

Multisymplectic Geometry and Noether Symmetries for the Field Theories and the Relativistic Mechanics

H. Loumi-Fergane, A. Belaidi

Abstract—The problem of symmetries in field theory has been analyzed using geometric frameworks, such as the multisymplectic models by using in particular the multivector field formalism. In this paper, we expand the vector fields associated to infinitesimal symmetries which give rise to invariant quantities as Noether currents for classical field theories and relativistic mechanics using the multisymplectic geometry where the Poincaré-Cartan form has thus been greatly simplified using the Second Order Partial Differential Equation (SOPDE) for multi-vector fields verifying Euler equations. These symmetries have been classified naturally according to the construction of the fiber bundle used. In this work, unlike other works using the analytical method, our geometric model has allowed us firstly to distinguish the angular moments of the gauge field obtained during different transformations while these moments are gathered in a single expression and are obtained during a rotation in the Minkowsky space. Secondly, no conditions are imposed on the Lagrangian of the mechanics with respect to its dependence in time and in q^i , the currents obtained naturally from the transformations are respectively the energy and the momentum of the system.

Keywords—Field theories, relativistic mechanics, Lagrangian formalism, multisymplectic geometry, symmetries, Noether theorem, conservation laws.

I. INTRODUCTION

THERE are different kinds of geometrical models. We have the so-called k -symplectic formalism which uses the k -symplectic structures introduced by Awane [1], [2] and which replaced the polysymplectic structures used by Günther [3]. In this polysymplectic formalism [4], a geometric Hamiltonian formalism for field theories was given by introduction of a vector-valued generalization of a symplectic form called a polysymplectic form. From this geometrical model, many of the characteristics of the autonomous Hamiltonian systems arise. The k -symplectic formalism is used to give a geometric description to field theories whose Lagrangian does not depend on the base coordinates denoted by (t_1, \dots, t_k) (said the space-time coordinates), which means that the k -symplectic formalism is verified for Lagrangians and Hamiltonians which depend only on fields (i.e. Lagrangians $L(\phi^j, \dots, \phi_\mu^j)$ and Hamiltonians $H(\phi^j, \dots, p_\mu^j)$). A natural extension of this is the k -cosymplectic formalism which is the generalization to field theories of the cosymplectic ($k=1$) description of non-autonomous mechanical systems [5], [6]. This formalism is

devoted to describing field theories involving the coordinates (t_1, \dots, t_k) on the Lagrangian $L(t^\mu, \phi^j, \dots, \phi_\mu^j)$ and on the Hamiltonian $H(t^\mu, \phi^j, \dots, p_\mu^j)$.

Another way to derive the field equations is to use the so-called multisymplectic formalism, developed by Tulczyjew's school in Warsaw [7]-[10], and independently by García and Pérez-Rendón [11], [12] and Goldschmidt and Sternberg [13]. This approach was revised by Martin [14], [15] and Gotay et al. [16]-[19] and more recently by Cantrijn et al. [20], [21]. A natural extension of this geometry was successfully operated to describe the dynamic for non-autonomous relativistic mechanical systems [22].

The study of symmetries and conservation laws of the k -symplectic first-order classical field theories in both Lagrangian and Hamiltonian formalisms was treated in [23], [24]. In these works, they introduced different kinds of symmetries and their relation; they associated to some of them the so-called Cartan symmetries. This problem of symmetries of the theories was extended to k -cosymplectic Hamiltonian system. In particular, those called the almost standard k -cosymplectic Hamiltonian system. To these, the authors associated Noether symmetries [25].

The problem of symmetries in field theory has also been treated using other geometric models such as the multisymplectic one by using in particular the multivector field formalism [26]. In this work, Noether's theorem is proved and generalized in order to include higher-order Cartan-Noether symmetries. Another subject of interest of the study of symmetries is to have different notions of infinitesimal symmetries. The work in [27] is devoted to classifying the different kind of infinitesimal symmetries and to study their relationship with conservation laws in the geometric context of multisymplectic geometry and Ehresmann connections.

In the present paper, we investigate some infinitesimal symmetries on the geometrical model already developed in [22] in order to retrieve Noether currents for classical field and mechanical theories by setting some particular multivector fields.

The paper is structured as follows: In Sections II and III, we review the Lagrangian formalism developed for multisymplectic geometry for hyper regular non-autonomous classical field theories and the relativistic mechanics respectively. Section IV is devoted to retrieve Noether current of the systems via solutions of the equations of motion by using the analytical method. In Sections V and VI, we introduce some particular multivector fields of infinitesimal

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symmetries to retrieve Noether currents for classical field theory and mechanical systems respectively, and finally we close the work with a conclusion.

II. MULTISYMPLECTIC GEOMETRY FOR CLASSICAL FIELD THEORIES

A. Lagrangian Formalism

The field theories are the classical limit of quantum fields' theories. Those are the fields, such as gauge fields of Yang-Mills which interact with matter fields. A geometric description has already been done [28] in building a principal fibre bundle $G \times S^{0,2}$ where $G \equiv$ Lie group associated in this case to the quantum fields of YM. This fibre is above a database the flat space: Minkowski space ($k=4$) which coincides with the form of the Lagrangian of fields that we studied (i.e. Lagrangian which is only explicit on fields, not on the database coordinates $(t^\mu)_{\mu=0,3}$). The classical limit of these Lagrangians corresponds to the study of fields without constraints (this coincides with the abstraction of ghosts which corresponds to the $S^{0,2}$ group). The favourable principal fibre of configuration is $E = G(G \equiv M)$, and the structure in this case is 4-symplectic (i.e. $L_0 \in (T_4^1 M)$).

In this section, we are going to summarize the multisymplectic geometry given for studying the dynamic of field theories [29]-[31]. In particular, we have concentrated on dynamic of most general case of field theories: theories whose Lagrangians are explicit on database coordinates $(t^\mu)_{\mu=0,3} = (t^0 = ct, (t^i)_{i=1,3})$ and which are hyper-regular [22]. So, we have followed the following steps:

Let $\pi : E \rightarrow M$ be a fibre bundle with M the base space which is a flat manifold, i.e. the Minkowski space with global coordinates $\{t^\mu\}$. π is the pull back of a section

$$\begin{aligned} \phi : R^4 &\rightarrow E \\ t^\mu &\rightarrow (t^\mu, y^A = \phi^A(t^\rho)) / \mu = \overline{0,3} \text{ et } A = \overline{1,d} \end{aligned}$$

where $\{\phi^A(t^\rho)\} \equiv$ physical fields. These fields are presented by a fibre above each (t^μ) of the base space R^4 . The set of fibres is denoted by the space M , so the fibre bundle E will be

$$E = R^4 \times M. \quad (1)$$

Let $\pi^1 : J^1\pi \rightarrow E$ be the first-order jet bundle of π . By using (1),

$$J^1\pi = R^4 \times T_4^1 M \quad (2)$$

where $T_4^1 M$ is the Whitney sum of 4-copies of the tangent space TM at the space M with local coordinates (y^A, v_v^A) .

π^1 is the pull back of a section which is a mapping $\psi : E \rightarrow J^1\pi$. If ψ is a global section of π^1 such that $\pi^1 \circ \psi = Id_E$, ψ is called a jet field. In this case, ϕ is an

integral section of ψ and $\psi \circ \phi = j^1\phi$ (where $j^1\phi : M \rightarrow J^1\pi$ denotes the canonical lifting of ϕ) and ψ is the integral jet field

$$\psi : E \rightarrow J^1\pi \left(\pi^1 \circ \psi = Id_E \right) \text{ and } \psi \circ \phi = j^1\phi / j^1\phi : R^4 \rightarrow J^1\pi$$

If (t_ν) is a natural local system on R^4 , (t_ν, y^A, v_v^A) is the induced local coordinates system on $J^1\pi$ where

$$j^1\phi(t_\nu) = (t_\nu, y^A, v_v^A) = (t_\nu, \phi^A(t_\rho), \partial_\nu \phi^A(t_\rho)) \quad (3)$$

with $\partial_\nu \phi^A = \frac{\partial \phi^A}{\partial t^\nu} = v_v^A \equiv$ velocity of field.

Let $\bar{\pi}^1 \equiv \pi \circ \pi^1 : J^1\pi \rightarrow M$, where $\bar{\pi}^1$ is the pull back of the section $j^1\phi$.

A Lagrangian density is usually written as $L = L(\bar{\pi}^{1*} \eta)$ where $L \in C^\infty(J^1\pi)$ is the Lagrangian function and η is the volume form on R^4 ($\eta \in \Omega^4(R^4)$) with

$$\eta = d^k t = dt^0 \wedge dt^1 \wedge dt^2 \wedge dt^3, k = 4$$

By using the natural system of coordinates defined on $J^1\pi$, the expression of the Lagrangian density is:

$$L d^k t = L(t_\nu, y^A, v_v^A) dt^0 \wedge dt^1 \wedge dt^2 \wedge dt^3 \quad (4)$$

The expressions of θ_L and Ω_L , the Poincaré-Cartan 4 and 5 forms, are respectively [23]:

$$\theta_L = \frac{\partial L}{\partial v_v^A} dy^A \wedge d^{k-1} t_\nu - \left(\frac{\partial L}{\partial v_v^A} v_v^A - L \right) d^k t \quad (5a)$$

$$\Omega_L = -d\theta_L \in \Omega^5(J^1\pi) \quad (5b)$$

where $d^{k-1} t_\alpha \equiv i\left(\frac{\partial}{\partial t^\alpha}\right) d^k t$.

Let $\Gamma(M, E)$ be the set of sections $\{\phi\}$ cited above and $(J^1\pi, \Omega_L)$ be the Lagrangian system. The Lagrangian field equations can be derived from a variational principle called the Hamilton principle associated to the Lagrangian formalism which is given by:

$$i(X_L)\Omega_L = 0 \quad (6)$$

where $\{X_L\} \subset \chi_L^4(J^1\pi)$ is a class of holonomic multivector fields associated to $j^1\phi$ (X_L is $\bar{\pi}^1$ -transverse, integrable and SOPDE).

The local expression of X_L is given by:

$$X_L = \wedge_{v=0}^3 \left(\frac{\partial}{\partial t^v} + F_v^A \frac{\partial}{\partial y^A} + G_{vp}^A \frac{\partial}{\partial v_p^A} \right) \quad (7)$$

where $F_v^A = v_v^A$ and $G_{vp}^A = \frac{\partial^2 y^A}{\partial t^v \partial t^p}$.

By substituting (7) and (5) in (6), the Euler-Lagrange equations for the fields satisfy:

$$\left(\frac{\partial L}{\partial y^A} - \frac{\partial}{\partial t^v} \left(\frac{\partial L}{\partial v_v^A} \right) \right) \circ j^1 \phi = 0, \quad \forall A = \overline{1, d} \quad (8)$$

In this case, $\frac{\partial^2 L}{\partial v_\mu^A \partial v_\nu^B} \neq 0 \quad \forall (\bar{y}) \in J^1 \pi$, the Lagrangian is hyper-regular (regular globally).

III. MULTISYMPLECTIC GEOMETRY FOR THE RELATIVISTIC MECHANICS

A. Lagrangian Formalism

By analogy with the work already done for the field theories, we have extended the idea to the relativistic mechanics [22].

Let $\pi: E = R \times M \rightarrow R$, where E is the configuration bundle, R as a base space spanned by (ct) as global coordinate and $M = R^3$ is the fibre above each point of the database ($\dim M = 3$ and $\dim E = 4$).

Let $(q^\mu)_{\mu=0,3} = (q^0 = ct, (q^i)_{i=1,3})$ be a natural coordinate defined in E . If the configuration bundle E can be equipped with a metric $\eta^{\mu\nu} = (1, -1, -1, -1)$ such that $q^\mu = \eta^{\mu\nu} q_\nu$, in this case, E coincides with the Minkowski space.

We note that “ c ” is speed of light, and $(q^i)_{i=1,3}$ are the generalized coordinates.

We note $J^1 \pi$ the first-order jet bundle of π associated to the section $j^1 \phi: R \rightarrow J^1 \pi := R \times TM$.

The natural coordinates defined on $J^1 \pi$ as done in (3) is (q^0, q^i, \dot{q}^i) and the global integral section $j^1 \phi$ such that:

$$\begin{aligned} i^1 \phi(q^0) &= (q^0, \phi^i(q^0)) = \phi^i(t) = q^i(t), \\ \frac{\partial \phi^i}{\partial q^0}(q^0) &= \frac{\partial \phi^i}{\partial ct}(t) = \frac{dq^i}{cdt}(t) = \frac{\dot{q}^i(t)}{c} = \bar{q}^i(t) \end{aligned} \quad (9)$$

We define the Lagrange function $L: R \times TM \rightarrow R$

$$L dt = L(q^0, q^i, \dot{q}^i) dt$$

The Poincaré-Cartan 1-form θ_L and 2-form Ω_L associated at L as in (5) by:

$$\theta_L = \frac{\partial L}{\partial \dot{q}^i} dq^i - \frac{1}{c} \left(\frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L \right) dq^0 \quad (10)$$

$$\Omega_L = -d\theta_L$$

$$\text{We put } \frac{dq^i}{cdt}(t) = \bar{q}^i(t) \text{ and } \frac{d^2 q^i}{c dt^2}(t) = \frac{\ddot{q}^i(t)}{c} = \bar{\ddot{q}}^i(t)$$

where $\dot{q}^i(t)$ and $\ddot{q}^i(t)$ are the velocity and the acceleration of the mechanical system, respectively.

For the relativistic mechanics, at the Hamilton principal (6), we can associate the following holonomic multivector field, (7) becomes:

$$\bar{X}_L = \frac{\partial}{\partial q^0} + \bar{q}^i \frac{\partial}{\partial q^i} + \bar{\ddot{q}}^i \frac{\partial}{\partial \dot{q}^i} \quad (11)$$

We can do the following remark that

$$\bar{X}_L = \frac{1}{c} \left(\frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} \right) = \frac{1}{c} X_L \quad (12)$$

For this dynamic, the first-order jet bundle $J^1 \pi$ is generated by the multivector field \bar{X}_L (i.e. the multivector field is a class of integrable and $\bar{\tau}$ -transverse $\{\bar{X}_L\} \subset \chi^1(J^1 \pi)$ which satisfy the Euler Lagrange equation

$$\left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) \right) = 0 \quad (13)$$

IV. NOETHER CURRENT VIA ANALYTICAL METHOD

The physical characteristics, in particular, the dynamical invariants, of the systems can be expressed via solutions of the equations of motion [32]-[34].

Let the infinitesimal transformations be

$$\begin{aligned} t &\rightarrow t' = t + \delta t \\ \phi_\alpha &\rightarrow \phi'_\alpha(t') = \phi_\alpha(t) + \delta \phi_\alpha(t) \end{aligned} \quad (14)$$

Note that the variation $\delta \phi_\alpha(t)$ is defined as

$$\delta \phi_\alpha(t) = \phi'_\alpha(t') - \phi_\alpha(t) \quad (15)$$

Equation (15) represents the change of the field due to both the transformation of the field and the coordinate transformation. One defines in a fixed point in space by

$$\delta_0 \phi_\alpha(t) = \phi'_\alpha(t) - \phi_\alpha(t) \quad (16)$$

Transformation of the integration measure limiting to first order is:

$$d^4 t' = \det \begin{pmatrix} \frac{d^4 t'}{d^4 t} \\ \frac{d^4 t'}{d^4 t} \\ \frac{d^4 t'}{d^4 t} \\ \frac{d^4 t'}{d^4 t} \end{pmatrix} d^4 t = \det \begin{pmatrix} \frac{\partial t^1}{\partial t^1} & \dots & \frac{\partial t^4}{\partial t^1} \\ \vdots & & \vdots \\ \frac{\partial t^1}{\partial t^4} & \dots & \frac{\partial t^4}{\partial t^4} \end{pmatrix} d^4 t \approx (1 + \partial_\mu \delta t^\mu) d^4 t \quad (17)$$

$$\begin{aligned} \delta \varphi_\alpha(t) &= \varphi_\alpha'(t') - \varphi_\alpha(t) = \varphi_\alpha'(t + \delta t) - \varphi_\alpha(t) \\ &= \varphi_\alpha'(t) + \delta t^\mu \partial_\mu \varphi_\alpha'(t) - \varphi_\alpha(t) \\ &= \varphi_\alpha'(t) + \delta t^\mu \partial_\mu (\delta_0 \varphi_\alpha(t) + \varphi_\alpha(t)) - \varphi_\alpha(t) \end{aligned}$$

The relationship between $\delta \phi_\alpha$ and $\delta_0 \phi_\alpha$ to first-order is:

where

$$\delta \phi_\alpha(t) = \delta_0 \phi_\alpha(t) + \partial_\mu \phi_\alpha(t) \delta t^\mu \quad (18)$$

The variation of the action to first-order

$$\begin{aligned} \delta S &= \int d^4 t' L(\phi_\alpha'(t'), \partial_\mu \phi_\alpha'(t'), t') - \int d^4 t L(\phi_\alpha(t), \partial_\mu \phi_\alpha(t), t) \\ &\approx \int d^4 t (1 + \partial_\mu \delta t^\mu) L(\phi_\alpha'(t'), \partial_\mu \phi_\alpha'(t'), t') - \int d^4 t L(\phi_\alpha(t), \partial_\mu \phi_\alpha(t), t) \\ &\approx \int d^4 t (\delta L + (\partial_\mu \delta t^\mu) L) = \int d^4 t (\delta_0 L + (\partial_\mu L) \delta t^\mu + (\partial_\mu \delta t^\mu) L) \\ &\approx \int d^4 t \left(\frac{\partial L}{\partial \phi_\alpha} \delta_0 \phi_\alpha + \frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} (\partial_\mu \delta_0 \phi_\alpha) + \partial_\mu (\delta t^\mu L) \right) \\ &= \int d^4 t \delta_0 \phi_\alpha \left(\frac{\partial L}{\partial \phi_\alpha} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} \right) + \int d^4 t \left[\partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} \delta_0 \phi_\alpha \right) + \partial_\mu (\delta t^\mu L) \right] \\ &= \int d^4 t \delta_0 \phi_\alpha \left(\frac{\partial L}{\partial \phi_\alpha} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} \right) + \int d^4 t \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} \delta_0 \phi_\alpha + \delta t^\mu L \right) \end{aligned} \quad (19)$$

By using (8), the first term of the last equation in (19) vanishes and substituting (18) in the last term of (19), it gives:

$$\begin{aligned} \delta S &= \int d^4 t \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} (\delta \phi_\alpha - \delta t^\mu \partial_\mu \phi_\alpha) + \delta t^\mu L \right) \\ &= \int d^4 t \partial_\mu \left[\frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} \delta \phi_\alpha - \left(\frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} \partial_\nu \phi_\alpha - \delta_\nu^\mu L \right) \delta t^\nu \right] \end{aligned}$$

The conserved current is therefore

$$J^\mu = - \frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} \delta \phi_\alpha + \left(\frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} \partial_\nu \phi_\alpha - \delta_\nu^\mu L \right) \delta t^\nu \quad (20)$$

V. NOETHER CURRENTS FOR FIELD THEORY

In this section, we are going to concentrate ourselves on fields' theory and use the geometrical model proposed in [22]. In particular, we are going to establish the physical transformations associated to diffeomorphisms [23]-[25] which give Noether currents; the result will be the same for the mechanic theory.

In Section II, we saw that the dynamic of field theories can be derived by the Hamilton principle which is given by:

$$i(X_L)\Omega_L = 0$$

where X_L is holonomic multivector fields integrable and SOPDE.

$$X_L = \frac{\partial}{\partial t^\nu} + F_\nu^A \frac{\partial}{\partial y^A} + G_{\nu\rho}^A \frac{\partial}{\partial v^{\rho A}} \quad (21)$$

where

$$F_\nu^A = v_\nu^A = \partial_\nu \phi^A = \phi_\nu^A \quad \text{and} \quad G_{\nu\rho}^A = \frac{\partial^2 y^A}{\partial t^\nu \partial t^\rho} = \phi_{\nu\rho}^A$$

In this condition, the Poincare-Cartan 4-form (5a) becomes:

$$\theta_L = L(t_\mu, \phi^i(t_\mu), \phi_\nu^i(t_\mu)) d^k t \quad (22)$$

A. A Translation in the Base Space "Space-Time"

Consider an infinitesimal space-time translations associated to diffeomorphisms:

$$\begin{aligned} \varphi: R^k &\rightarrow R^k \\ t^\mu &\rightarrow \varphi(t^\mu) = t^\mu + a^\mu = t'^\mu / a^\mu = \text{const} \quad \forall \mu = \overline{0,3} \end{aligned} \quad (23)$$

and let Φ be a symmetry of the multisymplectic lagrangian system for fields

$$\begin{aligned} \Phi: R^k \times M &\rightarrow R^k \times M \\ \Phi(t_\mu, \phi^i(t_\mu)) &= (t'_\mu = t_\mu + a_\mu, \phi^i(t'_\mu)) \end{aligned} \quad (24)$$

$$\Phi^* : R^k \times T_4^1 M \rightarrow R^k \times T_4^1 M$$

$$\Phi^*(t_\mu, \phi^i(t_\mu), \phi_v^i(t_\mu)) = (t'_\mu = t_\mu + a_\mu, \phi^i(t'_\mu), \phi_v^i(t'_\mu)) \quad (25)$$

During this transformation, the theory remains covariant, the Poincaré-Cartan 4-form (22) can be written:

$$\Phi^* \theta_L = \tilde{\theta}_L(t'_\mu, \phi^i(t'_\mu), \phi_v^i(t'_\mu)) = \tilde{L} d^k t' \quad (26)$$

A limited development of Taylor to first order of the Poincaré-Cartan 4-form (24) for this infinitesimal space-time translation gives (we treat θ_L as a function)

$$\Phi^* \theta_L = \tilde{\theta}_L(t'_\mu, \phi^i(t'_\mu), \phi_v^i(t'_\mu)) =$$

$$\theta_L(t_\mu, \phi^i(t_\mu), \phi_v^i(t_\mu)) +$$

$$\underbrace{(t'^\mu - t^\mu)}_{dt^\mu} \left[\frac{\partial \theta_L}{\partial t^\mu} + \frac{\partial \theta_L}{\partial \phi^i} \frac{\partial \phi^i}{\partial t^\mu} + \frac{\partial \theta_L}{\partial \phi_\rho^i} \frac{\partial \phi_\rho^i}{\partial t^\mu} \right]$$

where $\partial_\mu = \frac{\partial}{\partial t^\mu}$, $\phi_\mu^i = \partial_\mu \phi^i$

$$\tilde{\theta}_L(t'_\mu, \phi^i(t'_\mu), \phi_v^i(t'_\mu)) - \theta_L(t_\mu, \phi^i(t_\mu), \phi_v^i(t_\mu)) =$$

$$d\theta_L = a^\mu \left(\partial_\mu + \phi_\mu^i \frac{\partial}{\partial \phi^i} + \phi_{\mu\rho}^i \frac{\partial}{\partial \phi_\rho^i} \right) \theta_L = i_Y \theta_L \quad (27)$$

The term in bracket in (27) will be identified to an infinitesimal vector field associated to this symmetry

$$d i_Y \theta_L = a^\mu \left[\partial_\nu \partial_\mu L dt^\nu \wedge d^k t + \partial_\mu L \underbrace{\partial_\nu (d^k t)}_{d^k t} \wedge dt^\nu + \partial_\nu L dt^\nu \wedge \underbrace{d^{k-1} t_\mu}_{d^k t \delta_\mu^\nu} + L \underbrace{\partial_\nu (d^{k-1} t_\mu)}_{d^{k-1} t_\mu} \wedge dt^\nu \right.$$

$$+ \phi_{\nu\mu}^i \frac{\partial L}{\partial \phi^i} dt^\nu \wedge d^k t + \phi_\mu^i \frac{\partial^2 L}{\partial t^\nu \partial \phi^i} dt^\nu \wedge d^k t + \phi_\mu^i \frac{\partial L}{\partial \phi^i} \underbrace{\partial_\nu (d^k t)}_{d^k t} \wedge dt^\nu + \phi_{\nu\mu\gamma}^i \frac{\partial L}{\partial \phi_\gamma^i} dt^\nu \wedge d^k t$$

$$+ \phi_{\mu\gamma}^i \frac{\partial^2 L}{\partial t^\nu \partial \phi_\gamma^i} dt^\nu \wedge d^k t + \phi_{\mu\gamma}^i \frac{\partial L}{\partial \phi_\gamma^i} \underbrace{\partial_\nu (d^k t)}_{d^k t} \wedge dt^\nu + \frac{\partial^2 L}{\partial \phi^j \partial t^\mu} d\phi^j \wedge d^k t + \frac{\partial L}{\partial \phi^j} \underbrace{d\phi^j \wedge d^{k-1} t_\mu}_{\phi_\rho^j \delta_\mu^\rho d^k t}$$

$$+ \phi_\mu^i \frac{\partial^2 L}{\partial \phi^j \partial \phi^i} d\phi^j \wedge d^k t + \phi_{\mu\gamma}^i \frac{\partial^2 L}{\partial \phi^j \partial \phi_\gamma^i} d\phi^j \wedge d^k t + \frac{\partial^2 L}{\partial \phi_\rho^j \partial t^\mu} d\phi_\rho^j \wedge d^k t + \frac{\partial L}{\partial \phi_\rho^j} \underbrace{d\phi_\rho^j \wedge d^{k-1} t_\mu}_{\phi_\rho^j \delta_\mu^\rho d^k t}$$

$$\left. + \delta_j^i \delta_\rho^\mu \frac{\partial L}{\partial \phi^i} d\phi_\rho^j \wedge d^k t + \phi_\mu^i \frac{\partial^2 L}{\partial \phi_\rho^j \partial \phi^i} d\phi_\rho^j \wedge d^k t + \phi_{\mu\gamma}^i \frac{\partial^2 L}{\partial \phi_\rho^j \partial \phi_\gamma^i} d\phi_\rho^j \wedge d^k t \right]$$

Using that

$$Y = a^\mu \left(\partial_\mu + \phi_\mu^i \frac{\partial}{\partial \phi^i} + \phi_{\mu\rho}^i \frac{\partial}{\partial \phi_\rho^i} \right) \quad (28)$$

Unlike other authors [23]-[25], [27], [36], to this infinitesimal transformation, we must have (the result will be proved later)

$$\partial_\rho f^\rho d^k t = d i_Y \theta_L - i_Y d\theta_L$$

where $f^\rho \in C^\infty$ (29)

where $\eta = d^k t$ is the volume form on M and $k = \dim M$. The conservation law derived from (29) will be

$$\partial_\rho f^\rho = 0 \quad (30)$$

We identify f^ρ to Noether current. Contracting (22) by (28),

$$i_Y \theta_L = a^\mu \left[\partial_\mu L d^k t + L \underbrace{\partial_\mu (d^k t)}_{d^{k-1} t_\mu} + \phi_\mu^i \frac{\partial L}{\partial \phi^i} d^k t + \phi_{\mu\gamma}^i \frac{\partial L}{\partial \phi_\gamma^i} d^k t \right]$$

The differential “d” defined on the first-order jet bundle $J^1 \pi$ cited in Section II

$$d = \partial_\nu dt^\nu + \frac{\partial}{\partial \phi^j} d\phi^j + \frac{\partial}{\partial \phi_\rho^j} d\phi_\rho^j \quad (31)$$

$$\left\{ \begin{array}{l} dt^\rho \wedge d^k t = 0 \quad \forall \rho = \overline{0,3} \\ d\phi^j \wedge d^k t = \frac{d\phi^j}{dt^\rho} dt^\rho \wedge d^k t = \phi_\rho^j dt^\rho \wedge d^k t = 0 \\ d\phi_\rho^j \wedge d^k t = \frac{d\phi_\rho^j}{dt^\gamma} dt^\gamma \wedge d^k t = \phi_{\rho\gamma}^j dt^\gamma \wedge d^k t = 0 \\ d^{k-1} t_\nu \wedge dt^\sigma = \delta_\nu^\sigma d^k t \end{array} \right. \quad (33)$$

$$d\theta_L = \partial_\nu L dt^\nu \wedge d^k t + L \underbrace{\partial_\nu (d^k t)}_{d^k t} \wedge dt^\nu + \frac{\partial L}{\partial \phi^j} d\phi^j \wedge d^k t + \frac{\partial L}{\partial \phi_\rho^j} d\phi_\rho^j \wedge d^k t$$

Using that

$$d i_Y \theta_L = 2 \partial_\mu L d^k t + L d^{k-1} t_\mu + 2 \phi_\mu^j \frac{\partial L}{\partial \phi^j} d^k t + 2 \phi_{\mu\rho}^j \frac{\partial L}{\partial \phi_\rho^j} d^k t \quad \left\{ \begin{array}{l} \partial_\mu (d\phi^j \wedge d^k t) = -d\phi^j \wedge d^{k-1} t_\mu \\ \partial_\mu (d\phi_\rho^j \wedge d^k t) = -d\phi_\rho^j \wedge d^{k-1} t_\mu \end{array} \right. \quad (35)$$

Contracting (22) by (31),

$$\begin{aligned} i_Y d\theta_L = & a^\mu \left[\partial_\mu \partial_\nu L dt^\nu \wedge d^k t + \partial_\nu L \partial_\mu (dt^\nu) \wedge d^k t + \partial_\nu L dt^\nu \wedge \partial_\mu (d^k t) + \partial_\mu L d^k t + L \partial_\mu (d^k t) \right. \\ & + \frac{\partial^2 L}{\partial t^\mu \partial \phi^j} d\phi^j \wedge d^k t - \frac{\partial L}{\partial \phi^j} \underbrace{d\phi^j \wedge \partial_\mu (d^k t)}_{\phi_\mu^j d^k t} + \frac{\partial^2 L}{\partial t^\mu \partial \phi_\rho^j} d\phi_\rho^j \wedge d^k t - \frac{\partial L}{\partial \phi_\rho^j} \underbrace{d\phi_\rho^j \wedge \partial_\mu (d^k t)}_{\phi_{\mu\rho}^j d^k t} \\ & + \phi_\mu^i \frac{\partial^2 L}{\partial \phi^i \partial t^\nu} dt^\nu \wedge d^k t + \phi_\mu^i \frac{\partial L}{\partial \phi^i} d^k t + \phi_\mu^i \frac{\partial^2 L}{\partial \phi^i \partial \phi^j} d\phi^j \wedge d^k t + \phi_\mu^i \frac{\partial L}{\partial \phi^j} \delta_j^i d^k t \\ & + \phi_\mu^i \frac{\partial^2 L}{\partial \phi^i \partial \phi_\rho^j} d\phi_\rho^j \wedge d^k t + \phi_{\mu\rho}^i \frac{\partial^2 L}{\partial \phi_\rho^i \partial t^\nu} dt^\nu \wedge d^k t + \phi_{\mu\rho}^i \frac{\partial L}{\partial \phi_\rho^i} d^k t + \phi_{\mu\nu}^i \frac{\partial^2 L}{\partial \phi_\nu^i \partial \phi^j} d\phi^j \wedge d^k t \\ & \left. + \phi_{\mu\nu}^i \frac{\partial^2 L}{\partial \phi_\nu^i \partial \phi_\rho^j} d\phi_\rho^j \wedge d^k t + \phi_{\mu\nu}^i \frac{\partial L}{\partial \phi_\rho^j} \delta_j^i \delta_\rho^\nu d^k t \right] \quad (36) \end{aligned}$$

Using (35), (36) becomes

$$i_Y d\theta_L = a^\mu \left[3 \partial_\mu L d^k t + L d^{k-1} t_\mu + \phi_\mu^j \frac{\partial L}{\partial \phi^j} d^k t + \phi_{\mu\rho}^j \frac{\partial L}{\partial \phi_\rho^j} d^k t \right] \quad (37)$$

Inserting (34) and (37) in (29), it gives

By identification the members of (40), we find

$$d i_Y \theta_L - i_Y d\theta_L = a^\mu \left[-\partial_\mu L + \phi_\mu^j \frac{\partial L}{\partial \phi^j} + \phi_{\mu\rho}^j \frac{\partial L}{\partial \phi_\rho^j} \right] d^k t \quad \left\{ \begin{array}{l} \delta t^\rho = a^\mu \delta_\mu^\rho \\ \delta \phi^j = a^\mu \phi_\mu^j = \phi_\mu^j \delta t^\rho \end{array} \right. \quad (41)$$

Using the Euler-Lagrange equation (8), we get

From (40), the conserved current obtained is the second rank tensor T_μ^ρ called energy-momentum tensor or stress-energy tensor.

$$\frac{\partial L}{\partial \phi^j} = \partial_\rho \left(\frac{\partial L}{\partial \partial_\rho \phi^j} \right) = \partial_\rho \left(\frac{\partial L}{\partial \phi_\rho^j} \right) \quad (39)$$

Inserting (39) in (38) and using (19), it becomes

$$T_\mu^\rho = \frac{\partial L}{\partial \phi_\rho^j} \phi_\mu^j - L \delta_\mu^\rho \quad (42)$$

$$d i_Y \theta_L - i_Y d\theta_L = \partial_\rho \left[\frac{\partial L}{\partial \phi_\rho^j} a^\mu \phi_\mu^j - L a^\mu \delta_\mu^\rho \right] d^k t \quad (40)$$

The results (41) and (42) have been already established in [32]-[34].

$$= \partial_\rho \left[\frac{\partial L}{\partial \phi_\rho^j} \delta \phi^j + L \delta t^\rho \right] d^k t = \partial_\rho f^\rho d^k t$$

B. A Rotation in the Base Space "Space-Time"

Consider now an infinitesimal rotation in the space-time.

This transformation is an element $\Lambda_v^\mu = \delta_v^\mu + \delta\omega_v^\mu$ of Lorentz group denoted as SO(1, 3) which is associated to the following diffeomorphisms:

$$\varphi: R^k \rightarrow R^k \quad (43)$$

$$t'^\mu = \varphi(t^\mu) = (\delta_v^\mu + \delta\omega_v^\mu) t^\nu \quad / \quad \delta\omega^{\mu\nu} = \text{const in time}$$

The rotation being infinitesimal, the variation in space-time coordinates

$$t'^\mu - t^\mu = \delta t^\mu = \delta\omega^{\mu\nu} t_\nu \quad / \quad \delta\omega^{\mu\nu} \ll 1 \quad (44)$$

In this condition, (24) and (27) become

$$\Phi: R^k \times M \rightarrow R^k \times M \quad (45)$$

$$\Phi(t_\mu, \phi^i(t_\mu)) = (t'_\mu = t_\mu + \delta\omega^{\mu\nu} t_\nu, \phi^i(t'_\mu))$$

$$\Phi^*: R^k \times T_4^1 M \rightarrow R^k \times T_4^1 M \quad (46)$$

$$\Phi^*(t_\mu, \phi^i(t_\mu), \phi^j(t_\mu)) = (t'_\mu = t_\mu + \delta\omega^{\mu\nu} t_\nu, \phi^i(t'_\mu), \phi^j(t'_\mu))$$

$$d\theta_L = \delta\omega^{\mu\nu} t_\nu \left(\partial_\mu + \phi_\mu^i \frac{\partial}{\partial\phi^i} + \phi_{\mu\rho}^i \frac{\partial}{\partial\phi_\rho^i} \right) \theta_L = i_Y \theta_L$$

The infinitesimal vector field associated, to this symmetry, will be

$$Y = \delta\omega^{\mu\nu} t_\nu \left(\partial_\mu + \phi_\mu^i \frac{\partial}{\partial\phi^i} + \phi_{\mu\rho}^i \frac{\partial}{\partial\phi_\rho^i} \right) \quad (47)$$

$$i_Y \theta_L = \delta\omega^{\mu\nu} t_\nu \left(\partial_\mu L d^k t + L d^{k-1} t_\mu + \phi_\mu^i \frac{\partial L}{\partial\phi^i} d^k t + \phi_{\mu\rho}^i \frac{\partial L}{\partial\phi_\rho^i} d^k t \right) \quad (48)$$

$$\partial_\rho t_\mu = \frac{\partial t_\mu}{\partial t^\rho} = \frac{\partial \eta_{\mu\sigma} t^\sigma}{\partial t^\rho} = \eta_{\mu\sigma} \delta_\rho^\sigma = \eta_{\mu\rho} \quad (49)$$

By making the same calculus done in Section V.A and using (31), (33) and (49), we obtain

$$di_Y \theta_L = \delta\omega^{\mu\nu} \left[2 t_\nu \partial_\mu L + \eta_{\nu\mu} L + 2 t_\nu \phi_\mu^i \frac{\partial L}{\partial\phi^i} + 2 t_\nu \phi_{\mu\sigma}^i \frac{\partial L}{\partial\phi_\sigma^i} \right] d^k t + \delta\omega^{\mu\nu} t_\nu L d^{k-1} t_\mu \quad (50)$$

$$d\theta_L = \partial_\rho L dt^\rho \wedge d^k t + L \underbrace{d^{k-1} t_\rho \wedge dt^\rho}_{d^k t} + \frac{\partial L}{\partial\phi^i} d\phi^i \wedge d^k t + \frac{\partial L}{\partial\phi_\rho^i} d\phi_\rho^i \wedge d^k t$$

$$i_Y d\theta_L = \delta\omega^{\mu\nu} \left[3 t_\nu \partial_\mu L + t_\nu \phi_\mu^i \frac{\partial L}{\partial\phi^i} + t_\nu \phi_{\mu\rho}^i \frac{\partial L}{\partial\phi_\rho^i} \right] d^k t + \delta\omega^{\mu\nu} t_\nu L d^{k-1} t_\mu \quad (51)$$

Inserting (50) and (51) in (29)

$$di_Y \theta_L - i_Y d\theta_L = \delta\omega^{\mu\nu} \left[\eta_{\nu\mu} L - t_\nu \partial_\mu L + t_\nu \phi_\mu^j \frac{\partial L}{\partial\phi^j} + t_\nu \phi_{\mu\rho}^j \frac{\partial L}{\partial\phi_\rho^j} \right] d^k t$$

$$= \frac{1}{2} \delta\omega^{\mu\nu} \left[\frac{\partial L}{\partial\phi^j} \phi_\rho^j (t_\nu \delta_\mu^\rho - t_\mu \delta_\nu^\rho) + \frac{\partial L}{\partial\phi_\rho^j} \phi_{\gamma\rho}^j (t_\nu \delta_\mu^\gamma - t_\mu \delta_\nu^\gamma) \right] d^k t$$

$$\left[-\partial_\rho L (t_\nu \delta_\mu^\rho - t_\mu \delta_\nu^\rho) + L (\eta_{\nu\rho} \delta_\mu^\rho - \eta_{\mu\rho} \delta_\nu^\rho) \right] d^k t \quad (52)$$

Substituting (39), (19) and (49), (52) becomes

$$di_Y \theta_L - i_Y d\theta_L = \frac{1}{2} \delta\omega^{\mu\nu} \left[\partial_\rho \left(\frac{\partial L}{\partial\phi_\rho^j} \right) \phi_\gamma^j (t_\nu \delta_\mu^\gamma - t_\mu \delta_\nu^\gamma) + \frac{\partial L}{\partial\phi_\rho^j} \phi_{\gamma\rho}^j (t_\nu \delta_\mu^\gamma - t_\mu \delta_\nu^\gamma) \right] d^k t$$

$$\left[-\partial_\rho L (t_\nu \delta_\mu^\rho - t_\mu \delta_\nu^\rho) + L (\partial_\rho t_\nu \delta_\mu^\rho - \partial_\rho t_\mu \delta_\nu^\rho) \right]$$

$$= \frac{1}{2} \delta\omega^{\mu\nu} \partial_\rho \left[\frac{\partial L}{\partial\phi_\rho^j} \phi_\gamma^j (t_\nu \delta_\mu^\gamma - t_\mu \delta_\nu^\gamma) - L (t_\nu \delta_\mu^\rho - t_\mu \delta_\nu^\rho) \right] d^k t$$

$$\begin{aligned}
 & -\frac{1}{2}\delta\omega^{\mu\nu}\left[\frac{\partial L}{\partial\varphi_\rho^j}\varphi_\gamma^j\partial_\rho(t_\nu\delta_\mu^\gamma-t_\mu\delta_\nu^\gamma)-2L\partial_\rho(t_\nu\delta_\mu^\rho-t_\mu\delta_\nu^\rho)\right]d^k t \\
 & =\frac{1}{2}\delta\omega^{\mu\nu}\partial_\rho\left[\frac{\partial L}{\partial\varphi_\rho^j}\varphi_\gamma^j(t_\nu\delta_\mu^\gamma-t_\mu\delta_\nu^\gamma)-L(t_\nu\delta_\mu^\rho-t_\mu\delta_\nu^\rho)\right]d^k t \\
 & -\frac{1}{2}\delta\omega^{\mu\nu}\left[\frac{\partial L}{\partial\varphi_\rho^j}\varphi_\gamma^j(\eta_{\nu\rho}\delta_\mu^\gamma-\eta_{\mu\rho}\delta_\nu^\gamma)-2L(\eta_{\nu\rho}\delta_\mu^\rho-\eta_{\mu\rho}\delta_\nu^\rho)\right]d^k t \\
 & =\frac{1}{2}\delta\omega^{\mu\nu}\partial_\rho\left[\frac{\partial L}{\partial\varphi_\rho^j}\varphi_\gamma^j(t_\nu\delta_\mu^\gamma-t_\mu\delta_\nu^\gamma)-L(t_\nu\delta_\mu^\rho-t_\mu\delta_\nu^\rho)\right]d^k t \\
 & +\underbrace{\left[-\frac{1}{2}\delta\omega^{\mu\nu}\left(\frac{\partial L}{\partial\varphi^{\nu j}}\varphi_\mu^j-\frac{\partial L}{\partial\varphi^{\mu j}}\varphi_\nu^j\right)+\delta\omega^{\mu\nu}L(\eta_{\nu\mu}-\eta_{\mu\nu})\right]}_{=0}d^k t \\
 & =\frac{1}{2}\delta\omega^{\mu\nu}\partial_\rho\left[\frac{\partial L}{\partial\varphi_\rho^j}\varphi_\gamma^j(t_\nu\delta_\mu^\gamma-t_\mu\delta_\nu^\gamma)-L(t_\nu\delta_\mu^\rho-t_\mu\delta_\nu^\rho)\right]d^k t=\partial_\rho\left(\frac{\partial L}{\partial\varphi_\rho^j}\delta\varphi^j+L\delta t^\rho\right)d^k t
 \end{aligned} \tag{53}$$

By identification term by term and using (42), we find

$$\begin{aligned}
 \delta t^\rho & =\frac{1}{2}\delta\omega^{\mu\nu}(t_\nu\delta_\mu^\rho-t_\mu\delta_\nu^\rho) \\
 \delta\varphi^j & =\frac{1}{2}\delta\omega^{\mu\nu}\varphi_\gamma^j(t_\nu\delta_\mu^\gamma-t_\mu\delta_\nu^\gamma)=\varphi_\gamma^j\delta t^\gamma
 \end{aligned} \tag{54}$$

The Noether current will be

$$\begin{aligned}
 J_{\mu\nu}^\rho & =\frac{\partial L}{\partial\varphi_\rho^j}\varphi_\gamma^j(t_\nu\delta_\mu^\gamma-t_\mu\delta_\nu^\gamma)- \\
 L(t_\nu\delta_\mu^\rho-t_\mu\delta_\nu^\rho) & =t_\nu T_\mu^\rho-t_\mu T_\nu^\rho
 \end{aligned} \tag{55}$$

During a rotation in the base space, i.e. Minkowsy space, the Noether current (55) obtained by this symmetry will be identified to the angular momentum. The results (54) and (55) have already been developed in [32], [33].

C. A Translation along the Fibre

Having the geometric model of the fiber bundle for the fields theory proposed in [22], we define a new diffeomorphism $\tilde{\varphi}$, in addition to those already mentioned above [23]-[25]. This infinitesimal transformation is an element $U(\mathcal{G}^\alpha)=1+\mathcal{G}^\alpha T_\alpha/\alpha=1,\dots,m$ of special unitary group denoted SU(n) which allows to the transformation of field coordinates $(\phi^i/i=1,\dots,d)$ on the fiber.

$$\begin{aligned}
 \Phi: R^4 \times M & \rightarrow R^4 \times M \\
 (t_\mu, \phi^i(t_\mu)) & \rightarrow \Phi(t_\mu, \phi^i(t_\mu)) = (t_\mu, \phi'^i(t_\mu) = \phi^i(t_\mu) + \mathcal{G}_\alpha (T^\alpha)^i_j \phi^j(t_\mu))
 \end{aligned} \tag{59}$$

We note that (T_α) is a Hermitian matrix of SU (n). These generators (T_α) form a Lie algebra /

$$\begin{aligned}
 [T_\alpha, T_\beta] & = C_{\alpha\beta\gamma} T_\gamma \\
 \varphi: R^k & \rightarrow R^k \\
 t^\mu & \rightarrow \varphi(t^\mu) = t^\mu = t'^\mu \quad \forall \mu = \overline{0,3}
 \end{aligned}$$

and

$$\begin{aligned}
 \tilde{\varphi}: M & \rightarrow M \\
 \phi^i(t_\mu) & \rightarrow \tilde{\varphi}(\phi^i(t_\mu)) = \phi'^i(t_\mu) = U(\mathcal{G}^\alpha)^i_j \phi^j(t_\mu)
 \end{aligned} \tag{56}$$

The translation in the fiber being infinitesimal, the variation in field coordinates

$$\delta\phi^i = \phi'^i(t_\mu) - \phi^i(t_\mu) = \mathcal{G}_\alpha (T^\alpha)^i_j \phi^j(t_\mu) \tag{57}$$

We also note that infinitesimal (\mathcal{G}_α) is independent of coordinates (t_μ) , $\tilde{\varphi}$ is said to be global Gauge, using (57):

$$\delta\phi_v^i = \partial_\nu \delta\phi^i(t_\mu) = \mathcal{G}_\alpha (T^\alpha)^i_j \partial_\nu \phi^j(t_\mu) = \mathcal{G}_\alpha (T^\alpha)^i_j \phi^j(t_\mu) \tag{58}$$

$$\Phi^* : R^4 \times T_4^1 M \rightarrow R^4 \times T_4^1 M$$

$$(\mu, \phi^i(\mu), \phi_v^i(\mu)) \rightarrow \Phi^*(\mu, \phi^i(\mu), \phi_v^i(\mu)) = (\mu, \phi'^i(\mu), \phi_v'^i(\mu)) \quad (60)$$

$$d\theta_L = \frac{\partial \theta_L}{\partial \phi^i} d\phi^i + \frac{\partial \theta_L}{\partial \phi_v^i} d\phi_v^i = \mathcal{G}_\alpha (T^\alpha)_j \left(\phi^j \frac{\partial}{\partial \phi^i} + \phi_v^j \frac{\partial}{\partial \phi_v^i} \right) = i_Y \theta_L$$

$$i_Y \theta_L = \frac{\partial \theta_L}{\partial \phi^i} d\phi^i + \frac{\partial \theta_L}{\partial \phi_v^i} d\phi_v^i =$$

$$\mathcal{G}_\alpha (T^\alpha)_j \left(\phi^j \frac{\partial L}{\partial \phi^i} + \phi_v^j \frac{\partial L}{\partial \phi_v^i} \right) d^k t$$

The infinitesimal vector field associated to this Gauge symmetry is

$$Y = \mathcal{G}_\alpha (T^\alpha)_j \left(\phi^j \frac{\partial}{\partial \phi^i} + \phi_v^j \frac{\partial}{\partial \phi_v^i} \right) \quad (61)$$

The differential “d” defined on TM :

$$d = \frac{\partial}{\partial \phi^i} d\phi^i + \frac{\partial}{\partial \phi_v^i} d\phi_v^i \quad (62)$$

Contracting (22) by (61), we have

$$d i_Y \theta_L = \mathcal{G}_\alpha (T^\alpha)_j \left[\delta_i^j \frac{\partial L}{\partial \phi^i} d\phi^i \wedge d^k t + \phi^j \frac{\partial^2 L}{\partial \phi^i \partial \phi^i} d\phi^i \wedge d^k t + \phi_v^j \frac{\partial^2 L}{\partial \phi^i \partial \phi_v^i} d\phi^i \wedge d^k t \right.$$

$$\left. + \phi^j \frac{\partial^2 L}{\partial \phi_v^i \partial \phi^i} d\phi_v^i \wedge d^k t + \delta_i^j \delta_v^p \frac{\partial L}{\partial \phi_v^i} d\phi_v^i \wedge d^k t + \phi_v^j \frac{\partial^2 L}{\partial \phi_v^i \partial \phi_v^i} d\phi_v^i \wedge d^k t \right] \quad (63)$$

$$d\theta_L = \frac{\partial L}{\partial \phi^i} d\phi^i \wedge d^k t + \frac{\partial L}{\partial \phi_v^i} d\phi_v^i \wedge d^k t$$

$$i_Y d\theta_L = \mathcal{G}_\alpha (T^\alpha)_j \left[\phi^j \frac{\partial^2 L}{\partial \phi^i \partial \phi^i} d\phi^i \wedge d^k t + \delta_i^j \phi^j \frac{\partial L}{\partial \phi^i} d^k t + \phi_v^j \frac{\partial^2 L}{\partial \phi_v^i \partial \phi^i} d\phi^i \wedge d^k t \right.$$

$$\left. + \phi_v^j \frac{\partial^2 L}{\partial \phi_v^i \partial \phi_v^i} d\phi_v^i \wedge d^k t + \phi^j \frac{\partial^2 L}{\partial \phi^i \partial \phi_v^i} d\phi_v^i \wedge d^k t + \delta_i^j \delta_v^p \phi_v^j \frac{\partial L}{\partial \phi_v^i} d^k t \right] \quad (64)$$

Introducing (63) and (64) in (29),

$$d i_Y \theta_L - i_Y d\theta_L = \mathcal{G}_\alpha (T^\alpha)_j \left[\frac{\partial L}{\partial \phi^i} d\phi^j \wedge d^k t + \frac{\partial L}{\partial \phi_v^i} d\phi_v^j \wedge d^k t - \phi^j \frac{\partial L}{\partial \phi^i} d^k t - \phi_v^j \frac{\partial L}{\partial \phi_v^i} d^k t \right] \quad (65)$$

Using (57) and (58), (65) becomes

$$d i_Y \theta_L - i_Y d\theta_L = \mathcal{G}_\alpha (T^\alpha)_j \left[\mathcal{G}_\beta (T^\beta)_i \left(\phi^i \frac{\partial L}{\partial \phi^i} + \phi_v^i \frac{\partial L}{\partial \phi_v^i} \right) - \phi^j \frac{\partial L}{\partial \phi^i} - \phi_v^j \frac{\partial L}{\partial \phi_v^i} \right] d^k t$$

$$d i_Y \theta_L - i_Y d\theta_L = \mathcal{G}_\alpha (T^\alpha)_j \mathcal{G}_\beta (T^\beta)_i \left[\phi^i \frac{\partial L}{\partial \phi^i} + \phi_v^i \frac{\partial L}{\partial \phi_v^i} \right] d^k t - \mathcal{G}_\alpha (T^\alpha)_j \left[\phi^j \frac{\partial L}{\partial \phi^i} + \phi_v^j \frac{\partial L}{\partial \phi_v^i} \right] d^k t \quad (66)$$

The transformation being infinitesimal ($\mathcal{G}_\alpha \ll 1$), the first term in (66), vanishes

Using (39), (67) gives

$$d i_Y \theta_L - i_Y d\theta_L = -\mathcal{G}_\alpha (T^\alpha)_j \left[\phi^j \frac{\partial L}{\partial \phi^i} + \phi_v^j \frac{\partial L}{\partial \phi_v^i} \right] d^k t \quad (67)$$

$$\begin{aligned}
 di_Y \theta_L - i_Y d\theta_L &= -\mathcal{G}_\alpha (T^\alpha)_j^i \left[\varphi^j \partial_\nu \left(\frac{\partial L}{\partial \varphi_\nu^i} \right) + \varphi_\nu^j \frac{\partial L}{\partial \varphi_\nu^i} \right] d^k t \\
 &= -\partial_\nu \left[\mathcal{G}_\alpha (T^\alpha)_j^i \varphi^j \frac{\partial L}{\partial \varphi_\nu^i} \right] d^k t \\
 &= -\frac{1}{2} \partial_\nu \left[\mathcal{G}_\alpha (T^\alpha)_j^i \varphi^j \frac{\partial L}{\partial \varphi_\nu^i} + \mathcal{G}_\alpha (T^\alpha)_i^j \varphi^i \frac{\partial L}{\partial \varphi_\nu^j} \right] d^k t
 \end{aligned} \tag{68}$$

Knowing that the matrix elements of SU (n) verify

$$(T^\alpha)_j^i = - (T^\alpha)_i^j \quad / \quad i \neq j$$

So,

$$\begin{aligned}
 di_Y \theta_L - i_Y d\theta_L &= \mathcal{G}_\alpha \partial_\nu \left[-\frac{1}{2} (T^\alpha)_j^i \left(\varphi^j \frac{\partial L}{\partial \varphi_\nu^i} - \varphi^i \frac{\partial L}{\partial \varphi_\nu^j} \right) \right] d^k t \\
 &= \mathcal{G}_\alpha \partial_\nu \left[-\frac{1}{2} (T^\alpha)_i^j \left(\varphi^j P_i^\nu - \varphi^i P_j^\nu \right) \right] d^k t
 \end{aligned} \tag{69}$$

We put

$$J^{\alpha\nu} = -\frac{1}{2} (T^\alpha)_j^i (\varphi^j P_i^\nu - \varphi^i P_j^\nu) \tag{70}$$

By analogy with the mechanics as we can see below in the next section, we conclude that when translating along the fiber, the Noether current (70) can be identified to another type of angular momentum of the field which is related to the internal symmetry representing the Lorentz group in the field space (ϕ^i). This is proved by the natural appearance of the matrix elements $(T^\alpha)_j^i$ in (70). These terms are introduced

$$\begin{aligned}
 \Phi^* \theta_L &= \tilde{\theta}_L(t', q^i(t'), \dot{q}^i(t')) = \theta_L(t, q^i(t), \dot{q}^i(t)) + \underbrace{(t' - t)}_{dt} \left[\frac{\partial \theta_L}{\partial t} + \frac{\partial \theta_L}{\partial q^i} \frac{\partial q^i}{\partial t} + \frac{\partial \theta_L}{\partial \dot{q}^i} \frac{\partial \dot{q}^i}{\partial t} \right] \\
 \tilde{\theta}_L(t', q^i(t'), \dot{q}^i(t')) - \theta_L(t, q^i(t), \dot{q}^i(t)) &= a^0 \left[\frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} \right] \theta_L = i_Y \theta_L
 \end{aligned}$$

The associated vector field, in this case, is identified to

$$Y = a^0 \left[\frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} \right] \tag{75}$$

Contracting (71) by (75)

without any demonstration in [34]. We can also do the following remark that this angular momentum is obtained by a rotation in the space (1, 3) in [32], [34].

VI. NOETHER CURRENTS IN MECHANIC THEORY

In this section and by analogy with the work already developed for the field theories in the above section, we extend the idea to the relativistic mechanics by using the construction of the fiber bundle proposed in [22].

The Poincare-Cartan 1-form (10) becomes

$$\theta_L = L(t, q^i(t), \dot{q}^i(t)) dt \tag{71}$$

A. A Translation in the Base Space "Time"

Consider an infinitesimal time translation associated to the diffeomorphisms:

$$\begin{aligned}
 \Phi : R \times M &\rightarrow R \times M \quad / \quad M \cong R^3 \\
 \Phi(t, q^i(t)) &= (t' = t + a_0, q^i(t'))
 \end{aligned} \tag{72}$$

$\Phi^* : R \times TM \rightarrow R \times TM$

$$\Phi^*(t, q^i(t), \dot{q}^i(t)) = (t' = t + a_0, q^i(t'), \dot{q}^i(t')) \tag{73}$$

During this transformation, the Poincare-Cartan 1-form (71) can be written:

$$\Phi^* \theta_L = \tilde{\theta}_L(t', q^i(t'), \dot{q}^i(t')) = \tilde{L}(t', q^i(t'), \dot{q}^i(t')) dt' \tag{74}$$

A limited development of Taylor to first order of the Poincaré-Cartan 1-form (74) associated to this infinitesimal base time translation gives:

$$i_Y \theta_L = a^0 \left[\frac{\partial L}{\partial t} dt + \dot{q}^i \frac{\partial L}{\partial q^i} dt + \ddot{q}^i \frac{\partial L}{\partial \dot{q}^i} dt + L \right]$$

The differential "d" defined on the first-order jet bundle $J^1\pi$ cited in Section III.

$$d = \frac{\partial}{\partial t} dt + \frac{\partial}{\partial q^j} dq^j + \frac{\partial}{\partial \dot{q}^j} d\dot{q}^j \tag{76}$$

$$\begin{aligned}
 d i_Y \theta_L = a^0 & \left[\frac{\partial^2 L}{\partial t^2} dt \wedge dt + \frac{\partial L}{\partial t} dt + \ddot{q}^i \frac{\partial L}{\partial q^i} dt \wedge dt + \dot{q}^i \frac{\partial^2 L}{\partial t \partial q^i} dt \wedge dt + \dot{q}^i \frac{\partial L}{\partial q^i} dt \right. \\
 & + \ddot{q}^i \frac{\partial L}{\partial \dot{q}^i} dt \wedge dt + \ddot{q}^i \frac{\partial^2 L}{\partial t \partial \dot{q}^i} dt \wedge dt + \ddot{q}^i \frac{\partial L}{\partial \dot{q}^i} dt + \frac{\partial^2 L}{\partial q^j \partial t} dq^j \wedge dt \\
 & + \dot{q}^i \frac{\partial^2 L}{\partial q^j \partial q^i} dq^j \wedge dt + \ddot{q}^i \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} dq^j \wedge dt + \delta_j^i \frac{\partial L}{\partial q^i} d\dot{q}^j \wedge dt + \frac{\partial^2 L}{\partial \dot{q}^j \partial t} d\dot{q}^j \wedge dt \\
 & \left. + \dot{q}^i \frac{\partial^2 L}{\partial \dot{q}^j \partial q^i} d\dot{q}^j \wedge dt + \ddot{q}^i \frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i} d\dot{q}^j \wedge dt + \frac{\partial L}{\partial t} dt + \frac{\partial L}{\partial q^j} dq^j + \frac{\partial L}{\partial \dot{q}^j} d\dot{q}^j \right] \quad (77)
 \end{aligned}$$

Using that

$$\begin{aligned}
 dt \wedge dt &= 0 & d i_Y \theta_L &= 2a^0 \left[\frac{\partial L}{\partial t} dt + \dot{q}^i \frac{\partial L}{\partial q^i} dt + \ddot{q}^i \frac{\partial L}{\partial \dot{q}^i} dt \right] \quad (79) \\
 dq^j \wedge dt &= \dot{q}^j dt \wedge dt = 0 & d\theta_L &= \frac{\partial L}{\partial t} dt \wedge dt + \frac{\partial L}{\partial q^j} dq^j \wedge dt + \frac{\partial L}{\partial \dot{q}^j} d\dot{q}^j + L dt \\
 d\dot{q}^j \wedge dt &= \ddot{q}^j dt \wedge dt = 0
 \end{aligned}$$

Substituting (78) in (77), we obtain

$$\begin{aligned}
 i_Y d\theta_L = a^0 & \left[\frac{\partial^2 L}{\partial t^2} dt \wedge dt + \frac{\partial L}{\partial t} dt - \frac{\partial L}{\partial t} dt + \frac{\partial^2 L}{\partial t \partial q^j} dq^j \wedge dt - \frac{\partial L}{\partial q^j} dq^j + \frac{\partial^2 L}{\partial t \partial \dot{q}^j} d\dot{q}^j \wedge dt - \frac{\partial L}{\partial \dot{q}^j} d\dot{q}^j \right. \\
 & + \dot{q}^i \frac{\partial^2 L}{\partial q^i \partial q^j} dq^j \wedge dt + \dot{q}^i \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} d\dot{q}^j \wedge dt + \ddot{q}^i \frac{\partial^2 L}{\partial \dot{q}^i \partial t} dt \wedge dt + \ddot{q}^i \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} dq^j \wedge dt \\
 & \left. + \ddot{q}^i \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} d\dot{q}^j \wedge dt + \dot{q}^i \frac{\partial L}{\partial q^j} \delta_j^i dt + \ddot{q}^i \frac{\partial L}{\partial \dot{q}^j} \delta_j^i dt + \frac{\partial L}{\partial t} dt + \dot{q}^i \frac{\partial L}{\partial q^i} dt + \ddot{q}^i \frac{\partial L}{\partial \dot{q}^i} dt + L \right] \quad (80)
 \end{aligned}$$

Using (78), (80) gives

$$T^{00} = H = \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L \quad (85)$$

$$i_Y d\theta_L = a^0 \left[\frac{\partial L}{\partial t} dt + \dot{q}^i \frac{\partial L}{\partial q^i} dt + \ddot{q}^i \frac{\partial L}{\partial \dot{q}^i} dt + L \right] \quad (81)$$

Gather (79) and (80) in (29)

$$d i_Y \theta_L - i_Y d\theta_L = a^0 \left[\frac{\partial L}{\partial t} dt + \dot{q}^i \frac{\partial L}{\partial q^i} dt + \ddot{q}^i \frac{\partial L}{\partial \dot{q}^i} dt - L \right] \quad (82)$$

Using (13) and (19), (82) becomes

$$d i_X \theta_L - i_X d\theta_L = a^0 dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L \right) = dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i + L \delta t \right) \quad (83)$$

By identification the members of (83), we find

$$\begin{aligned}
 \delta t &= a^0 \\
 \delta q^i &= a^0 \dot{q}^i = \dot{q}^i \delta t
 \end{aligned} \quad (84)$$

The conserved current T^{00} appearing in (84) is the density of the Hamiltonian

Here, in our model, the Lagrangian one treated is explicitly dependent on time $\left(\frac{\partial L}{\partial t} \neq 0\right)$ and a translation in time along the base leads to a conservation of energy which is the integral density of the Hamiltonian

$$P^0 = \int T^{00} dt = \int \left(\frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L \right) dt$$

Equation (85) has already been found in [35] with the condition that the Lagrangian is not explicit in time $\left(\frac{\partial L}{\partial t} = 0\right)$

B. A Translation along the Fiber

To a translation of coordinates on the fiber, we associate the following application:

$$\begin{aligned}
 \varphi : R &\rightarrow R \\
 t &\rightarrow \varphi(t) = t \quad \text{and}
 \end{aligned}$$

$$\begin{aligned} \tilde{\varphi} : M &\rightarrow M \\ q^i(t) &\rightarrow q'^i(t) = q^i(t) + a^i \end{aligned} \quad (86)$$

$$\begin{aligned} \Phi = \varphi \otimes \tilde{\varphi} : R \times M &\rightarrow R \times M \\ \Phi(t, q^i(t)) &= (t' = t, q'^i(t) = q^i(t) + a^i) \end{aligned} \quad (87)$$

$$\begin{aligned} \Phi^* : R \times TM &\rightarrow R \times TM \\ \Phi^*(t, q^i(t), \dot{q}^i(t)) &= (t' = t, q'^i(t) = q^i(t) + a^i, \dot{q}'^i(t) = \dot{q}^i(t)) \end{aligned} \quad (88)$$

$$\begin{aligned} \Phi^* \theta_L &= \tilde{\theta}_L(t, q'^i(t), \dot{q}'^i(t)) = \\ &\theta_L(t, q^i(t), \dot{q}^i(t)) + \frac{\partial \theta_L}{\partial q^i} dq^i + \frac{\partial \theta_L}{\partial \dot{q}^i} d\dot{q}^i \end{aligned}$$

$$\tilde{\theta}_L(t, q'^i(t), \dot{q}'^i(t)) - \theta_L(t, q^i(t), \dot{q}^i(t)) = a^i \frac{\partial \theta_L}{\partial q^i} = i_Y \theta_L$$

The vector field associated to this infinitesimal transformation

$$Y = a^i \frac{\partial}{\partial q^i} \quad (89)$$

The differential “d” defined on the TM

$$d = \frac{\partial}{\partial q^j} dq^j + \frac{\partial}{\partial \dot{q}^j} d\dot{q}^j \quad (90)$$

$$d\theta_L = \frac{\partial L}{\partial q^j} dq^j \wedge dt + \frac{\partial L}{\partial \dot{q}^j} d\dot{q}^j \wedge dt$$

$$i_Y d\theta_L = a^i \left[\frac{\partial^2 L}{\partial q^i \partial q^j} dq^j \wedge dt + \frac{\partial L}{\partial q^i} \delta_i^j dt + \frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j} d\dot{q}^j \wedge dt \right] \quad (91)$$

$$i_Y \theta_L = a^i \frac{\partial L}{\partial q^i} dt$$

$$d i_Y \theta_L = a^i \left[\frac{\partial^2 L}{\partial q^j \partial q^i} dq^j \wedge dt + \frac{\partial L}{\partial \dot{q}^j \partial q^i} d\dot{q}^j \wedge dt \right] \quad (92)$$

Inserting (91) and (92) in (29), we get

$$d i_Y \theta_L - i_Y d\theta_L = -a^i \frac{\partial L}{\partial q^i} dt \quad (93)$$

Using (13) and (19), (93) becomes

$$\begin{aligned} d i_Y \theta_L - i_Y d\theta_L &= -a^i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} dt = \\ -a^i dt \frac{d}{dt} (P_i) &= dt \frac{d}{dt} (\frac{\partial L}{\partial \dot{q}^i} \delta q^i + L \delta t) \end{aligned} \quad (94)$$

By identification the members of (94), we find

$$\begin{aligned} \delta q^i &= a^i \\ \delta t &= 0 \end{aligned} \quad (95)$$

Using the Lagrangian (71), the conserved current T_i^0 appearing in (94) is the momentum vector $P_i = c_{st} \forall t$ obtained during a translation along the fiber, and the result has already been found in [35] by supposing that the Lagrange function L does not explicitly involve the coordinate q^i (i.e. $\frac{\partial L}{\partial q^i} = 0$),

which means that is a cyclic coordinate.

C. A Rotation in the Fiber

Consider now an infinitesimal rotation in the fiber: the space $M = R^3$ above each point of the database: time. This transformation is an element $R_i^j = \exp(\delta \omega^i) =$

$\delta_i^j + \delta \omega_i^j / \delta \omega_i^j \ll 1$ of the group denoted $SO(3)$ which is associated to the following diffeomorphisms:

$$\begin{aligned} \varphi : R &\rightarrow R \\ t &\rightarrow \varphi(t) = t \end{aligned}$$

and

$$\begin{aligned} \tilde{\varphi} : M &\rightarrow M \\ q^i(t) &\rightarrow q'^i(t) = \exp(\delta \omega^{ij}) q_j(t) \quad / \delta \omega^{ij} = \text{const in time} \end{aligned} \quad (96)$$

$$\begin{aligned} \Phi &\equiv \varphi \otimes \tilde{\varphi} : R \times M \rightarrow R \times M \\ \Phi(t, q^i(t)) &= (t' = t, q'^i(t) = \exp(\delta \omega^{ij}) q_j(t)) \end{aligned} \quad (97)$$

The rotation is infinitesimal, and the variation in space coordinates

$$q'^i - q^i = \delta q^i = \delta \omega^{ij} q_j \quad / \quad \delta \omega^{ij} \ll 1 \quad (98)$$

$$\begin{aligned} \Phi^* : R \times TM &\rightarrow R \times TM \\ \Phi^*(t, q^i(t), \dot{q}^i(t)) &= (t' = t, q'^i(t) = \exp(\delta \omega^{ij}) q_j(t), \dot{q}'^i(t) = \exp(\delta \omega^{ij}) \dot{q}_j(t)) \end{aligned} \quad (99)$$

$$\Phi^* \theta_L = \tilde{\theta}_L(t, q'^i(t), \dot{q}'^i(t)) = \theta_L(t, q^i(t), \dot{q}^i(t)) + \frac{\partial \theta_L}{\partial q^i} dq^i + \frac{\partial \theta_L}{\partial \dot{q}^i} d\dot{q}^i \quad (100)$$

Inserting (99) in (100), it becomes

$$\begin{aligned} \Phi^* \theta_L &= \tilde{\theta}_L(t, q^i(t), \dot{q}^i(t)) - \theta_L(t, q^i(t), \dot{q}^i(t)) && \text{Contracting (69) by (101)} \\ &= \delta \omega^{ij} \left[q_j \frac{\partial}{\partial q^i} + \dot{q}_j \frac{\partial}{\partial \dot{q}^i} \right] \theta_L = i_Y \theta_L && i_Y \theta_L = \delta \omega^{ij} \left(q_j \frac{\partial L}{\partial q^i} + \dot{q}_j \frac{\partial L}{\partial \dot{q}^i} \right) dt \\ Y &= \delta \omega^{ij} \left(q_j \frac{\partial}{\partial q^i} + \dot{q}_j \frac{\partial}{\partial \dot{q}^i} \right) && \text{Using (90)} \end{aligned} \quad (101)$$

$$\begin{aligned} d i_Y \theta_L &= \delta \omega^{ij} \left[\delta_i^j \frac{\partial L}{\partial q^i} dq^j \wedge dt + q_j \frac{\partial^2 L}{\partial q^i \partial q^j} dq^i \wedge dq^j + \dot{q}_j \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} dq^i \wedge dt \right. \\ &\quad \left. + q_j \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} d\dot{q}^j \wedge dt + \delta_i^j \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^j \wedge dt + \dot{q}_j \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} d\dot{q}^i \wedge dt \right] \end{aligned} \quad (102)$$

$$d\theta_L = \frac{\partial L}{\partial q^i} dq^i \wedge dt + \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i \wedge dt$$

$$\begin{aligned} i_Y d\theta_L &= \delta \omega^{ij} \left[q_j \frac{\partial^2 L}{\partial q^i \partial q^j} dq^j \wedge dt + q_j \frac{\partial L}{\partial q^i} \delta_i^j dt + q_j \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} d\dot{q}^j \wedge dt \right. \\ &\quad \left. + q_j \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} d\dot{q}^j \wedge dt + \dot{q}_j \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} d\dot{q}^j \wedge dt + \dot{q}_j \frac{\partial L}{\partial \dot{q}^i} \delta_i^j dt \right] \end{aligned} \quad (103)$$

Putting (102) and (103) in (29), we obtain

$$d i_Y \theta_L - i_Y d\theta_L = \delta \omega^{ij} \left[\frac{\partial L}{\partial q^i} dq^j \wedge dt + \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^j \wedge dt - q_j \frac{\partial L}{\partial q^i} dt - \dot{q}_j \frac{\partial L}{\partial \dot{q}^i} dt \right] \quad (104)$$

Inserting (98) in (105), we get

$$\begin{aligned} d i_Y \theta_L - i_Y d\theta_L &= \delta \omega^{ij} \left[\frac{\partial L}{\partial q^i} \delta \omega^{jl} q_l dt + \frac{\partial L}{\partial \dot{q}^i} \delta \omega^{jl} \dot{q}_l dt - q_j \frac{\partial L}{\partial q^i} dt - \dot{q}_j \frac{\partial L}{\partial \dot{q}^i} dt \right] \\ &= \delta \omega^{ij} \delta \omega^{jl} \left[\frac{\partial L}{\partial q^i} q_l + \frac{\partial L}{\partial \dot{q}^i} \dot{q}_l \right] dt - \delta \omega^{ij} \left(q_j \frac{\partial L}{\partial q^i} dt - \dot{q}_j \frac{\partial L}{\partial \dot{q}^i} dt \right) \end{aligned} \quad (105)$$

The rotation is infinitesimal (i.e.: $\delta \omega^{ij} \ll 1$), and (105) gives

$$d i_Y \theta_L - i_Y d\theta_L = \delta \omega^{ij} \left(-q_j \frac{\partial L}{\partial q^i} dt - \dot{q}_j \frac{\partial L}{\partial \dot{q}^i} dt \right) \quad (106)$$

$$\begin{aligned} d i_Y \theta_L - i_Y d\theta_L &= \delta \omega^{ij} \left(-q_j \frac{\partial^2 L}{\partial t \partial \dot{q}^i} dt - \dot{q}_j \frac{\partial L}{\partial \dot{q}^i} dt \right) \\ &= -dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} (\delta \omega^{ij} q_j) \right) = -dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right) \end{aligned} \quad (107)$$

Using (13) and (20), (106) becomes

By identification of the two last terms in (107), we find

$$\delta q^i = \delta \omega^{ij} q_j \quad (108)$$

By using the skew symmetry of $\delta \omega^{ij}$

$$\begin{aligned} d i_Y \theta_L - i_Y d\theta_L &= -\frac{1}{2} \delta \omega^{ij} dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} q_j - \frac{\partial L}{\partial \dot{q}^j} q_i \right) = -\frac{1}{2} \delta \omega^{ij} dt \frac{d}{dt} (p_i q_j - p_j q_i) \\ &= -dt \frac{d}{dt} \left[\delta \omega^{ij} \times \frac{1}{2} (q_i p_j - q_j p_i) \right] = \bar{M}^{ext} dt \end{aligned} \quad (109)$$

The conserved current in (109) between brackets, will be identified to the angular momentum \vec{L} in mechanics whose components are:

$$L_{ij} = \frac{1}{2}(q_i p_j - q_j p_i) \quad (110)$$

This will be contracted

$$\vec{L} = \vec{q} \wedge \vec{p}$$

and \vec{M}^{ext} is only the moment of the external forces which can contribute to the rotation of the system.

In our calculation, the system is invariant in an arbitrary rotation about any axis in the space R^3 (i.e. we did not impose conditions), which is the case of an isolated system $\vec{M}^{ext} = 0$, it follows that the angular momentum \vec{L} is a constant of movement, this result has been already found in [35].

VII. CONCLUSION

In this work, the multisymplectic model of the fiber bundle [22], the Poincaré-Cartan form has been greatly simplified using the SOPDE condition for multi-vector fields verifying Euler equations. This allows to expand easily the vector fields associated to infinitesimal symmetries which gave successfully the Noether currents for classical fields, in particular, those called Gauge fields and relativistic mechanical fields that are in good agreement with the results already provided by the analytical method. The remarks that can be made are that our geometric model [22] also allowed us firstly to distinguish the angular moments of the Gauge field obtained during a transformation while these moments are gathered in a single expression and are obtained during a rotation in the Minkowsky space by the analytic method.

Secondly, the Lagrangian of the mechanics that has been treated in our calculation, is explicit in time and q^i ; there was no need to pose the conditions $\left(\frac{\partial L}{\partial t} = 0\right)$ and $\left(\frac{\partial L}{\partial q^i} = 0\right)$ respectively as other works. The currents obtained naturally from the transformations are respectively the energy of the system and the momentum.

And finally, we remark that these symmetries have been classified naturally according to the construction of the fiber bundle [22]. The Noether currents associated to the transformations along the fibers do not depend on the base coordinates unlike those associated moving along the base space.

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