Abstract—The aim of this paper is to introduce the concepts of generalized fuzzy subalgebras, generalized fuzzy ideals and generalized fuzzy quotient algebras of BCI-algebras with operators, and to investigate their basic properties.

Keywords—BCI-algebras with operators, generalized fuzzy subalgebras, generalized fuzzy ideals, generalized fuzzy quotient algebras.

I. INTRODUCTION

The fuzzy set is a generalization of the classical set. After the introduction of fuzzy sets, there have been a number of generalizations of this fundamental concept, especially, in the branches of mathematics. Imai and Iseki [1], [2] introduced the concept of BCK/BCI-algebras, which are generalizations of BCK-algebras. In 1980, Ming et al. [13] introduced the concept of BCI-algebra. In 1981, Liu and Sun [10] introduced the concept of BCI-algebras with operators and gave several results about it.

Definition 1. [5] Let \( \langle X; *, 0 \rangle \) be a BCI-algebra, \( M \) is a non-empty set, if there exists a mapping \( (m, x) \rightarrow m x \) from \( M \times X \) to \( X \) which satisfies \( m(x * y) = (m x) * (m y) \), \( \forall x, y \in X, m \in M \). Then \( M \) is called a left operator of \( X \), \( X \) is called a BCI-algebra with left operator \( M \), or \( M - BCI \)-algebra for short.

Definition 2. [13] Let \( \langle X; *, 0 \rangle \) be a BCI-algebra, a fuzzy subset \( A \) of \( X \) of the form

\[ A(y) = \{ t(0), y = x, \} \]

is said to be a fuzzy point with support \( x \) and value \( t \), and is denoted by \( x_t \).

Proposition 1. [10] Let \( \langle X; *, 0 \rangle \) be a BCI-algebra, if \( A \) is a fuzzy generalized ideal of it, then \( x * y \leq z \), then

\[ A(x) \land \land A(z) \land \land \mu, x, y, z \in X. \]

Definition 3. [5] Let \( \langle X; *, 0 \rangle \) and \( \langle X; *, 0 \rangle \) be two \( M - BCI \)-algebras, if \( f \) is a homomorphism from \( \langle X; *, 0 \rangle \) to \( \langle X; *, 0 \rangle \), and

\[ f(m x) = m f(x) \]

for all \( x \in X, m \in M \), then \( f \) is called a homomorphism with operators.
Definition 4. If \( \{X; \ast, 0\} \) is a BCI-algebra, \( A \) is a non-empty subset of \( X \), and \( mx \in A \) for all \( x \in A, m \in M \), then \( \{A; \ast, 0\} \) is called an \( M \)-subalgebra of \( \{X; \ast, 0\} \).

In the following parts, \( X \) always means a \( M \)-BCI-algebra unless otherwise specified.

III. GENERALIZED FUZZY SUBALGEBRAS OF BCI-ALGEBRAS WITH OPERATORS

Definition 5. \( \{X; \ast, 0\} \) is a BCI-algebra, let \( A \) be a fuzzy subset of \( X \), \( t, \lambda, \mu \in [0,1] \) and \( \lambda < \mu \). If \( A(x) \geq t \), we denoted \( x_{i} \in A; \) if \( t > \lambda \) and \( A(x) > t + 2\mu \), we denoted \( x_{q}(x, \lambda, \mu) \in A \); if \( x_{j} \in A \) or \( x_{k}(x, \lambda, \mu) \in A \), we denoted \( x_{i} \in q(x, \lambda, \mu) \).

Definition 6. \( \{X; \ast, 0\} \) is an \( M \)-BCI-algebra, let \( A \) be a fuzzy subset of \( X \), if it satisfies:
1. \( x_{i} \in A \) and \( y_{j} \in A \) implies \( (x \ast y)_{i,j} \in q(x, \lambda, \mu), \forall x, y \in X \), \( t, r \in [0,1] \);
2. \( x_{i} \in A \) implies \( (mx)_{i} \in q(x, \lambda, \mu), \forall x \in X, t \in [0,1] \).

Then \( A \) is called an \( M \)-fuzzy subalgebra or a generalized \( M \)-fuzzy subalgebra for short.

Proposition 2. A fuzzy subset \( A \) of \( X \) is a generalized \( M \)-fuzzy subalgebra of \( X \) if and only if it satisfies:
1. \( A(x \ast y) \geq \lambda \geq A(x) \wedge A(y) \wedge \mu, \forall x, y \in X \);
2. \( A(mx) \geq \lambda \geq A(x) \wedge \mu, \forall x \in X \).

Proof. Suppose that \( A \) is a generalized \( M \)-fuzzy subalgebra of \( X \). We first verify that
\[ A(x \ast y) \geq \lambda \geq A(x) \wedge A(y) \wedge \mu, \forall x, y \in X. \]

Suppose there exists \( x_{i}, y_{j} \in X \) such that \( A(x_{i} \ast y_{j}) < \lambda \) \( A(x_{i}) \wedge A(y_{j}) \wedge \mu \), choose \( t \) such that \( A(x_{i} \ast y_{j}) < t < A(x_{i}) \wedge A(y_{j}) \wedge \mu \), then \( A(x_{i} \ast y_{j}) < t \) \( A(x_{i}) \wedge A(y_{j}) \wedge \mu \), therefore \( (x_{i}, y_{j}) \in A \). Based on Definition 6, \( (x_{i} \ast y_{j}) \in q(x, \lambda, \mu) \), but we have \( A(x_{i} \ast y_{j}) < \mu \), therefore \( A(x_{i} \ast y_{j}) < t \) \( 0 < t \leq 2\mu \), this is a contradiction, therefore we have \( A(x \ast y) \geq \lambda \geq A(x) \wedge A(y) \wedge \mu, \forall x, y \in X \). We shall now show that \( A(mx) \geq \lambda \geq A(x) \wedge \mu, \forall x \in X \).

Suppose there exists \( x_{0} \in X \) such that \( A(mx_{0}) < \lambda \) \( A(x_{0}) \wedge \mu \). choose \( t \) such that \( A(mx_{0}) < t < A(x_{0}) \wedge \mu \), then \( A(mx_{0}) < t \) \( A(x_{0}) \wedge \mu \), therefore \( (x_{0}, t) \in A \). Based on Definition 6, \( (mx_{0}) \in q(x, \lambda, \mu) \), but we have \( A(mx_{0}) < t \), therefore \( A(mx_{0}) \geq t \) \( 0 < t \leq 2\mu \), this is a contradiction, therefore we have \( A(mx) \geq \lambda \geq A(x) \wedge \mu, \forall x \in X \).

Conversely, assume that \( A \) satisfies condition 1, 2, 1). If \( (x, y) \in A(x, y) \in A, \forall x, y \in X, t, t, t \in [0,1] \), then \( A(x) \geq t, A(y) \geq t \), choose \( t = t \wedge t \), since \( A \) is a generalized \( M \)-fuzzy subalgebra of \( X \), we have
\[ A(x \ast y) \geq \lambda \geq A(x) \wedge A(y) \wedge \mu, \forall x, y \in X, t, t, t \in [0,1] \].

Example 1. If \( A \) is a generalized \( M \)-fuzzy subalgebra of \( X \), then \( X_{A} \) is a generalized \( M \)-fuzzy subalgebra of \( X \), define \( X_{A} \) by
\[ X_{A} : X \rightarrow [0,1], X_{A}(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases} \]

Proof. (1) For all \( x, y \in X \), if \( x, y \in A \), then \( x \ast y \in A \), thus
\[ X_{A}(x \ast y) = 1 \geq X_{A}(x) \wedge X_{A}(y) \wedge \mu. \]

If there exists at least one which does not belong to \( A \) between \( x \) and \( y \), for example \( x \notin A \), thus
\[ X_{A}(x \ast y) = 0 \geq X_{A}(x) \wedge X_{A}(y) \wedge \mu. \]

(2) For all \( x \in X, m \in M \), if \( x \in A \), then \( mx \in A \), therefore
\[ X_{A}(mx) = 1 \geq X_{A}(x) \wedge \mu. \]

Example 1. If \( A \) is a generalized \( M \)-fuzzy subalgebra of \( X \), where \( A \) is a non-empty set, define \( X_{A} \) by
\[ X_{A}(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases} \]

Proof. Suppose \( A \) is a generalized \( M \)-fuzzy subalgebra of \( X \), where \( A \) is a non-empty set, \( t \in (\lambda, \mu) \), then
\[ A(x \ast y) \geq \lambda \geq A(x) \wedge A(y) \wedge \mu. \]

If \( x \in A \), then \( A(x) \geq \lambda \geq A(x) \wedge A(y) \wedge \mu \), thus \( A(x \ast y) \geq \lambda \geq A(x) \wedge A(y) \wedge \mu \). Therefore, \( x \ast x \in A \).

For all \( x \in X, m \in M \), if \( A \) is a generalized \( M \)-fuzzy subalgebra of \( X \), we have \( x \ast x \in A \). Let
Proof. Let \( y \in Y \), suppose \( f \) is an epimorphism, then there exists \( x \in X \), we have \( y = f(x) \). If \( A \) is a generalized \( M \)-fuzzy subalgebra of \( Y \), then we have
\[
A(x)\wedge \lambda \geq A\left( x \wedge A(y) \wedge \mu; A(m) \wedge \lambda \geq A(x) \wedge \mu.
\]

Therefore \( f^{-1}(A) \) is a generalized \( M \)-fuzzy subalgebra of \( X \).

IV. GENERALIZED FUZZY IDEALS OF BCI-ALGEBRAS WITH OPERATORS

Definition 7. \( (X; \ast, 0) \) is an \( M \)-BCI-algebra, let \( A \) be a fuzzy subset of \( X \), if it satisfies:
1. \( x \in A \) implies \( 0, \in \mathcal{V}_{q_{(\lambda,\mu)}}A, \forall x \in X, t \in [0,1] \);
2. \( (x \ast y) \in A \) and \( y \in A \) implies \( x, r, \in \mathcal{V}_{q_{(\lambda,\mu)}}A, \forall x, y \in X, t, r \in [0,1] \);
3. \( x \in A \) implies \( (m) \in \mathcal{V}_{q_{(\lambda,\mu)}}A, \forall x \in X, t, \in [0,1] \).

Then \( A \) is called a \( M \)-\( \{e_i \in \mathcal{V}_{q_{(\lambda,\mu)}}\} \)-fuzzy subalgebra or a generalized \( M \)-fuzzy subalgebra for short.

Proposition 5. A fuzzy subset \( A \) of \( X \) is a generalized \( M \)-fuzzy ideal of \( X \) if and only if it satisfies:
1. \( A(0) \wedge \lambda \geq A(x) \wedge \mu, \forall x \in X \);
2. \( A(x) \wedge \lambda \geq A(x \ast y) \wedge (y) \wedge \mu, \forall x, y \in X \);
3. \( A(m) \wedge \lambda \geq A(x) \wedge \mu, \forall x \in X \).

Proof. Suppose that \( A \) is a generalized \( M \)-fuzzy ideal of \( X \). We first verify that \( A(0) \wedge \lambda \geq A(x) \wedge \mu, \forall x \in X \). Suppose there exists \( x_0 \in X \) such that \( A(0) \wedge \lambda < A(x_0) \wedge \mu \), choose \( t \) such that \( A(0) \wedge \lambda < A(x_0) \wedge \mu \) and \( \lambda < t \), therefore \( (x_0) \in A \). Based on Definition 7, \( 0, \in \mathcal{V}_{q_{(\lambda,\mu)}}A, \) but we have \( 0 < t \leq \mu \), therefore \( A(0) + t \leq t + t \leq 2\mu \), this is a contradiction, therefore we have \( A(0) \wedge \lambda \geq A(x) \wedge \mu, \forall x \in X \). We shall now show that
\[
A(x) \wedge \lambda \geq A(x \ast y) \wedge (y) \wedge \mu, \forall x, y \in X.
\]

Suppose there exists \( x_0, y_0 \in X \) such that \( A(x_0) \wedge \lambda < A(x_0 \ast y_0) \wedge (y_0) \wedge \mu \), choose \( t \) such that \( A(x_0) \wedge \lambda < A(x_0 \ast y_0) \wedge (y_0) \wedge \mu \) and \( \lambda < t \), therefore \( (x_0, y_0) \in A \). Based on Definition 7, \( (x_0, y_0) \in \mathcal{V}_{q_{(\lambda,\mu)}}A \), we have \( A(x_0) < t \), therefore \( A(x_0) + t < t + t < 2\mu \), this is a contradiction, therefore we have \( A(x) \wedge \lambda \geq A(x \ast y) \wedge (y) \wedge \mu, \forall x, y \in X \).

Example 2. If \( A \) is a generalized \( M \)-fuzzy ideal of \( X \), then \( X_A \) is a generalized \( M \)-fuzzy ideal of \( X \), define \( X_A \) by
\[
X_A : X \rightarrow [0,1], X_A (x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}
\]

Proof. (1) For all \( x, y \in X \), if \( x, y \in A \), then \( x \ast y \in A \), thus
if there exists at least one which does not belong to $A$ between $x$ and $y$, for example $x \not\in A$, thus

$$X_A(0) \lor \lambda = 1 \geq X_A(x) \land \mu,$$

$$X_A(x) \lor \lambda = 1 \geq X_A(x \lor y) \land X_A(y) \lor \mu,$$

(2)For all $x \in X$, $m \in M$, if $x \in A$, then $mx \in A$, thus $X_A(mx) \lor \lambda = 1 \geq X_A(x) \lor \mu$. If $x \not\in A$, then $X_A(mx) \lor \lambda \geq 0 = X_A(x) \lor \mu$, therefore $X_A$ is a generalized $M$–fuzzy ideal of $X$.

**Proposition 6.** $A$ is a generalized $M$–fuzzy ideal of $X$ if only if $A_1$ is an $M$–ideal of $X$, where $A_1$ is non-empty set, define $A_1$ by $A_1 = \{x \subseteq X | A(x) \geq \lambda, \forall x \in (\lambda, \mu)\}$.

**Proof.** Suppose $A$ is a generalized $M$–fuzzy ideal of $X$, $A_1$ is non-empty set, $e \in (\lambda, \mu)$, then we have $A(0) \lor \lambda \geq A(x) \lor \mu \geq e \lor \lambda$, thus $0 \in A_1$. If $x \lor y \in A, y \in A$, then $A(x \lor y) \geq e \lor \lambda$, thus $A(x) \lor A(y) \geq e \lor \lambda$; hence $A(x) \lor A(y) \geq e \lor \lambda$, thus $mx \in A$, therefore $A_1$ is an $M$–ideal of $X$. Conversely, suppose $A_1$ is an $M$–ideal of $X$, then we have $0 \in A_1, A(0) \geq e$. Let $A(x) = e$, thus $x \in A$, we have $A(0) \lor \lambda \geq A(x) \lor \mu$, suppose there is no $X_A(x) \lor \lambda \geq A(x) \lor \mu$; then there exist $x_0, y_0 \in X$, we have $X_A(x_0) \lor \lambda < A(x_0, y_0) \land X_A(y_0) \lor \mu$, then $A(x_0) \lor A(y_0) \geq e \lor \lambda$, if $x_0 \not\in A$, then we have $x_0 \lor y_0 \in A$, then $A(x_0) \lor A(y_0) \geq e \lor \lambda$, therefore $A$ is a generalized $M$–fuzzy ideal of $X$.

**Proposition 7.** Suppose $X, Y$ are $M$–BCI-algebras, $f$ is a mapping from $X$ to $Y$, $A$ is a generalized $M$–fuzzy ideal of $Y$, then $f^{-1}(A)$ is a generalized $M$–fuzzy ideal of $X$.

**Proof.** Let $y \in Y$, suppose $f$ is an epimorphism, then there exists $x \in X$, we have $y = f(x)$. If $A$ is a generalized $M$–fuzzy ideal of $Y$, then we have

$$A(0) \lor \lambda \geq A(y) \lor \mu;$$

$$A(x) \lor \lambda \geq A(x \lor y) \land A(y) \lor \mu;$$

$$A(mx) \lor \lambda \geq A(x) \lor \mu.$$
that is \( A_i \geq A_i \). Therefore, \( A_i = A_i \). We complete the proof.

**Proposition 9.** Let \( A_i = A_i, A_j = A_j \), then \( A_i \cap A_j = A_i \cup A_j \).

**Proof.** Since
\[
((a \cdot b) \cdot (a' \cdot b')) \cdot (a \cdot a') = ((a \cdot b) \cdot (a \cdot a')) \cdot (a' \cdot b') \\
\leq (a \cdot b) \cdot (a' \cdot b') \leq b' \cdot b, \\
((a' \cdot b') \cdot (a \cdot b')) \cdot (a \cdot b) = ((a' \cdot b') \cdot (b \cdot b')) \cdot (a \cdot b) \\
\leq (a' \cdot b') \cdot (a \cdot b) \leq a' \cdot a.
\]
Hence
\[
A((a \cdot b) \cdot (a' \cdot b')) = A((a \cdot b) \cdot (a' \cdot b')) \cup \lambda \\
\geq A(a \cdot a') \cdot A(b' \cdot b) \cdot \mu, \\
A((a' \cdot b') \cdot (a \cdot b)) = A((a' \cdot b') \cdot (a \cdot b)) \cup \lambda \\
\geq A(b' \cdot b') \cdot A(a' \cdot a) \cdot \mu.
\]
Therefore
\[
A((a \cdot b) \cdot (a' \cdot b')) \cdot A((a' \cdot b') \cdot (a \cdot b)) \cdot \mu = \\
A(a \cdot a') \cdot A(a' \cdot a) \cdot \mu \cdot A(b' \cdot b') \cdot b \cdot A(b' \cdot b) \cdot b \cdot \mu \\
= A(0) \cdot \mu.
\]
it follows from Proposition 8 that \( A_i \cap A_j = A_i \cap A_j \), we completed the proof. Let \( A \) be a generalized \( M \)-fuzzy ideal of \( X \), the operation \( \mu \cdot \) of \( R / A \) is defined as follows:
\[
\forall A_i, A_j \in R / A, A_i \cdot A_j = \mu(A_i \cdot A_j).
\]
By Proposition 8, the above operation is reasonable.

**Proposition 10.** Let \( A \) be a generalized \( M \)-fuzzy ideal of \( X \), then \( R / A = \{ R(A) \cdot A \} \) is an \( M \)-BCI-algebra.

**Proof.** For all \( A_i, A_j, A_k \in R / A \),
\[
\left((A_i \cdot A_j) \cdot (A_k \cdot A_j) \cdot (A_k \cdot A_j) \right) = A_i; \\
\left(A_i \cdot (A_i \cdot A_j) \cdot (A_i \cdot A_j) \right) = A_i; \\
A_i \cdot A_j = A_i \cdot A_j.
\]
if \( A_i \cdot A_j = A_i, A_j \cdot A_k = A_k, \) then \( A_i \cdot A_j = A_i, A_j \cdot A_k = A_k, \) it follows from Proposition 8 that \( A(x \cdot y) = A(0), A(y \cdot x) = A(0) \), hence
\[
A(x \cdot y) \cdot A(y \cdot x) \cdot \mu = A(0) \cdot \mu,
\]
then \( A_i = A_i \). Therefore \( R / A = \{ R(A) \cdot A \} \) is a BCI-algebra. For all \( A_i \in R / A, m \in M \), we define \( mA_i = mA_i \). Firstly, we verify that \( mA_i = mA_i \), is reasonable. If \( A_i = A_i \), then we verify \( mA_i = mA_i \), that is to verify \( mA_i = mA_i \). We have
\[
A(mx \cdot my) \cdot \mu = A(m(x \cdot y)) \cdot \mu \geq A(x \cdot y) \cdot \mu = A(0) \cdot \mu, \\
A(my \cdot mx) \cdot \mu = A(m(y \cdot x)) \cdot \mu \geq A(y \cdot x) \cdot \mu = A(0) \cdot \mu,
\]
so we have \( A(mx \cdot my) \cdot A(my \cdot mx) \cdot \mu = A(0) \cdot \mu, \) that is,
\[
A_{mx} = A_{my}. \quad \text{In addition, for all } m \in M, A_i, A_j \in R / A,
\]
\[
m \cdot A_i \cdot A_j = mA_{i \cdot j}, \\
A_{i \cdot j} = A_{i \cdot j} = mA_{i \cdot j}.
\]
Therefore \( R / A = \{ R(A) \cdot A \} \) is an \( M \)-BCI-algebra.

**Definition 9.** Let \( \mu \) be a generalized \( M \)-fuzzy subalgebra of \( X \), and \( A \) be a generalized \( M \)-fuzzy ideal of \( X \), we define a fuzzy set of \( X / A \) as follows:
\[
\mu(A) = [0, 1], \\
\mu / A = \mu / A(x) \cdot \mu = \sup \mu(x) \cdot \mu, \quad \forall A_i \in X / A.
\]

**Proposition 11.** \( \mu / A \) is a generalized \( M \)-fuzzy subalgebra of \( X / A \).

**Proof.** For all \( A_i, A_j \in X / A \), we have
\[
\mu(A_i \cdot A_j) \cdot \mu = \mu(A_i) \cdot \mu = \sup \mu(z) \cdot \mu, \\
\geq \sup \mu(s) \cdot \mu \geq \sup \mu(s) \cdot \mu \cdot \mu = \mu / A(A_i) \cdot \mu / A(A_j) \cdot \mu.
\]
For all \( m \in M, A_i \in R / A \), we have
\[
\mu / A(A_i) \cdot \mu = \sup \mu(mz) \cdot \mu, \\
\geq \sup \mu(z) \cdot \mu = \mu / A(A_i) \cdot \mu.
\]
Therefore \( \mu / A \) is a generalized \( M \)-fuzzy subalgebra of \( X / A \).

**REFERENCES**


