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Geometric Properties and Neighborhood for Certain Subclasses of Multivalent Functions

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Abstract**—**By using the two existing operators, we have defined an operator, which is an extension for them. In this paper, first the operator is introduced. Then, using this operator, the subclasses of multivalent functions are defined. These subclasses of multivalent functions are utilized in order to obtain coefficient inequalities, extreme points, and integral means inequalities for functions belonging to these classes.

Keywords**—**Coefficient inequalities, extreme points, integral means, multivalent functions, Al-Oboudi operator, and Sãlãgean operator.

I. Introduction

We have defined the operator $\boldsymbol{D_p(\lambda,q,n)}$, which is an extension of operators of Al-Oboudi Operator [1] and **Sãlãgean** Operator [9]. Also we define two subclasses $\mathcal{N}_p^{\lambda}(m,n,\alpha,\beta,b,q)$ and $\widetilde{\mathcal{N}}_p^{\lambda}(m,n,\alpha,\beta,b,q)$ using the above mentioned operator. These operator's subclasses mentioned are extension of $\mathcal{N}_p^{\lambda}(m,n,\alpha,\beta)$ and $\widetilde{\mathcal{N}}_p^{\lambda}(m,n,\alpha,\beta)$, which are introduced by Eker and Seker [5].

Let $\mathcal{A}_p$ be the class of functions f of the form

$$f(z)=z^p+\sum_{j=p+1}^{\infty}a_j z^j,\quad (p\in\mathbb{N}=\{1,2,3,\ldots.\}) \quad (1)$$

which are analytic and p -valent in the open unit disc $\mathcal{U}=\{z\in\mathbb{C}: |z|<1\}$.

Definition 1. Let $\lambda,\ q\in\mathbb{R},\ \lambda\geq 0,\ q\geq 0,\ p,\ n\in\mathbb{N}$ and

$$f(z)=z^p+\sum_{j=p+1}^{\infty}a_j z^k.$$

Then, we define the operator as: $D_p(\lambda,q,n):\mathcal{A}_p\longrightarrow\mathcal{A}_p$ by

$$D_p(\lambda,q,n)f(z)=z^p+\sum_{j=p+1}^{\infty}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n a_j z^j,\quad (z\in\mathcal{U}). \quad (2)$$

Definition 2. The function $f\in\mathcal{A}_p$ is to be in the class $\mathcal{N}_p^{\lambda}(m,n,\alpha,\beta,b,q)$ if

$$\Re\left\{1+\frac{1}{b}\left[\frac{D_p(\lambda,q,m)f(z)}{D_p(\lambda,q,n)f(z)}-1\right]\right\}>\beta\left|\frac{1}{b}\left[\frac{D_p(\lambda,q,m)f(z)}{D_p(\lambda,q,n)f(z)}-1\right]\right|+\alpha \quad (3)$$

for some $0\leq\alpha<1,\ \beta\geq 0,\ \lambda\geq 0,\ q\geq 0,\ m\in\mathbb{N},\ n\in\mathbb{N}_0,\ b\in\mathbb{C}\setminus\{0\}$ and $z\in\mathcal{U}$.

Let $\mathcal{T}_p$ be the subclass of $\mathcal{A}_p$ consisting of

$$f(z)=z^p-\sum_{j=p+1}^{\infty}a_j z^j,\quad (a_j\geq 0,\ p\in\mathbb{N}), \quad (4)$$

which are p - valent in $\mathcal{U}$. Let Ω be the class of function ω analytic in $\mathcal{U}$ such that $\omega(0)=0,\ |\omega(z)|<1$.

For any two functions f and g in $\mathcal{T}_p$, f is said to be subordinate to g denoted $f\prec g$ if there exists an analytic function $\omega\in\Omega$ such that $f(z)=g(\omega(z))$ [2]. Further, if $f(z)=z^p-\sum_{j=p+1}^{\infty}a_j z^j$, then we have :

$$D_p(\lambda,q,n)f(z)=z^p-\sum_{j=p+1}^{\infty}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n a_j z^j,\quad (z\in\mathcal{U}). \quad (5)$$

For any function $f\in\mathcal{T}_p$ and $\delta\geq 0$, the (δ) - neighborhood of f is defined s :

$$\mathcal{N}_\delta(f)=\left\{g(z)=z^p-\sum_{j=p+1}^{\infty}b_j z^j\in\mathcal{T}_p:\ \sum_{j=p+1}^{\infty}j\,|a_j-b_j|\leq\delta\right\}. \quad (6)$$

In particular, for the function $e(z)=z^p$, we see that :

$$\mathcal{N}_\delta(e)=\left\{g(z)=z^p-\sum_{j=p+1}^{\infty}b_j z^j\in\mathcal{T}_p:\ \sum_{j=p+1}^{\infty}j\,|b_j|\leq\delta\right\}. \quad (7)$$

The concept of neighborhoods was first introduced by Goodman [6] and then generalized by Ruscheweyh [8].

Definition 3. A function $f\in\mathcal{T}_p$ is said to be in the class $\mathcal{T}(\beta,n,m,p,q,A,B,\lambda)$ if

$$\frac{D_p(\lambda,q,m+n)f(z)}{D_p(\lambda,q,n)f(z)}\prec\frac{1+\gamma z}{1+Bz},\quad z\in\mathcal{U}, \quad (8)$$

where, $\gamma=(1-\beta)A+\beta B,\ 0\leq\beta<1,\ q,n\in\mathbb{N}_0,\ m\in\mathbb{N},\ \lambda\geq 1$ and $-1\leq B<A\leq 1$.

II. Coefficient Inequalities for the Class $\mathcal{N}_p^{\lambda}(m,n,\alpha,\beta,b,q)$

Theorem 1. If $f\in\mathcal{A}_p$ satisfies, then

$$\sum_{j=p+1}^{\infty}\psi_p(\lambda,m,n,j,\alpha,\beta,b,q)\left|a_j\right|\leq 2|b|(1-\alpha) \quad (9)$$

where,

$$\psi_p(\lambda,m,n,j,\alpha,\beta,b,q)=\left|(1+b\alpha)\left[1+\frac{(j-p)\lambda}{p+q}\right]^n-\left[1+\frac{(j-p)\lambda}{p+q}\right]^m\right|+\left([b(2-\alpha)-1]\left[1+\frac{(j-p)\lambda}{p+q}\right]^n+\left[1+\frac{(j-p)\lambda}{p+q}\right]^m\right)+2\beta\left|\left[1+\frac{(j-p)\lambda}{p+q}\right]^m-\left[1+\frac{(j-p)\lambda}{p+q}\right]^n\right|\ \left|a_j\right|\geq 0.$$

III. Relation for $\widetilde{\mathcal{N}}_p^{\lambda}(m,n,\alpha,\beta,b,q)$

By Theorem 1 we introduce the class $\widetilde{\mathcal{N}}_p^{\lambda}(m,n,\alpha,\beta,b,q)$

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as the subclass of $\mathcal{N}_p^{\lambda}(m,n,\alpha,\beta,b,q)$ consisting of f satisfying

$$\sum_{j=p+1}^{\infty} \psi_p\,(\lambda,m,n,j,\alpha,\beta,b,q)\left|a_j\right| \leq 2|b|(1-\alpha) \quad (10)$$

where,

$$\psi_p(\lambda,m,n,j,\alpha,\beta,b,q) = \left|(1+b\alpha)\left[1+\frac{(j-p)\lambda}{p+q}\right]^n - \left[1+\frac{(j-p)\lambda}{p+q}\right]^m\right|$$

$$+\left([b(2-\alpha)-1]\left[1+\frac{(j-p)\lambda}{p+q}\right]^n + \left[1+\frac{(j-p)\lambda}{p+q}\right]^m\right)$$

$$+2\beta\left|\left[1+\frac{(j-p)\lambda}{p+q}\right]^m - \left[1+\frac{(j-p)\lambda}{p+q}\right]^n\right|$$

for some $\alpha\,(0 \leq \alpha < 1)$, $\beta \geq 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $b \in \mathbb{C} \setminus \{0\}$.

Theorem 2. If $f \in \mathcal{A}_p$, then $\widetilde{\mathcal{N}}_p^{\lambda}(m,n,\alpha,\beta_2,b,q) \subset \widetilde{\mathcal{N}}_p^{\lambda}(m,n,\alpha,\beta_1,b,q)$ for some β_1 and β_2, such that

$$0 \leq \beta_1 \leq \beta_2.$$

Proof. For $0 \leq \beta_1 \leq \beta_2$, we have

$$\sum_{j=p+1}^{\infty} \psi_p\,(\lambda,m,n,j,\alpha,\beta_1,b,q)\left|a_j\right| \leq \sum_{j=p+1}^{\infty} \psi_p\,(\lambda,m,n,j,\alpha,\beta_2,b,q)\left|a_j\right|.$$

Therefore, if $f \in \widetilde{\mathcal{N}}_p^{\lambda}(m,n,\alpha,\beta_2,b,q)$, then $f \in \widetilde{\mathcal{N}}_p^{\lambda}(m,n,\alpha,\beta_1,b,q)$.

IV. Extreme Points

The determination of extreme points of a family $\mathcal{F}$ of univalent functions enables us to solve many external problems for $\mathcal{F}$.

Theorem 3. Let $f_p(z) = z^p$ and

$$f_j(z) = z^p + \frac{2|b|(1-\alpha)\varepsilon_j}{\psi_p(\lambda,m,n,j,\alpha,\beta,b,q)}\, z^j, \quad (j = p+1, p+2, \ldots.;\ \left|\varepsilon_j\right| = 1).$$

Then, $f \in \widetilde{\mathcal{N}}_p^{\lambda}(m,n,\alpha,\beta,b,q)$ if and only if it can be expressed in the form

$$f(z) = \lambda_p f_p(z) + \sum_{j=p+1}^{\infty} \lambda_j\, f_j(z),$$

where, $\lambda_j > 0$ and $\lambda_p = 1 - \sum_{j=p+1}^{\infty} \lambda_j$.

Proof. Suppose that

$$f(z) = \lambda_p f_p(z) + \sum_{j=p+1}^{\infty} \lambda_j\, f_j(z) = z^p + \sum_{j=p+1}^{\infty} \lambda_j\, \frac{2|b|(1-\alpha)\varepsilon_j}{\psi_p(\lambda,m,n,j,\alpha,\beta,b,q)}\, z^j.$$

Then,

$$\sum_{j=p+1}^{\infty} \psi_p\,(\lambda,m,n,j,\alpha,\beta,b,q)\left|\frac{2|b|(1-\alpha)\varepsilon_j}{\psi_p(\lambda,m,n,j,\alpha,\beta,b,q)}\,\lambda_j\right|$$

$$= \sum_{j=p+1}^{\infty} 2\,|b|(1-\alpha)\lambda_j$$
$$= 2|b|(1-\alpha)\sum_{j=p+1}^{\infty} \lambda_j$$

$$= 2|b|(1-\alpha)(1-\lambda_p)$$

$$\leq 2|b|(1-\alpha).$$

Thus, $f \in \widetilde{\mathcal{N}}_p(m,n,\alpha,\beta,b)$. Conversely, let $f \in \widetilde{\mathcal{N}}_p^{\lambda}(m,n,\alpha,\beta,b,q)$. Since

$$\left|a_j\right| \leq \frac{2|b|(1-\alpha)}{\psi_p(\lambda,m,n,j,\alpha,\beta,b,q)}, \quad (j = p+1, p+2, \ldots.)$$

we put

$$\lambda_j = \frac{\psi_p(\lambda,m,n,j,\alpha,\beta,b,q)}{2|b|(1-\alpha)\varepsilon_j}\, a_j, \quad \left(\left|\varepsilon_j\right| = 1\right)$$

and

$$\lambda_p = 1 - \sum_{j=p+1}^{\infty} \lambda_j.$$

Then,

$$f(z) = \lambda_p f_p(z) + \sum_{j=p+1}^{\infty} \lambda_j\, f_j(z).$$

Corollary 1. The extreme points of $\widetilde{\mathcal{N}}_p^{\lambda}(m,n,\alpha,\beta,b,q)$ are the functions $f_p(z) = z^p$ and

$$f_j(z) = z^p + \frac{2|b|(1-\alpha)\varepsilon_j}{\psi_p(\lambda,m,n,j,\alpha,\beta,b,q)}\, z^j, \quad (j = p+1, p+2, \ldots.;\ \left|\varepsilon_j\right| = 1).$$

V. Distortion and Covering Theorems

Theorem 4. If $f \in \mathrm{T}(\beta,n,m,p,q,A,B,\lambda)$, then

$$r^p - \frac{\gamma - B}{(1+\frac{1}{p+q}\lambda)^n\left\{(1-B)(1+\frac{1}{p+q}\lambda)^m - (1-\gamma)\right\}}\, r^{1+p} \leq |f(z)| \leq$$

$$r^p + \frac{\gamma - B}{(1+\frac{1}{p+q}\lambda)^n\left\{(1-B)(1+\frac{1}{p+q}\lambda)^m - (1-\gamma)\right\}}\, r^{1+p}.3cm(0<|z|=r<1),$$

with equality for

$$f(z) = z^p - \frac{\gamma - B}{(1+\frac{1}{p+q}\lambda)^n\left\{(1-B)(1+\frac{1}{p+q}\lambda)^m - (1-\gamma)\right\}}\, r^{1+p}.3cm(z=\pm r) \quad (11)$$

Proof. In Theorem 1, we have

$$-3.5cm(1+\frac{1}{p+q}\lambda)^n\left\{(1-B)(1+\frac{1}{p+q}\lambda)^m - (1-\gamma)\right\}\sum_{k=1+p}^{\infty} a_k$$

$$\leq \sum_{k=1+p}^{\infty}\left[1+\frac{(k-p)\lambda}{p+q}\right]^n\left\{(1-B)\left[1+\frac{(k-p)\lambda}{p+q}\right]^m - (1-\gamma)\right\}a_k \leq \gamma - B.$$

Hence

$$-3.7cm\,|f(z)|\le r^p+\sum_{k=1+p}^{\infty}a_k r^k\le r^p+r^{1+p}\sum_{k=1+p}^{\infty}a_k$$

$$1cm\le r^p+\frac{\gamma-B}{(1+\frac{1}{p+q}\lambda)^n\left\{(1-B)(1+\frac{1}{p+q}\lambda)^m-(1-\gamma)\right\}}r^{1+p}$$

and

$$-3.7cm\,|f(z)|\ge r^p-\sum_{k=1+p}^{\infty}a_k r^k\ge r^p-r^{1+p}\sum_{k=1+p}^{\infty}a_k$$

$$1cm\ge r^p-\frac{\gamma-B}{(1+\frac{1}{p+q}\lambda)^n\left\{(1-B)(1+\frac{1}{p+q}\lambda)^m-(1-\gamma)\right\}}r^{1+p}.$$

This completes the proof.

Theorem 5. Any function $f\in \mathrm{T}(\beta,n,m,p,q,A,B,\lambda)$ maps the disk $|z|<1$ on to a domain that contains the disk

$$|w|<1-\frac{\gamma-B}{(1+\frac{j}{p+q}\lambda)^n\left\{(1-B)(1+\frac{j}{p+q}\lambda)^m-(1-\gamma)\right\}}.$$

Proof. The proof follows upon letting $r\to 1$ in Theorem 4.

Theorem 6. If $f\in \mathrm{T}(\beta,n,m,p,q,A,B,\lambda)$, then

$$1\quad-\frac{(\gamma-B)}{(1+\frac{1}{p+q}\lambda)^{n-1}\left\{(1-B)(1+\frac{1}{p+q}\lambda)^m-(1-\gamma)\right\}}r\le|f'(z)|\le$$

$$1\quad+\frac{\gamma-B}{(1+\frac{1}{p+q}\lambda)^{n-1}\left\{(1-B)(1+\frac{1}{p+q}\lambda)^m-(1-\gamma)\right\}}r^1.3cm(0<|z|=r<1),$$

with the equality for

$$f(z)=z^p-\frac{\gamma-B}{(1+\frac{1}{p+q}\lambda)^{n-1}\left\{(1-B)(1+\frac{1}{p+q}\lambda)^m-(1-\gamma)\right\}}z^{1+p}.3cm(z=\pm r) \tag{12}$$

Proof. We have

$$|f'(z)|\le 1+\sum_{k=1+p}^{\infty}ka_k|z|^{k-1}\le 1+r\sum_{k=1+p}^{\infty}ka_k.$$

In Theorem 5, we have

$$\sum_{k=1+p}^{\infty}ka_k\le\frac{\gamma-B}{(1+\frac{1}{p+q}\lambda)^{n-1}\left\{(1-B)(1+\frac{1}{p+q}\lambda)^m-(1-\gamma)\right\}}$$

Thus

$$|f'(z)|\le 1+\frac{\gamma-B}{(1+\frac{1}{p+q}\lambda)^{n-1}\left\{(1-B)(1+\frac{1}{p+q}\lambda)^m-(1-\gamma)\right\}}r.$$

On the other hand,

$$|f'(z)|\quad\ge 1-\sum_{k=1+p}^{\infty}ka_k|z|^{k-p}\ge 1-r\sum_{k=1+p}^{\infty}ka_k$$

$$\ge 1-\frac{\gamma-B}{(1+\frac{1}{p}\lambda)^{n-1}\left\{(1-B)(1+\frac{1}{p+q}\lambda)^m-(1-\gamma)\right\}}r.$$

This completes the proof.

VI. New Subclasses of Radii of Starlikeness and Convexity

We calculate the radius of starlikeness of order ρ and the radius of convexity of order ρ for functions $\mathrm{T}(\beta,n,m,p,q,A,B,\lambda)$.

Theorem 7. Let $f\in \mathrm{T}(\beta,n,m,p,q,A,B,\lambda)$, then we have f as starlike of order $\rho,(0\le\rho<p)$ in $|z|<r_1(\beta,n,m,p,q,A,B,\lambda,\rho)$ where

$$r_1(\beta,n,m,p,q,A,B,\lambda,\rho)=\inf_k\left[\frac{\left[1+\frac{(k-p)\lambda}{p+q}\right]^n-.1cm\left\{(1-B)\left[1+\frac{(k-p)\lambda}{p+q}\right]^m-.18cm--.18cm(1-\gamma)\right\}(p-\rho)}{(k-\rho)(\gamma-B)}-.1cm\right]^{\frac{1}{k-p}}$$

Proof. We show that $\left|z\frac{f'(z)}{f(z)}-p\right|\le p-\rho(0\le\rho<p)$ for $|z|<r_1(\beta,n,m,p,q,A,B,\lambda,\rho)$.

We have

$$\left|z\frac{f'(z)}{f(z)}-p\right|\le\frac{\sum_{k=1+p}^{\infty}(k-p)a_k|z|^{k-p}}{1-\sum_{k=1+p}^{\infty}a_k|z|^{k-p}}$$

Thus

$$\left| z\frac{f'(z)}{f(z)} - p \right| \le p - \rho$$

if

$$\sum_{k=1+p}^{\infty} \frac{(k-\rho)a_k |z|^{k-p}}{(p-\rho)} \le 1. \qquad (13)$$

Then by Theorem 1, will be true if

$$\frac{(k-\rho)|z|^{k-p}}{(p-\rho)} \le \frac{\left[1+\frac{(k-p)\lambda}{p+q}\right]^n \left\{(1-B)\left[1+\frac{(k-p)\lambda}{p+q}\right]^m -(1-\gamma)\right\}}{(\gamma-B)}$$

or if

$$|z| \le \left[\frac{\left[1+\frac{(k-p)\lambda}{p+q}\right]^n -.1cm \left\{(1-B)\left[1+\frac{(k-p)\lambda}{p+q}\right]^m -.18cm--.18cm(1-\gamma)\right\}(p-\rho)}{(k-\rho)(\gamma-B)} -.1cm\right]^{\frac{1}{k-p}}$$

$$(k \ge 1+p). \qquad (14)$$

Theorem 8. Let $f \in T(\beta,n,m,p,q,A,B,\lambda)$, we proof that f is convex of order $\rho,(0 \le \rho < p)$ in $|z| < r_2(\beta,n,m,p,q,A,B,\lambda,\rho)$ where

$$r_2(\beta,n,m,p,q,A,B,\lambda,\rho) = \inf_k \left[\frac{\left[1+\frac{(k-p)\lambda}{p+q}\right]^n -.1cm \left\{(1-B)\left[1+\frac{(k-p)\lambda}{p+q}\right]^m -.18cm--.18cm(1-\gamma)\right\}(p-\rho)}{k(k-\rho)(\gamma-B)} -.1cm\right]^{\frac{1}{k-p}}$$

$$(k \ge 1+p).$$

Proof. We show that $\left|1+z\frac{f''(z)}{f'(z)} - p\right| \le p-\rho (0 \le \rho < p)$ for $|z| < r_2(\beta,n,m,p,q,A,B,\lambda,\rho)$. We have

$$\left|1+z\frac{f''(z)}{f'(z)} - p\right| \le \frac{\sum_{k=1+p}^{\infty} k(k-p)a_k |z|^{k-p}}{1-\sum_{k=1+p}^{\infty} k a_k |z|^{k-p}}$$

Thus

$$\left|1+z\frac{f''(z)}{f'(z)} - p\right| \le p-\rho$$

if

$$\sum_{k=1+p}^{\infty} \frac{k(k-\rho)a_k |z|^{k-p}}{(p-\rho)} \le 1. \qquad (15)$$

Then by Theorem 1, will be true if

$$\frac{k(k-\rho)|z|^{k-p}}{(p-\rho)} \le \frac{\left[1+\frac{(k-p)\lambda}{p+q}\right]^n \left\{(1-B)\left[1+\frac{(k-p)\lambda}{p+q}\right]^m -(1-\gamma)\right\}}{(\gamma-B)}$$

or if

$$|z| \le \left[\frac{\left[1+\frac{(k-p)\lambda}{p+q}\right]^n -.1cm \left\{(1-B)\left[1+\frac{(k-p)\lambda}{p+q}\right]^m -.18cm--.18cm(1-\gamma)\right\}(p-\rho)}{k(k-\rho)(\gamma-B)} -.1cm\right]^{\frac{1}{k-p}}$$

VII. Neighborhoods for the Class $T(\beta,n,m,p,q,A,B,\lambda)$

Theorem 9 A function $f \in T_p$ belongs to the class $T(\beta,n,m,p,q,A,B,\lambda)$ if and only if

$$\sum_{j=p+1}^{\infty} \left[1+\frac{(j-p)\lambda}{p+q}\right]^n \left\{(1-B)\left[1+\frac{(j-p)\lambda}{p+q}\right]^m -(1-\gamma)\right\} a_j \le \gamma - B \qquad (16)$$

where,

$$\gamma = (1-\beta)A + \beta B, 0 \le \beta < 1, q,n \in N_0, m \in N, \lambda \ge 1$$

and

$$-1 \le B < A \le 1.$$

Proof. Let $f \in T(\beta,n,m,p,q,A,B,\lambda)$. Then,

$$\frac{D_p(\lambda,q,m+n)f(z)}{D_p(\lambda,q,n)f(z)} \prec \frac{1+\gamma z}{1+Bz}, \quad z \in U. \qquad (17)$$

Therefore, there exists an analytic function ω such that

$$\omega(z) = \frac{D_p(\lambda,q,n)f(z) - D_p(\lambda,q,m+n)f(z)}{BD_p(\lambda,q,m+n)f(z) - \gamma D_p(\lambda,q,n)f(z)} \qquad (18)$$

Hence,

$$|\omega(z)| = \left|\frac{D_p(\lambda,q,n)f(z)-D_p(\lambda,q,m+n)f(z)}{BD_p(\lambda,q,m+n)f(z)-\gamma D_p(\lambda,q,n)f(z)}\right|$$

$$=\left|\frac{\sum_{j=p+1}^{\infty}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n\left\{\left[1+\frac{(j-p)\lambda}{p+q}\right]^m-1\right\}a_jz^j}{(\gamma-B)z^p+\sum_{j=p+1}^{\infty}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n\left\{B\left[1+\frac{(j-p)\lambda}{p+q}\right]^m-\gamma\right\}a_jz^j}\right|<1.$$

Thus,

$$\Re\left\{\frac{\sum_{j=p+1}^{\infty}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n\left\{\left[1+\frac{(j-p)\lambda}{p+q}\right]^m-1\right\}a_jz^j}{(\gamma-B)z^p+\sum_{j=p+1}^{\infty}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n\left\{B\left[1+\frac{(j-p)\lambda}{p+q}\right]^m-\gamma\right\}a_jz^j}\right\}<1. \tag{19}$$

Taking $|z|=r$, for sufficiently small r with $0<r<1$, the denominator of (19) is positive and so it is positive for all r with $0<r<1$, since $\omega(z)$ is analytic for $|z|<1$. Then, the inequality (19) yields

$$\sum_{j=p+1}^{\infty}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n\left\{\left[1+\frac{(j-p)\lambda}{p+q}\right]^m-1\right\}a_jr^j$$

$$<(\gamma-B)r^p+B\sum_{j=p+1}^{\infty}\left[1+\frac{(j-p)\lambda}{p+q}\right]^{m+n}a_jr^j-\gamma\sum_{j=p+1}^{\infty}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n a_jr^j.$$

Equivalently,

$$\sum_{j=p+1}^{\infty}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n\left\{(1-B)\left[1+\frac{(j-p)\lambda}{p+q}\right]^m-(1-\gamma)\right\}a_jr^j\le(\gamma-B)r^p$$

and (16) follows upon letting $r\to 1$. Conversely, for $|z|=r, 0<r<1$, we have $r^j<r^p$. That is,

$$\sum_{j=p+1}^{\infty}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n\left\{(1-B)\left[1+\frac{(1-p)\lambda}{p+q}\right]^m-(1-\gamma)\right\}a_jr^j$$

$$\le\sum_{j=p+1}^{\infty}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n\left\{(1-B)\left[1+\frac{(j-p)\lambda}{p+q}\right]^m-(1-\gamma)\right\}a_jr^p\le(\gamma-B)r^p.$$

From (16) we have

$$\left|\sum_{j=p+1}^{\infty}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n\left\{\left[1+\frac{(j-p)\lambda}{p+q}\right]^m-1\right\}a_jz^j\right|$$

$$\le\sum_{j=p+1}^{\infty}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n\left\{\left[1+\frac{(j-p)\lambda}{p+q}\right]^m-1\right\}a_jr^j$$

$$<(\gamma-B)r^p+\sum_{j=p+1}^{\infty}\left\{B\left[1+\frac{(j-p)\lambda}{p+q}\right]^m-\gamma\right\}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n a_jr^j$$

$$<\left|(\gamma-B)z^p+\sum_{j=p+1}^{\infty}\left\{B\left[1+\frac{(j-p)\lambda}{p+q}\right]^m-\gamma\right\}\left[1+\frac{(j-p)\lambda}{p+q}\right]^n a_jz^j\right|.$$

This proves that

$$\frac{D_p(\lambda,q,m+n)f(z)}{D_p(\lambda,q,n)f(z)}\prec\frac{1+\gamma z}{1+Bz},\quad z\in\mathrm{U}$$

and hence $f\in\mathrm{T}(\beta,n,m,p,q,A,B,\lambda)$.

Theorem 10. If

$$\delta=\frac{(\gamma-B)}{(1+\frac{1}{p+q}\lambda)^{n-1}\left[(1-B)(1+\frac{1}{p+q}\lambda)^m-(1-\gamma)\right]}, \tag{20}$$

then $\mathrm{T}(\beta,n,m,p,q,A,B,\lambda)\subset N_\delta(e)$.

Proof. It follows from (16) that if $f\in\mathrm{T}(\beta,n,m,p,q,A,B,\lambda)$, then

$$(1+\frac{1}{p+q}\lambda)^{n-1}\left[(1-B)(1+\frac{1}{p+q}\lambda)^m-(1-\gamma)\right]\sum_{j=p+1}^{\infty}ja_j\le(\gamma-B), \tag{21}$$

which implies,

$$\sum_{j=p+1}^{\infty}ja_j\le\frac{(\gamma-B)}{(1+\frac{1}{p+q}\lambda)^{n-1}\left[(1-B)(1+\frac{1}{p+q}\lambda)^m-(1-\gamma)\right]}=\delta. \tag{22}$$

Solving (22) we get the result.

VIII. Conclusion

The following are the special cases of the class $\mathcal{N}_p^\lambda(m,n,\alpha,\beta,b,q)$:

1. $\mathcal{N}_p^1(m,n,\alpha,\beta,1,0)\equiv\mathcal{N}_p(m,n,\alpha,\beta)$, the class introduced by Eker and Seker [5].
2. $\mathcal{N}_1^1(m,n,\alpha,\beta,1,0)\equiv\mathcal{N}_{m,n}(\alpha,\beta)$, the class studied by Eker and Owa [3].
3. $\mathcal{N}_1^1(1,0,\alpha,\beta,1,0)\equiv\mathcal{SD}(\alpha,\beta)$ and $\mathcal{N}_1^1(2,1,\alpha,\beta,1,0)\equiv\mathcal{KD}(\alpha,\beta)$, the classes studied by Shams et al.[10].
4. $\mathcal{N}_1^1(1,0,\alpha,0,1,0)\equiv\mathcal{S}^\star(\alpha)$ and $\mathcal{N}_1^1(2,1,\alpha,0,1,0)\equiv\mathcal{K}(\alpha)$, the classes suggested by Robertson [7].
5. $\mathcal{N}_1^1(m,n,\alpha,0,1,0)\equiv\mathcal{K}_{m,n}(\alpha)$, the class studied by Eker

and Owa [4].

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