Geometric Properties and Neighborhood for Certain Subclasses of Multivalent Functions
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Abstract—By using the two existing operators, we have defined an operator, which is an extension for them. In this paper, first the operator is introduced. Then, using this operator, the subclasses of multivalent functions are defined. These subclasses of multivalent functions are utilized in order to obtain coefficient inequalities, extreme points, and integral means inequalities for functions belonging to these classes.

Keywords—Coefficient inequalities, extreme points, integral means, multivalent functions, Al-Oboudi operator, and Sălăgean operator.

I. INTRODUCTION
WE have defined the operator $D_p(\lambda, q, n)$, which is an extension of operators of Al-Oboudi Operator [1] and Sălăgean Operator [9]. Also we define two subclasses $N_p^\lambda(m, n, \alpha, \beta, b, q)$ and $\tilde{N}_p^\lambda(m, n, \alpha, \beta, b, q)$ using the above mentioned operator. These operator’s subclasses mentioned are extension of $N_p^\lambda(m, n, \alpha, \beta, b)$ and $\tilde{N}_p^\lambda(m, n, \alpha, \beta, b)$, which are introduced by Eker and Seker [5].

Let $\mathcal{A}_p$ be the class of functions $f$ of the form

$$f(z) = z^P + \sum_{j=p+1}^{\infty} a_j z^j, \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}) \tag{1}$$

which are analytic and $p$-valent in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Definition 1. Let $\lambda, q \in \mathbb{R}$, $\lambda \geq 0$, $q \geq 0$, $p, n \in \mathbb{N}$ and

$$f(z) = z^P + \sum_{j=p+1}^{\infty} a_j z^j \quad (z \in \mathcal{U}). \tag{2}$$

Then, we define the operator as: $D_p(\lambda, q, n): \mathcal{A}_p \to \mathcal{A}_p$ by

$$D_p(\lambda, q, n)f(z) = z^P + \sum_{j=p+1}^{\infty} a_j z^j. \quad (x \in \mathcal{U}). \tag{2}$$

Definition 2. The function $f \in \mathcal{A}_p$ is to be in the class $N_p^\lambda(m, n, \alpha, \beta, b, q)$ if

$$R \left\{ 1 + \frac{1}{b} \left[ \frac{D_p(\lambda, q, m)(f)(z)}{D_p(\lambda, q, n)(f)(z)} - 1 \right] \right\} \geq \beta \left[ \frac{D_p(\lambda, q, m)(f)(z)}{D_p(\lambda, q, n)(f)(z)} - 1 \right] + \alpha \tag{3}$$

for some $0 \leq \alpha < 1$, $\beta \geq 0$, $\lambda \geq 0$, $q \geq 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $b \in \mathbb{C} \setminus \{0\}$ and $z \in \mathcal{U}$.

Let $\mathcal{T}_p$ be the subclass of $\mathcal{A}_p$ consisting of

$$f(z) = z^P - \sum_{j=p+1}^{\infty} a_j z^j, \quad (a_j \geq 0, \ p \in \mathbb{N}), \tag{4}$$

which are $p$ - valent in $\mathcal{U}$. Let $\mathcal{U}$ be the class of function $\omega$ analytic in $\mathcal{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$.

For any two functions $f$ and $g$ in $\mathcal{T}_p$, $f$ is said to be subordinate to $g$ denoted $f < g$ if there exists an analytic function $\omega \in \mathcal{U}$ such that $f(z) = g(\omega(z))$ [2]. Further, if $f(z) = z^P - \sum_{j=p+1}^{\infty} a_j z^j$, then we have:

$$D_p(\lambda, q, n)f(z) = z^P - \sum_{j=p+1}^{\infty} a_j z^j, \quad (z \in \mathcal{U}). \tag{5}$$

For any function $f \in \mathcal{T}_p$ and $\delta \geq 0$, the (6) - neighborhood of $f$ is defined s:

$$N_\delta(f) = \{ g(z) = z^P - \sum_{j=p+1}^{\infty} b_j z^j \in \mathcal{T}_p : \Sigma_{j=p+1}^{\infty} |a_j - b_j| \leq \delta \}. \tag{6}$$

In particular, for the function $e(z) = z^P$, we see that:

$$N_\delta(e) = \{ g(z) = z^P - \sum_{j=p+1}^{\infty} b_j z^j \in \mathcal{T}_p : \Sigma_{j=p+1}^{\infty} |b_j| \leq \delta \}. \tag{7}$$

The concept of neighborhoods was first introduced by Goodman [6] and then generalized by Ruscheweyh [8].

Definition 3. A function $f \in \mathcal{T}_p$ is said to be in the class $\mathcal{T}_p(\beta, n, m, p, q, A, B, \lambda)$ if

$$\frac{D_p(\lambda, q, m+n)(f)(z)}{D_p(\lambda, q, n)(f)(z)} < \frac{1+qz(1+B^2)}{1+B^2}, \quad z \in \mathcal{U}, \tag{8}$$

where, $\gamma = (1 - \beta)A + \beta B$, $0 \leq \beta < 1$, $q, n \in \mathbb{N}_0$, $m \in \mathbb{N}$, $\lambda \geq 1$ and $-1 < B < A \leq 1$.

II. COEFFICIENT INEQUALITIES FOR THE CLASS $N_p^\lambda(m, n, \alpha, \beta, b, q)$

Theorem 1. If $f \in \mathcal{A}_p$ satisfies, then

$$\sum_{j=p+1}^{\infty} \psi_p(\lambda, m, n, j, \alpha, \beta, b, q) |a_j| \leq 2 |b|(1 - \alpha) \tag{9}$$

where,

$$\psi_p(\lambda, m, n, j, \alpha, \beta, b, q) = \left[ 1 + \frac{1}{b} \left[ \frac{(\beta^p)^m}{(\beta^{p+1})^m} \right] \right] \left[ 1 + \frac{1}{b} \left[ \frac{(\beta^p)^m}{(\beta^{p+1})^m} \right] \right] + 2 \beta \left[ \frac{1}{b} \left[ \frac{(\beta^p)^m}{(\beta^{p+1})^m} \right] \right] a_j \geq 0. \tag{9}$$

III. RELATION FOR $\tilde{N}_p^\lambda(m, n, \alpha, \beta, b, q)$
By Theorem 1 we introduce the class $\tilde{N}_p^\lambda(m, n, \alpha, \beta, b, q)$

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as the subclass of $N_{p}^{\lambda}(m,n,\alpha,\beta,b,q)$ consisting of $f$ satisfying

$$
\sum_{j=p+1}^{\infty} \psi_p(\lambda, m, n, j, \alpha, \beta, b, q) |a_j| \leq 2|b|(1-\alpha)
$$

(10)

where,

$$
\psi_p(\lambda, m, n, j, \alpha, \beta, b, q) = \frac{2b(2-\alpha) - 1}{1 + \frac{(j-p)a_n}{p+q}}
$$

for some $\alpha (0 \leq \alpha < 1)$, $\beta \geq 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $b \in \mathbb{C} \setminus \{0\}$

**Theorem 2.** If $f \in \mathcal{A}_p$, then $N_{p}^{\lambda}(m, n, \alpha, \beta, b, q) \subseteq \tilde{N}_{p}^{\lambda}(m, n, \alpha, \beta, b, q)$ for some $\beta_1$ and $\beta_2$, such that

$$0 \leq \beta_1 \leq \beta_2.$$

**Proof.** For $0 \leq \beta_1 \leq \beta_2$, we have

$$
\sum_{j=p+1}^{\infty} \psi_p(\lambda, m, n, j, \alpha, \beta_1, b, q) a_j \leq \sum_{j=p+1}^{\infty} \psi_p(\lambda, m, n, j, \alpha, \beta_2, b, q) a_j.
$$

Therefore, if $f \in \tilde{N}_{p}^{\lambda}(m, n, \alpha, \beta, b, q)$, then $f \in N_{p}^{\lambda}(m, n, \alpha, \beta, b, q)$.

**IV. EXTREME POINTS**

The determination of extreme points of a family $\mathcal{F}$ of univalent functions enables us to solve many external problems for $\mathcal{F}$.

**Theorem 3.** Let $f_p(z) = z^p$ and

$$f_j(z) = z^p + \frac{2b(1-\alpha)x_j}{\psi_p(\lambda, m, n, j, \alpha, \beta, b, q)} z^j, \quad (j = p + 1, p + 2, \ldots; |\epsilon| = 1).$$

Then, $f \in \tilde{N}_{p}^{\lambda}(m, n, \alpha, \beta, b, q)$ if and only if it can be expressed in the form

$$f(z) = \lambda_p f_p(z) + \sum_{j=p+1}^{\infty} \lambda_j f_j(z),$$

where, $\lambda_j > 0$ and $\lambda_p = 1 - \sum_{j=p+1}^{\infty} \lambda_j$.

**Proof.** Suppose that

$$f(z) = \lambda_p f_p(z) + \sum_{j=p+1}^{\infty} \lambda_j f_j(z) = z^p + \sum_{j=p+1}^{\infty} \lambda_j \frac{2b(1-\alpha)x_j}{\psi_p(\lambda, m, n, j, \alpha, \beta, b, q)} z^j.$$

Then,

$$\sum_{j=p+1}^{\infty} \psi_p(\lambda, m, n, j, \alpha, \beta, b, q) \frac{2b(1-\alpha)x_j}{\psi_p(\lambda, m, n, j, \alpha, \beta, b, q)} \lambda_j \leq 2|b|(1-\alpha).$$

Thus, $f \in \tilde{N}_{p}^{\lambda}(m, n, \alpha, \beta, b, q)$. Conversely, let $f \in N_{p}^{\lambda}(m, n, \alpha, \beta, b, q)$. Since

$$|a_j| \leq \frac{2|b|(1-\alpha)}{\psi_p(\lambda, m, n, j, \alpha, \beta, b, q)}, \quad (j = p + 1, p + 2, \ldots)$$

we put

$$\lambda_j = \frac{\psi_p(\lambda, m, n, j, \alpha, \beta, b, q)}{2|b|(1-\alpha)} a_j, \quad (|\epsilon| = 1).$$

and

$$\lambda_p = 1 - \sum_{j=p+1}^{\infty} \lambda_j.$$

Then,

$$f(z) = \lambda_p f_p(z) + \sum_{j=p+1}^{\infty} \lambda_j f_j(z).$$

**Corollary 1.** The extreme points of $\tilde{N}_{p}^{\lambda}(m, n, \alpha, \beta, b, q)$ are the functions $f_p(z) = z^p$ and

$$f_j(z) = z^p + \frac{2b(1-\alpha)x_j}{\psi_p(\lambda, m, n, j, \alpha, \beta, b, q)} z^j, \quad (j = p + 1, p + 2, \ldots; |\epsilon| = 1).$$

**V. DISTORTION AND COVERING THEOREMS**

**Theorem 4.** If $f \in T(\beta, n, m, p, q, A, B, \lambda)$, then

$$r^{\gamma - B} \leq |f(z)| ^{\gamma - B} \leq \gamma - B \left(1 + \frac{1}{p+q}\right) \left(1 - B(1 + \frac{1}{p+q})^{-\gamma}(1 - \gamma) \right) r^{\gamma - B} .$$

with equality for

$$f(z) - z^p = \frac{1}{(1 + \frac{1}{p+q}) \left(1 - B(1 + \frac{1}{p+q})^{-\gamma}(1 - \gamma) \right) r^{\gamma - B} .$$

(11)

**Proof.** In Theorem 1, we have

$$-3.5cm \left(1 + \frac{1}{p+q}\right) \left(1 - B(1 + \frac{1}{p+q})^{-\gamma}(1 - \gamma) \right) \sum_{j=p+1}^{\infty} a_j \leq \sum_{j=p+1}^{\infty} \left(1 + \frac{1}{p+q}\right) \left(1 - B(1 + \frac{1}{p+q})^{-\gamma}(1 - \gamma) \right) a_j \leq \gamma - B.$$

Hence
\[ -3.7 \text{cm} \left| f(z) \right| \leq r^p + \sum_{k=1}^{\infty} a_k r^k \leq r^p + r^{1+p} \sum_{k=1}^{\infty} a_k \]

\[ 1 \text{cm} \leq r^p + \frac{\gamma - B}{(1 + \frac{r}{p+q})^\lambda \left( 1 - B \right) (1 + \frac{r}{p+q})^n - \left( 1 - \gamma \right)} r^{1+p}, \]

and

\[ -3.7 \text{cm} \left| f(z) \right| \geq r^p - \sum_{k=1}^{\infty} a_k r^k \geq r^p - r^{1+p} \sum_{k=1}^{\infty} a_k \]

\[ 1 \text{cm} \geq r^p - \frac{\gamma - B}{(1 + \frac{r}{p+q})^\lambda \left( 1 - B \right) (1 + \frac{r}{p+q})^n - \left( 1 - \gamma \right)} r^{1+p}, \]

This completes the proof.

**Theorem 5.** Any function \( f \in \mathcal{T}(\beta, n, m, p, q, A, B, \lambda) \) maps the disk \( |z| < 1 \) onto a domain that contains the disk \( |w| < 1 - \frac{\gamma - B}{(1 + \frac{r}{p+q})^\lambda \left( 1 - B \right) (1 + \frac{r}{p+q})^n - \left( 1 - \gamma \right)} r \).

**Proof.** The proof follows upon letting \( r \to 1 \) in Theorem 4.

**Theorem 6.** If \( f \in \mathcal{T}(\beta, n, m, p, q, A, B, \lambda) \), then

\[ 1 - \frac{\left( \frac{1}{p+q} \right)^\lambda - \left( \frac{1}{p+q} \right)^n - \left( 1 - \gamma \right)}{r^{1+p}} \leq \left| f(z) \right| \leq \frac{1}{r^{1+p}} \sum_{k=1}^{\infty} a_k \left| k a_k \right| \]

with the equality for

\[ r_{\beta}(\beta, n, m, p, q, A, B, \lambda) = \inf_{1} \left[ 1 - \frac{\left( \frac{1}{p+q} \right)^\lambda - \left( \frac{1}{p+q} \right)^n - \left( 1 - \gamma \right)}{r^{1+p}} \right]. \]

**Proof.** We show that \( \left| \frac{f'(z)}{f(z)} - p \right| \leq p - \rho (0 \leq \rho < p) \) for \( |z| < r_{\beta}(\beta, n, m, p, q, A, B, \lambda, \rho) \).

We have

\[ \left| \frac{f'(z)}{f(z)} - p \right| \leq \frac{\sum_{k=1}^{\infty} (k-p) a_k |z|^{k-p}}{1 - \sum_{k=1}^{\infty} a_k |z|^{k-p}} \]

**VI.** **New Subclasses of Radii of Starlikeness and Convexity**

We calculate the radius of starlikeness of order \( \rho \) and the radius of convexity of order \( \rho \) for functions \( \mathcal{T}(\beta, n, m, p, q, A, B, \lambda) \).

**Theorem 7.** Let \( f \in \mathcal{T}(\beta, n, m, p, q, A, B, \lambda) \), then we have \( f \) as starlike of order \( \rho, (0 \leq \rho < p) \) in \( |z| < r_{\beta}(\beta, n, m, p, q, A, B, \lambda, \rho) \) where

\[ r_{\beta}(\beta, n, m, p, q, A, B, \lambda, \rho) = \inf_{1} \left[ \frac{1}{(k-p)(\gamma-B)} - \frac{\frac{1}{p+q}}{\frac{1}{p+q}^\lambda - \frac{1}{p+q}^n - \left( 1 - \gamma \right)} \right]. \]

**Proof.** We show that \( \left| \frac{f'(z)}{f(z)} - p \right| \leq p - \rho (0 \leq \rho < p) \) for \( |z| < r_{\beta}(\beta, n, m, p, q, A, B, \lambda, \rho) \).

We have

\[ \left| \frac{f'(z)}{f(z)} - p \right| \leq \frac{\sum_{k=1}^{\infty} (k-p) a_k |z|^{k-p}}{1 - \sum_{k=1}^{\infty} a_k |z|^{k-p}} \]
Thus
\[
\left| z f'(z) - p \right| \leq p - \rho
\]
if
\[
\sum_{k=1+\rho}^{\infty} (k-\rho)a_k \left| z \right|^{k-\rho} \leq 1.
\] (13)

Then by Theorem 1, will be true if
\[
\frac{(k-\rho)z^{k-\rho}}{(p-\rho)} \leq \frac{1 + (k-p)\lambda}{p+q} \left(1 - B\right) \frac{1 + (k-p)\lambda}{p+q} - (1-\gamma)
\]

Proof. We show that
\[
1 + z f''(z) - p \leq p - \rho(0 \leq \rho < p)
\]
for
\[|z| < r_2(\beta,n,m,p,q,A,B,\lambda,\rho).
\]
We have
\[
1 + z f''(z) - p \leq \sum_{k=1+\rho}^{\infty} \frac{k(k-p)a_k \left| z \right|^{k-p}}{1 - \sum_{k=1+\rho}^{\infty} k a_k \left| z \right|^{k-p}}
\]
Thus
\[
1 + z f''(z) - p \leq p - \rho
\]
if
\[
\sum_{k=1+\rho}^{\infty} k(k-p)a_k \left| z \right|^{k-\rho} \leq 1.
\] (15)

Then by Theorem 1, will be true if
\[
\frac{k(k-\rho)z^{k-\rho}}{(p-\rho)} \leq \frac{1 + (k-p)\lambda}{p+q} \left(1 - B\right) \frac{1 + (k-p)\lambda}{p+q} - (1-\gamma)
\]

or if
\[
1 + z f''(z) - p \leq \frac{1 + (k-p)\lambda}{p+q} \left(1 - B\right) \frac{1 + (k-p)\lambda}{p+q} - (1-\gamma)
\]

\[
|z| \leq \frac{1 + (k-p)\lambda}{p+q} \left(1 - B\right) \frac{1 + (k-p)\lambda}{p+q} - (1-\gamma)
\]

\[
(k \geq 1 + \rho).
\] (14)

Theorem 8. Let \( f \in \mathcal{T}(\beta,n,m,p,q,A,B,\lambda) \), we prove that \( f \) is convex of order \( \rho, (0 \leq \rho < p) \) in
\[|z| < r_2(\beta,n,m,p,q,A,B,\lambda,\rho)\]
where
\[
\gamma = (1-\beta)A + \beta B, 0 \leq \beta < 1, q,n \in \mathbb{N}_0, m \in \mathbb{N}, \lambda \geq 1
\]

and
\[-1 \leq B < A \leq 1.
\]

Proof. Let \( f \in \mathcal{T}(\beta,n,m,p,q,A,B,\lambda) \). Then,
\[
\frac{D_\omega(\lambda,q,m+n+f(z))}{D_\omega(\lambda,q,n) f(z)} < \frac{1 + \gamma z}{1 + Bz}, \quad z \in \mathbb{U}.
\] (17)

Therefore, there exists an analytic function \( \alpha \) such that
\[
\frac{D_\omega(\lambda,q,m+n+f(z))}{BD_\omega(\lambda,q,m+n) f(z)} \leq \frac{1}{BD_\omega(\lambda,q,m+n) f(z)} \alpha(z) + \gamma D_\omega(\lambda,q,n) f(z)
\]
\[
\alpha(z) = \frac{D_\omega(\lambda,q,n) f(z) - D_\omega(\lambda,q,m+n) f(z)}{BD_\omega(\lambda,q,m+n) f(z) - \gamma D_\omega(\lambda,q,n) f(z)}
\] (18)

Hence,
Proof. Taking \( |z| = r \), for sufficiently small \( r \) with \( 0 < r < 1 \), the denominator of (19) is positive and so it is positive for all \( r \) with \( 0 < r < 1 \), since \( \sigma(z) \) is analytic for \( |z| < 1 \). Then, the inequality (19) yields

\[
\sum_{j=p+1}^{\infty} \left[ \left( 1 + \frac{(j-p)p}{p+q} \right)^m \right] \left[ \left( 1 + \frac{(j-p)\lambda}{p+q} \right)^m \right] -1 \right] a_j \leq \sum_{j=p+1}^{\infty} \left[ \left( 1 + \frac{(j-p)p}{p+q} \right)^m \right] \left[ \left( 1 + \frac{(j-p)\lambda}{p+q} \right)^m \right] -1 \right] a_j. 
\]

This proves that

\[
\frac{D_r(\lambda, q, m+n) f(z)}{D_r(\lambda, q, n) f(z)} < 1 + z^2, \quad z \in \mathbb{U}
\]

and hence \( f \in T(\beta, n, m, p, q, A, B, \lambda) \).

\[\text{Theorem 10.} \quad \delta = \frac{(\gamma-B)}{(1+1/p+q)} - (1-\gamma) \sum_{j=1}^{\infty} a_j \leq (\gamma-B) \]

and

\[
\chi(\beta, n, m, p, q, A, B, \lambda) \subset N_\delta(e).
\]

Proof. It follows from (16) that if \( f \in T(\beta, n, m, p, q, A, B, \lambda) \), then

\[
(1+1/p+q) - (1-\gamma) \sum_{j=1}^{\infty} a_j \leq (\gamma-B) \]

which implies,

\[
\chi(\beta, n, m, p, q, A, B, \lambda) \subset N_\delta(e).
\]

Solving (22) we get the result.

\[\text{VIII. CONCLUSION} \]

The following are the special cases of the class \( N_\delta^2(m, n, a, \beta, b, q) \):

1. \( N_\delta^2(m, n, a, \beta, 1,0) \equiv N_\delta^2(m, n, a, \beta) \), the class introduced by Eker and Seker [5].
2. \( N_\delta^2(m, n, a, \beta, 1,0) \equiv N_{mn}(a, \beta) \), the class studied by Eker and Owa [3].
3. \( N_\delta^2(1,0, a, \beta, 1,0) \equiv S_D(a, \beta) \) and \( N_\delta^2(1,0, a, \beta, 1,0) \equiv K_D(a, \beta) \), the classes studied by Shams et al.[10].
4. \( N_\delta^2(1,0, a, 0,1,0) \equiv S^*(a) \) and \( N_\delta^2(1,0, a, 0,1,0) \equiv K(a) \), the classes suggested by Robertson [7].
5. \( N_\delta^2(m, n, a, 0,1,0) \equiv K_{mn}(a) \), the class studied by Eker.
and Owa [4].

REFERENCES