. In 2017, Hu et al. [14] introduced the fuzzy subalgebras and fuzzy ideals of BCI-algebras with operators.

In this paper, we give the notions of generalized fuzzy subalgebras, generalized fuzzy ideals and generalized fuzzy quotient algebras of BCI-algebras with operators, in particular, discuss the basic properties of generalized fuzzy BCI-algebras with operators and give several results about it.

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II. Preliminaries

We recall some definitions and propositions which may be needed.

An algebra $\langle X;*,0\rangle$ of type (2,0) is called a BCI-algebra, if for all $x,y,z\in X$, it satisfies the following conditions:

1. $((x*y)*(x*z))*(z*y)=0;$
2. $(x*(x*y))*y=0;$
3. $x*x=0;$
4. $x*y=0$ and $y*x=0$ imply $x=y$.

We can define $x*y=0$ if and only if $x\le y$, then the above conditions can be written as:

1. $(x*y)*(x*z)\le z*y;$
2. $x*(x*y)\le y;$
3. $x\le x;$
4. $x\le y$ and $y\le x$ imply $x=y$.

If a BCI-algebra satisfies $0*x=0$, then it is called a BCK-algebra.

Definition 1. [5] $\langle X;*,0\rangle$ is a BCI-algebra, M is a non-empty set, if there exists a mapping $(m,x)\to mx$ from $M\times X$ to X which satisfies $m(x*y)=(mx)*(my),\forall x,y\in X,m\in M$. then M is called a left operator of X, X is called a BCI-algebra with left operator M, or $M-$BCI-algebra for short.

Definition 2. [13] $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy subset A of X of the form

$$A(y)=\begin{cases}t(\ne 0), y=x,\\ 0, y\ne x,\end{cases}$$

is said to be a fuzzy point with support x and value t, and is denoted by x_t.

Proposition 1. [10] Let $\langle X;*,0\rangle$ be a BCI-algebra, if A is a fuzzy generalized ideal of it, and $x*y\le z$, then

$$A(x)\vee\lambda\ge A(y)\wedge A(z)\wedge\mu, x,y,z\in X.$$

Definition 3. [5] Let $\langle X;*,0\rangle$ and $\langle \bar{X};*,0\rangle$ be two $M-$BCI-algebras, if f is a homomorphism from $\langle X;*,0\rangle$ to $\langle \bar{X};*,0\rangle$, and $f(mx)=mf(x)$ for all $x\in X$, $m\in M$, then f is called a homomorphism with operators.

Definition 4. If $\langle X;*,0\rangle$ is a BCI-algebra, A is a non-empty subset of X, and $mx \in A$ for all $x \in A, m \in M$, then $\langle A;*,0\rangle$ is called an $M-$subalgebra of $\langle X;*,0\rangle$.

In the following parts, X always means a $M-$BCI-algebra unless otherwise specified.

III. Generalized Fuzzy Subalgebras of BCI-Algebras with Operators

Definition 5. $\langle X;*,0\rangle$ is a BCI-algebra, let A be a fuzzy subset of X, $t,\lambda,\mu \in [0,1]$ and $\lambda < \mu$. if $A(x) \geq t$, we denoted $x_t \in A$; if $t > \lambda$ and $A(x)+t > 2\mu$, we denoted $x_t q_{(\lambda,\mu)} A$; if $x_t \in A$ or $x_t q_{(\lambda,\mu)} A$, we denoted $x_t \in \vee q_{(\lambda,\mu)} A$.

Definition 6. $\langle X;*,0\rangle$ is an $M-$BCI-algebra, let A be a fuzzy subset of X, if it satisfies:

1. $x_t \in A$ and $y_r \in A$ implies $(x*y)_{t\wedge r} \in \vee q_{(\lambda,\mu)} A$, $\forall x,y \in X$, $t,r \in [0,1]$;
2. $x_t \in A$ implies $(mx)_t \in \vee q_{(\lambda,\mu)} A, \forall x \in X, t \in [0,1]$.

Then A is called an $M-(\in, \in \vee q_{(\lambda,\mu)})-$fuzzy subalgebra or a generalized $M-$fuzzy subalgebra for short.

Proposition 2. A fuzzy subset A of X is a generalized $M-$fuzzy subalgebra of X if and only if it satisfies:

1. $A(x*y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu, \forall x,y \in X$;
2. $A(mx) \vee \lambda \geq A(x) \wedge \mu, \forall x \in X$.

Proof. Suppose that A is a generalized $M-$fuzzy subalgebra of X. We first verify that

$$A(x*y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu, \forall x,y \in X.$$

Suppose there exists $x_0, y_0 \in X$ such that $A(x_0 * y_0) \vee \lambda < A(x_0) \wedge A(y_0) \wedge \mu$, choose t such that $A(x_0 * y_0) \vee \lambda < t < A(x_0) \wedge A(y_0) \wedge \mu$, then $A(x_0 * y_0) < t$, $\lambda < t < \mu$, $A(x_0) > t$ and $A(y_0) > t$, therefore $(x_0)_t \in A, (y_0)_t \in A$. Based on Definition 6, $(x_0 * y_0)_t \in \vee q_{(\lambda,\mu)} A$, but we have $A(x_0 * y_0) < t$, therefore $A(x_0 * y_0) + t \leq t + t < 2\mu$, this is a contradiction, therefore we have $A(x*y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu, \forall x,y \in X$. We shall now show that $A(mx) \vee \lambda \geq A(x) \wedge \mu, \forall x \in X$.

Suppose there exists $x_0 \in X$ such that $A(mx_0) \vee \lambda < A(x_0) \wedge \mu$, choose t such that $A(mx_0) \vee \lambda < t < A(x_0) \wedge \mu$, then $A(x_0) > t$, therefore $(x_0)_t \in A$. Based on Definition 6, $(mx_0)_t \in \vee q_{(\lambda,\mu)} A$, but we have $A(mx_0) < t$, therefore $A(mx_0) + t \leq t + t < 2\mu$, this is a contradiction, therefore we have $A(mx) \vee \lambda \geq A(x) \wedge \mu, \forall x \in X$.

Conversely, assume that A satisfies condition 1, 2.

1). If $(x)_{t_1} \in A, (y)_{t_2} \in A, \forall x,y \in X, t_1, t_2 \in [0,1]$, then $A(x) \geq t_1, A(y) \geq t_2$, choose $T = t_1 \wedge t_2$, since A is a generalized $M-$fuzzy subalgebra of X, we have

$$A(x*y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu > t_1 \wedge t_2 \wedge \mu,$$

if $T \leq \mu$, then $A(x*y) \geq T$, so we have $(x*y)_T \in A$, if $T > \mu$, then $A(x*y) \geq \mu$, thus $A(x*y) + T \geq \mu + T > 2\mu$, then $(x*y)_T \, q_{(\lambda,\mu)} A$, therefore we have $(x*y)_T \in \vee q_{(\lambda,\mu)} A$.

2). If $x_t \in A, \forall x \in X, t \in [0,1]$, then $A(x) \geq t$, since A is a generalized $M-$fuzzy subalgebra of X, then $A(mx) \vee \lambda \geq A(x) \wedge \mu$, if $t \leq \mu$, then $A(mx) \vee \lambda \geq t$, since $\lambda < t$, so we have $A(mx) \geq t$, hence $(mx)_t \in A$, if $t > \mu$, then $A(mx) \vee \lambda \geq \mu$, since $\lambda < \mu$, so we have $A(mx) \geq \mu$, hence $A(mx) + t \geq \mu + t > 2\mu$, thus $(mx)_t \, q_{(\lambda,\mu)} A$, therefore we have $(mx)_t \in \vee q_{(\lambda,\mu)} A$. So A is a generalized $M-$fuzzy subalgebra of X.

Example 1. If A is a generalized $M-$fuzzy subalgebra of X, then X_A is a generalized $M-$fuzzy subalgebra of X, define X_A by

$$X_A : X \to [0,1], X_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A. \end{cases}$$

Proof. (1) For all $x, y \in X$, if $x, y \in A$, then $x*y \in A$, thus

$$X_A(x*y) \vee \lambda = 1 \geq X_A(x) \wedge X_A(y) \wedge \mu,$$

if there exists at least one which does not belong to A between x and y, for example $x \notin A$, thus

$$X_A(x*y) \vee \lambda \geq 0 = X_A(x) \wedge X_A(y) \wedge \mu.$$

(2) For all $x \in X$, $m \in M$, if $x \in A$, then $mx \in A$, therefore

$$X_A(mx) \vee \lambda = 1 \geq X_A(x) \wedge \mu,$$

if $x \notin A$, then $X_A(mx) \vee \lambda \geq 0 = X_A(x) \wedge \mu$, therefore X_A is a generalized $M-$fuzzy subalgebra of X.

Proposition 3. A is a generalized $M-$fuzzy subalgebra of X if only if A_t is a $M-$subalgebra of X, where A_t is a non-empty set, define X_A by $A_t = \{x \mid x \in X, A(x) \geq t\}, \forall t \in (\lambda, \mu]$.

Proof. Suppose A is a generalized $M-$fuzzy subalgebra of X, A_t is a non-empty set, $t \in (\lambda,\mu]$, then we have $A(x*y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$. If $x \in A_t, y \in A_t$, then $A(x) \geq t, A(y) \geq t$, thus $A(x*y) \lambda \geq A(x) \wedge A(y) \wedge \mu \geq t$, thus we have $x*y \in A_t$.

For all $x \in X, m \in M$, if A is a generalized $M-$fuzzy subalgebra of X, hence $A(mx) \vee \lambda \geq A(x) \wedge \mu \geq t$, thus $mx \in A_t$, therefore A_t is an $M-$subalgebra of X. Conversely, suppose A_t is an $M-$subalgebra of X, then we have $x*y \in A_t$. Let

$A(x)=t,$ then $A(x*y)\vee\lambda \geq t = A(x) \geq A(x)\wedge A(y)\wedge\mu.$ For all $x\in X$, $m\in M$, if A_t is an $M-$subalgebra of X, then we have $A(mx)\vee\lambda\geq t = A(x)\geq A(x)\wedge\mu,$ therefore A is a generalized $M-$fuzzy subalgebra of X.

Proposition 4. Suppose X, Y are $M-$BCI-algebras, f is a mapping from X to Y, if A is a generalized $M-$fuzzy subalgebra of the Y, then $f^{-1}(A)$ is a generalized $M-$fuzzy subalgebra of X.

Proof. Let $y\in Y$, suppose f is a epimorphism, then there exists x in X, we have $y=f(x)$. If A is a generalized $M-$fuzzy subalgebra of Y, then we have

$$A(x*y)\vee\lambda\geq A(x)\wedge A(y)\wedge\mu; A(mx)\vee\lambda\geq A(x)\wedge\mu.$$

For all $x,y\in X, m\in M$, we have

$$(1) f^{-1}(A)(x*y)\vee\lambda = A(f(x)*f(y))\vee\lambda$$
$$\geq A(f(x))\wedge A(f(y))\wedge\mu = f^{-1}(A)(x)\wedge f^{-1}(A)(y)\wedge\mu;$$
$$(2) f^{-1}(A)(mx)\vee\lambda = A(f(mx))\vee\lambda = A(mf(x))\vee\lambda$$
$$\geq A(f(x))\wedge\mu = f^{-1}(A)(x)\wedge\mu.$$

Therefore $f^{-1}(A)$ is a generalized $M-$fuzzy subalgebra of X.

IV. Generalized Fuzzy Ideals of BCI-Algebras with Operators

Definition 7. $\langle X;*,0\rangle$ is an $M-$BCI-algebra, let A be a fuzzy subset of X, if it satisfies:

1. $x_t\in A$ implies $0_t\in\vee q_{(\lambda,\mu)}A, \forall x\in X, t\in[0,1]$;
2. $(x*y)_t\in A$ and $y_r\in A$ implies $x_{t\wedge r}\in\vee q_{(\lambda,\mu)}A, \forall x,y\in X$, $t,r\in[0,1]$;
3. $x_t\in A$ implies $(mx)_t\in\vee q_{(\lambda,\mu)}A, \forall x\in X, t,\in[0,1]$.

Then A is called a $M-\left(\in,\in\vee q_{(\lambda,\mu)}\right)-$fuzzy subalgebra or a generalized $M-$fuzzy subalgebra for short.

Proposition 5. A fuzzy subset A of X is a generalized $M-$fuzzy ideal of X if and only if it satisfies:

1. $A(0)\vee\lambda\geq A(x)\wedge\mu, \forall x\in X;$
2. $A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu, \forall x,y\in X;$
3. $A(mx)\vee\lambda\geq A(x)\wedge\mu, \forall x\in X.$

Proof. Suppose that A is a generalized $M-$fuzzy ideal of X. We first verify that $A(0)\vee\lambda\geq A(x)\wedge\mu,\forall x\in X.$ Suppose there exists $x_0\in X$ such that $A(0)\vee\lambda < A(x_0)\wedge\mu,$ choose t such that $A(0)\vee\lambda<t<A(x_0)\wedge\mu,$ then $A(x_0)>t$ and $\lambda<t<\mu,$ therefore $(x_0)_t\in A.$ Based on Definition 7, $0_t\in\vee q_{(\lambda,\mu)}A,$ but we have $A(0)<t\leq\mu,$ therefore $A(0)+t\leq t+t\leq 2\mu,$ this is a contradiction, therefore we have $A(0)\vee\lambda\geq A(x)\wedge\mu,\forall x\in X.$ We shall now show that

$$A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu, \forall x,y\in X.$$

Suppose there exists $x_0, y_0\in X$ such that $A(x_0)\vee\lambda<A(x_0*y_0)\wedge A(y_0)\wedge\mu,$ choose t such that $A(x_0)\vee\lambda<t<A(x_0*y_0)\wedge A(y_0)\wedge\mu,$ then $A(x_0)<t,$ $\lambda<t<\mu,$ $A(x_0*y_0)>t$ and $A(y_0)>t,$ therefore $(x_0*y_0)_t\in A, (y_0)_t\in A.$ Based on Definition 7, $(x_0)_t\in\vee q_{(\lambda,\mu)}A,$ but we have $A(x_0)<t,$ therefore $A(x_0)+t\leq t+t\leq 2\mu,$ this is a contradiction, therefore we have $A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu,\forall x,y\in X.$

Next, we shall show that $A(mx)\vee\lambda\geq A(x)\wedge\mu,\forall x\in X.$ Suppose there exists $x_0\in X$ such that $A(mx_0)\vee\lambda<A(x_0)\wedge\mu,$ choose t such that $A(mx_0)\vee\lambda<t<A(x_0)\wedge\mu,$ then $A(x_0)>t,$ therefore $(x_0)_t\in A.$ Based on Definition 7, $(mx_0)_t\in\vee q_{(\lambda,\mu)}A,$ but we have $A(mx_0)<t,$ therefore $A(mx_0)+t\leq t+t<2\mu,$ this is a contradiction, therefore we have $A(mx)\vee\lambda\geq A(x)\wedge\mu,\forall x\in X.$

Conversely, assume that A satisfies condition 1, 2, 3.

1). If $x_t\in A,\forall x\in X, t\in(0,1],$ then $A(x)\geq t,$ since A is a generalized $M-$fuzzy ideal of X, we have $A(0)\vee\lambda\geq A(x)\wedge\mu\geq t\wedge\mu,$ if $t\leq\mu,$ then $A(0)\geq t,$ so we have $0_t\in A,$ if $t>\mu,$ then $A(0)\geq\mu,$ thus $A(0)+t\geq t+\mu>2\mu,$ then $0_t q_{(\lambda,\mu)}A,$ therefore we have $0_t\in\vee q_{(\lambda,\mu)}A.$

2). If $(x*y)_{t_1}\in A, y_{t_2}\in A,\forall x,y\in X, t_1,t_2\in(\lambda,1],$ then $A(x*y)\geq t_1, A(y)\geq t_2,$ choose $T=t_1\wedge t_2,$ since A is a generalized $M-$fuzzy ideal of X. We have $A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu>t_1\wedge t_2\wedge\mu,$ if $T\leq\mu,$ then $A(x)\geq T,$ so we have $x_T\in A,$ if $T>\mu,$ then $A(x)\geq\mu,$ thus $A(x)+T\geq\mu+T>2\mu,$ then $x_T q_{(\lambda,\mu)}A,$ therefore we have $x_T\in\vee q_{(\lambda,\mu)}A.$

3). If $x_t\in A,\forall x\in X, t\in(\lambda,1],$ then $A(x)\geq t,$ since A is a generalized $M-$fuzzy ideal of X. We have $A(mx)\vee\lambda\geq A(x)\wedge\mu,$ if $t\leq\mu,$ then $A(mx)\vee\lambda\geq t,$ since $\lambda<t,$ so we have $A(mx)\geq t,$ hence $(mx)_t\in A,$ if $t>\mu,$ then $A(mx)\vee\lambda\geq\mu,$ since $\lambda<\mu,$ so we have $A(mx)\geq\mu,$ hence $A(x)+t\geq\mu+t>2\mu,$ thus $(mx)_t q_{(\lambda,\mu)}A,$ therefore we have $(mx)_t\in\vee q_{(\lambda,\mu)}A.$ So, A is a generalized $M-$fuzzy ideal of X.

Example 2. If A is a generalized $M-$fuzzy ideal of X, then X_A is a generalized $M-$fuzzy ideal of X, define X_A by

$$X_A: X\rightarrow[0,1], X_A(x)=\begin{cases}1, x\in A\\ 0, x\notin A.\end{cases}$$

Proof. (1) For all $x,y\in X$, if $x,y\in A,$ then $x*y\in A,$ thus

$$X_A(0)\vee\lambda=1\geq X_A(x)\wedge\mu,$$
$$X_A(x)\vee\lambda=1\geq X_A(x*y)\wedge X_A(y)\wedge\mu,$$

if there exists at least one which does not belong to A between x and y, for example $x\notin A$, thus

$$X_A(0)\vee\lambda=1\geq X_A(x)\wedge\mu,$$

$$X_A(x)\vee\lambda\geq X_A(x*y)\wedge X_A(y)\wedge\mu=0;$$

(2)For all $x\in X$, $m\in M$, if $x\in A$, then $mx\in A$, thus $X_A(mx)\vee\lambda=1\geq X_A(x)\wedge\mu.$ If $x\notin A$, then $X_A(mx)\vee\lambda\geq 0=X_A(x)\wedge\mu$, therefore X_A is a generalized $M-$fuzzy ideal of X.

Proposition 6. A is a generalized $M-$fuzzy ideal of X if only if A_t is an $M-$ideal of X, where A_t is non-empty set, define A_t by $A_t=\{x\mid x\in X, A(x)\geq t\},\forall t\in(\lambda,\mu]$.

Proof. Suppose A is a generalized $M-$fuzzy ideal of X, A_t is non-empty set, $t\in(\lambda,\mu]$, then we have $A(0)\vee\lambda\geq A(x)\wedge\mu\geq t$, thus $0\in A_t$. If $x*y\in A_t, y\in A_t$, then $A(x*y)\geq t, A(y)\geq t$, thus $A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu\geq t$, thus we have $x\in A_t$. For all $x\in X$, $m\in M$, if A is a generalized $M-$ fuzzy ideal of X, hence $A(mx)\vee\lambda\geq A(x)\wedge\mu\geq t$, thus $mx\in A_t$, therefore A_t is an $M-$ideal of X. Conversely, suppose A_t is an $M-$ideal of X, then we have $0\in A_t, A(0)\geq t$. Let $A(x)=t$, thus $x\in A_t$, we have $A(0)\vee\lambda\geq t=A(x)\wedge\mu$, suppose there is no $A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu$, then there exist $x_0,y_0\in X$, we have $A(x_0)\vee\lambda<A(x_0*y_0)\wedge A(y_0)\wedge\mu$, let $t_0=A(x_0*y_0)\wedge A(y_0)\wedge\mu$, then $A(x_0)\vee\lambda<t_0=A(x_0*y_0)\wedge A(y_0)\wedge\mu$, if $x_0*y_0\in A_{t_0}, y_0\in A_{t_0}$, then we have $x_0\in A_{t_0}$, then $A(x_0)\geq t_0$, which is inconsistent with $A(x_0)\vee\lambda<t_0=A(x_0*y_0)\wedge A(y_0)\wedge\mu$, then we have $A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu.$ For all $x\in X, m\in M$, if A_t is an $M-$ideal of X, then we have $A(mx)\vee\lambda\geq t=t\wedge\mu=A(x)\wedge\mu$, therefore A is a generalized $M-$fuzzy ideal of X.

Proposition 7. Suppose X,Y are $M-$BCI-algebras, f is a mapping from X to Y, A is a generalized $M-$fuzzy ideal of Y, then $f^{-1}(A)$ is a generalized $M-$fuzzy ideal of X.

Proof. Let $y\in Y$, suppose f is an epimorphism, then there exists $x\in X$, we have $y=f(x)$. If A is a generalized $M-$fuzzy ideal of Y, then we have

$$A(0)\vee\lambda\geq A(y)\wedge\mu; A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu;$$
$$A(mx)\vee\lambda\geq A(x)\wedge\mu.$$

For all $x,y\in X, m\in M$, we have

$$(1)\, f^{-1}(A)(0)\vee\lambda=A(f(0))\vee\lambda=A(0)\vee\lambda$$
$$\geq A(f(x))\wedge\mu=f^{-1}(A)(x)\wedge\mu;$$
$$(2)\, f^{-1}(A)(x)\vee\lambda=A(f(x))\vee\lambda\geq A(f(x)*f(y))\wedge A(f(y))\wedge\mu$$
$$=A(f(x*y))\wedge A(f(y))\wedge\mu=f^{-1}(A)(x*y)\wedge f^{-1}(A)(y)\wedge\mu;$$
$$(3)\, f^{-1}(A)(mx)\vee\lambda=A(f(mx))\vee\lambda=A(mf(x))\vee\lambda$$
$$\geq A(f(x))\wedge\mu=f^{-1}(A)(x)\wedge\mu.$$

Therefore $f^{-1}(A)$ is a generalized $M-$fuzzy ideal of X.

V. Generalized Fuzzy Quotient BCI-Algebras with Operators

Definition 8. Let A be an $M-\left(\in,\in\vee q_{(\lambda,\mu)}\right)$-fuzzy ideal of X, for all $a\in X$, fuzzy set A_a on X defined as: $A_a: X\to[0,1]$ $A_a(x)=A(a*x)\wedge A(x*a)\wedge\mu,\forall x\in X.$ Denote $X/A=\{A_a: a\in X\}; A(x)\geq\lambda.$

Proposition 8. Let $A_a,A_b\in X/A$, then $A_a=A_b$ if only if $A(a*b)\wedge A(b*a)\wedge\mu=A(0)\wedge\mu.$

Proof. Let $A_a=A_b$, then we have $A_a(b)=A_b(b)$, thus $A(a*b)\wedge A(b*a)\wedge\mu=A(b*b)\wedge A(b*b)\wedge\mu=A(0)\wedge\mu$, that is $A(a*b)\wedge A(b*a)\wedge\mu=A(0)\wedge\mu.$ Conversely, suppose that $A(a*b)\wedge A(b*a)\wedge\mu=A(0)\wedge\mu.$ For all $x\in X$, since $(a*x)*(b*x)\leq a*b,(x*a)*(x*b)\leq b*a.$ It follows from Proposition 1 that

$$A(a*x)=A(a*x)\vee\lambda\geq A(b*x)\wedge A(a*b)\wedge\mu,$$
$$A(x*a)=A(x*a)\vee\lambda\geq A(x*b)\wedge A(b*a)\wedge\mu.$$

Hence

$$A_a(x)=A(a*x)\wedge A(x*a)\wedge\mu$$
$$\geq A(b*x)\wedge A(x*b)\wedge A(a*b)\wedge A(b*a)\wedge\mu$$
$$=A(b*x)\wedge A(x*b)\wedge A(0)\wedge\mu=A(b*x)\wedge A(x*b)\wedge\mu$$
$$=A_b(x),$$

that is $A_a\geq A_b$. Similarly, for all $x\in X$, since

$$(b*x)*A(a*x)\leq b*a,(x*b)*A(x*a)\leq a*b.$$

It follows from Proposition 1 that

$$A(b*x)=A(b*x)\vee\lambda\geq A(a*x)\wedge A(b*a)\wedge\mu,$$
$$A(x*b)=A(x*b)\vee\lambda\geq A(x*a)\wedge A(a*b)\wedge\mu.$$

Hence

$$A_b(x)=A(b*x)\wedge A(x*b)\wedge\mu$$
$$\geq A(a*x)\wedge A(x*a)\wedge A(b*a)\wedge A(a*b)\wedge\mu$$
$$=A(a*x)\wedge A(x*a)\wedge A(0)\wedge\mu$$
$$=A(a*x)\wedge A(x*a)\wedge\mu$$
$$=A_a(x),$$

that is $A_b \geq A_a$. Therefore, $A_a = A_b$. We complete the proof.

Proposition 9. Let $A_a = A_{a'}, A_b = A_{b'}$, then $A_{a*b} = A_{a'*b'}$.

Proof. Since

$$\begin{aligned}
&((a*b)*(a'*b'))*(a*a') = ((a*b)*(a*a'))*(a'*b') \\
&\leq (a'*b)*(a'*b') \leq b'*b, \\
&((a'*b')*(a*b))*(b*b') = ((a'*b')*(b*b'))*(a*b) \\
&\leq (a'*b)*(a*b) \leq a'*a.
\end{aligned}$$

Hence

$$\begin{aligned}
&A((a*b)*(a'*b')) = A((a*b)*(a'*b')) \vee \lambda \\
&\geq A(a*a') \wedge A(b'*b) \wedge \mu, \\
&A((a'*b')*(a*b)) = A((a'*b')*(a*b)) \vee \lambda \\
&\geq A(b*b') \wedge A(a'*a) \wedge \mu.
\end{aligned}$$

Therefore

$$\begin{aligned}
&A((a*b)*(a'*b')) \wedge A((a'*b')*(a*b)) \wedge \mu \\
&= A(a*a') \wedge A(a'*a) \wedge \mu \wedge A(b*b') \wedge A(b'*b) \wedge \mu \wedge \mu \\
&= A(0) \wedge \mu,
\end{aligned}$$

it follows from Proposition 8 that $A_{a*b} = A_{a'*b'}$, we completed the proof. Let A be a generalized $M-$fuzzy ideal of X, the operation "*" of R/A is defined as follows: $\forall A_a, A_b \in R/A, A_a * A_b = A_{a*b}$. By Proposition 8, the above operation is reasonable.

Proposition 10. Let A be a generalized $M-$fuzzy ideal of X, then $R/A = \{R/A; *, A_0\}$ is an $M-$BCI-algebra.

Proof. For all $A_x, A_y, A_z \in R/A$,

$$((A_x * A_y) * (A_x * A_z)) * (A_z * A_y) = A_{((x*y)*(x*z))*(z*y)} = A_0;$$

$$(A_x * (A_x * A_y)) * A_y = A_{(x*(x*y))*y} = A_0;$$
$$A_x * A_x = A_{x*x} = A_0;$$

if $A_x * A_y = A_0, A_y * A_x = A_0$, then $A_{x*y} = A_0, A_{y*x} = A_0$, it follows from Proposition 8 that $A(x*y) = A(0), A(y*x) = A(0)$, hence $A(x*y) \wedge A(y*x) \wedge \mu = A(0) \wedge \mu$, then $A_x = A_y$. Therefore $R/A = \{R/A; *, A_0\}$ is a BCI-algebra. For all $A_x \in R/A, m \in M$, we define $mA_x = A_{mx}$. Firstly, we verify that $mA_x = A_{mx}$ is reasonable. If $A_x = A_y$, then we verify $mA_x = mA_y$, that is to verify $A_{mx} = A_{my}$. We have

$$A(mx*my) \wedge \mu = A(m(x*y)) \wedge \mu \geq A(x*y) \wedge \mu = A(0) \wedge \mu,$$

$$A(my*mx) \wedge \mu = A(m(y*x)) \wedge \mu \geq A(y*x) \wedge \mu = A(0) \wedge \mu,$$

so we have $A(mx*my) \wedge A(my*mx) \wedge \mu = A(0) \wedge \mu$, that is, $A_{mx} = A_{my}$. In addition, for all $m \in M, A_x, A_y \in R/A$,

$$\begin{aligned}
m(A_x * A_y) &= mA_{x*y} = A_{m(x*y)} = A_{(mx)*(my)} \\
&= A_{mx} * A_{my} = mA_x * mA_y.
\end{aligned}$$

Therefore $R/A = \{R/A; *, A_0\}$ is an $M-$BCI-algebra.

Definition 9. Let μ be a generalized $M-$fuzzy subalgebra of X, and A be a generalized $M-$fuzzy ideal of X, we define a fuzzy set of X/A as follows:

$$\mu/A : X/A \to [0,1],$$
$$\mu/A(A_i) \vee \lambda = \sup_{A_x = A_i} \mu(x) \wedge \mu, \forall A_i \in X/A.$$

Proposition 11. μ/A is a generalized $M-$fuzzy subalgebra of X/A.

Proof. For all $A_x, A_y \in X/A$, we have

$$\begin{aligned}
&\mu/A(A_x * A_y) \vee \lambda = \mu/A(A_{x*y}) \vee \lambda = \sup_{A_z = A_{x*y}} \mu(z) \wedge \mu \\
&\geq \sup_{A_s = A_x, A_t = A_y} \mu(s*t) \wedge \mu \geq \sup_{A_s = A_x, A_t = A_y} \mu(s) \wedge \mu(t) \wedge \mu \\
&= \sup_{A_s = A_x,} \mu(s) \wedge \sup_{A_t = A_y,} \mu(t) \wedge \mu = \mu/A(A_x) \wedge \mu/A(A_y) \wedge \mu.
\end{aligned}$$

For all $m \in M, A_x \in R/A$, we have

$$\begin{aligned}
&\mu/A(A_{mx}) \vee \lambda = \sup_{A_{mz} = A_{mx}} \mu(mz) \wedge \mu \\
&\geq \sup_{A_z = A_x} \mu(z) \wedge \mu = \mu/A(A_x) \wedge \mu.
\end{aligned}$$

Therefore μ/A is a generalized $M-$fuzzy subalgebra of X/A.

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