Limit State of Heterogeneous Smart Structures under Unknown Cyclic Loading

M. Chen, S-Q. Zhang, X. Wang, D. Tate

***Abstract*—**This paper presents a numerical solution, namely limit and shakedown analysis, to predict the safety state of smart structures made of heterogeneous materials under unknown cyclic loadings, for instance, the flexure hinge in the micro-positioning stage driven by piezoelectric actuator. In combination of homogenization theory and finite-element method (FEM), the safety evaluation problem is converted to a large-scale nonlinear optimization programming for an acceptable bounded loading as the design reference. Furthermore, a general numerical scheme integrated with the FEM and interior-point-algorithm based optimization tool is developed, which makes the practical application possible.

***Keywords*—**Limit state, shakedown analysis, homogenization, heterogeneous structure.

I. Introduction

SMART structures are usually partly composed with components made of heterogeneous materials which provide feedback for a controlled loop. They are widely used in some advanced engineering fields, like the flexure hinge in the micro-positioning stage driven by a piezoelectric actuator. Those components are usually suffer as a result of cyclic loadings, yet are varying with time and amplitude. The local deformation will tend to lead stress concentration and fatigue failure. Direct method, namely limit and shakedown analysis, represent the most convenient tools to determine the loading bearing capacity of structures subjected to varying loadings [1]. In combination of the homogenization theory and FEM, this paper presents a general numerical approach to deal with the homogenized parameters prediction of heterogeneous material for further structure safety analysis, which may largely reduce the design expense either in material or structure level.

II. Direct Methods Applied to Composites

A. Multiscale Analysis

For heterogeneous media, two different scales are adopted, as shown in Fig. 1: the macroscopic (or global) scale system $\boldsymbol{x}$ and the mesoscopic (or local) scale system $\boldsymbol{\xi}$. The link between them is homogenization theory [2], whereas θ is a small scale parameter which determines the size of the Representative Volume Element (RVE). It plays an important role in studying the heterogeneous material, especially for non-uniform structures.

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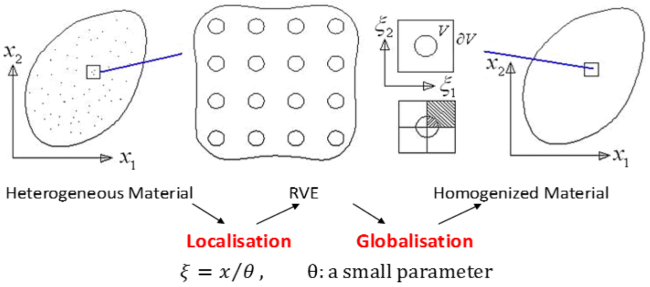


Fig. 1 Illustration of homogenization theory

The macroscopic stress $\boldsymbol{\Sigma}$ and strain $\boldsymbol{E}$ and their counterparts in microscopic level stratify the relationship,

$$\boldsymbol{\Sigma} = \frac{1}{V}\int_V \boldsymbol{\sigma}(\xi)dV = \langle\boldsymbol{\sigma}(\xi)\rangle \tag{1}$$

$$\boldsymbol{E} = \frac{1}{V}\int_V \boldsymbol{\varepsilon}(\xi)dV = \langle\boldsymbol{\varepsilon}(\xi)\rangle \tag{2}$$

Here $\langle\cdot\rangle$ stands for the averaging operator. V is the volume of RVE. The local strain $\boldsymbol{\varepsilon}$ can be decomposed into two parts: the average value $\boldsymbol{E}$ and the fluctuating part $\boldsymbol{\varepsilon}^*$,

$$\boldsymbol{\varepsilon} = \boldsymbol{E} + \boldsymbol{\varepsilon}^* \tag{3}$$

Over RVE, it becomes

$$\langle\boldsymbol{\varepsilon}^*\rangle = 0 \tag{4}$$

Therefore, one obtains

$$\boldsymbol{u} = \boldsymbol{E}\cdot\boldsymbol{\xi} + \boldsymbol{u}^* \qquad n \tag{5}$$

in which $\boldsymbol{u}^*$ is the fluctuating displacement. Referring to (4) and (5), the numerical implementation of boundary condition of RVE is:

$$u_i' - u_i + u_i^d = 0 \tag{6}$$

in which u_i' and u_i are the displacements of relative opposite periodic node pairs, while u_i^d is the displacement of the dummy node in the macroscopic level [3], as illustrated in Fig. 2. The deformed RVE should fully satisfy the periodic conditions, as shown in Fig. 3.

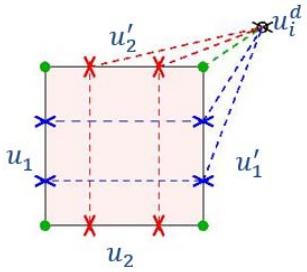


Fig. 2 Implementation of boundary condition

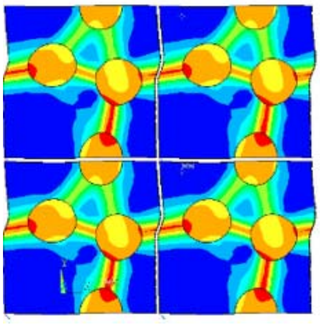


Fig. 3 Numerical illustration of deformed RVE

B. Static Limit and Shakedown Formulation

In static limit and shakedown theory, the structure shakedown will occur, if there exists a loading factor α>1 and a time-independent residual stress field $\boldsymbol{\rho}$, whose superposition with elastic stress field $\boldsymbol{\sigma^e}$ does not exceed the material yield condition $F \le 0$ at any time $t > 0$ and at all points in volume,

$$\boldsymbol{F}[\alpha\boldsymbol{\sigma^e} + \boldsymbol{\rho}, \boldsymbol{\sigma_Y}] \le 0 \tag{7}$$

$\boldsymbol{\rho}$ is the assumed the variables to be optimized for maximal load factor α. Limit analysis is a special case of shakedown analysis, when the loads increase proportionally. For heterogeneous media, the macroscopic stress is decomposed into elastic part and residual part. The residual stress field satisfies the periodicity condition,

$$\frac{1}{V}\int_V \boldsymbol{\rho} dV = 0 \tag{8}$$

The periodicity conditions for elastic stress filed $\boldsymbol{\sigma^e}$ and residual stress field $\boldsymbol{\rho}$ in RVE are given in the following equations, respectively [4],

$$\boldsymbol{\mathcal{P}}^e \begin{cases} \mathrm{div}\, \boldsymbol{\sigma}^e = 0 & \text{in } V \\ \boldsymbol{\sigma}^e = \boldsymbol{d}:(\boldsymbol{E} + \boldsymbol{\varepsilon}^*) & \text{in } V \\ \boldsymbol{\sigma}^e \cdot \boldsymbol{n} & \text{anti}-\text{periodic on } \partial V \\ \boldsymbol{\varepsilon}^* & \text{periodic on } \partial V \\ \langle \boldsymbol{\varepsilon} \rangle = \boldsymbol{E} \ \text{ or } \langle \boldsymbol{\sigma} \rangle = \boldsymbol{\Sigma} \end{cases} \tag{9}$$

$$\boldsymbol{\mathcal{P}}^{res} \begin{cases} \mathrm{div}\, \boldsymbol{\rho} = \boldsymbol{0} & \text{in } V \\ \boldsymbol{\rho} \cdot \boldsymbol{n} & \text{anti}-\text{periodic on } \partial V \\ \langle \boldsymbol{\varepsilon}^* \rangle = \boldsymbol{0} \end{cases} \tag{10}$$

III. Numerical Implementation

Based on the principle of virtual work and periodicity condition (8), the work done by the local stress and strain can be expressed as the global stress and strain,

$$\int_V \{\delta\boldsymbol{\varepsilon}\}^T \{\alpha\boldsymbol{\sigma}^e + \boldsymbol{\rho}\}\, dV = V\, \boldsymbol{\Sigma}: \delta\boldsymbol{E} \tag{11}$$

The purely elastic reference solution $\boldsymbol{\sigma^e}$ is calculated for each loading vertex P_k by means of conventional FE-Analysis, like ANSYS, ABAQUS.

After the finite element discretization of the residual stress is filed, the shakedown problem can be formulated as a mathematical programming:

$$\max \alpha$$
$$\begin{cases} [\boldsymbol{C}]\{\boldsymbol{\rho}\} = 0 \\ F[\alpha\boldsymbol{\sigma}_i^e(P_k) + \boldsymbol{\rho}_i, \sigma_{Yi}] \le 0 \\ i \in [1, NGS], k \in [1, 2^n] \end{cases} \tag{12}$$

$[\boldsymbol{C}]$ is the equilibrium matrix, derived from (8). NGS is the total number of Gaussian points, NGS is related to stress vector at each Gaussian point and the number of nodes. n is the number of independent loads, for plane loads $n = 2$, which means, for Limit Analysis, $k = 1$, loads increase proportionally; for Shakedown Analysis, there are four load vertices. F here is the von Mises yield criterion used for individual material phase in this work.

For composite materials, due to the shear stress in the interface of two phases, the 8-node solid element with linear shape functions may not overcome the "shear lock". A 20-node solid element with 2nd order shape functions leads to a much more precise result, but cause a huge number of variables in the further mathematical programming. Here, the 8-node non-conforming element [6] is proved to be a feasible solution, since additional nonlinear equations are included in the shape functions.

$$\begin{cases} u = \sum_{i=1}^{8} N_i u_i + \alpha_1(1-r^2) + \alpha_2(1-s^2) + \alpha_3(1-t^2) \\ v = \sum_{i=1}^{8} N_i v_i + \alpha_4(1-r^2) + \alpha_5(1-s^2) + \alpha_6(1-t^2) \\ w = \sum_{i=1}^{8} N_i w_i + \alpha_7(1-r^2) + \alpha_8(1-s^2) + \alpha_9(1-t^2) \end{cases} \tag{13}$$

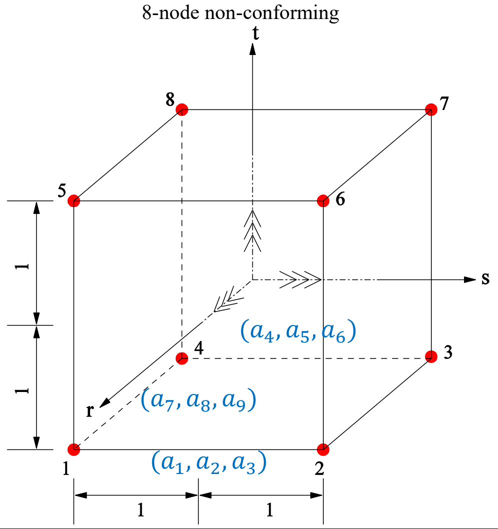
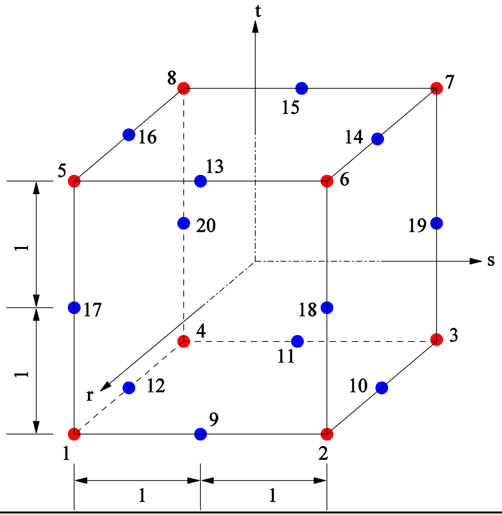
$\{u, v, w\}$ is the nodal displacement, N_i $(i = 1, \dots, 8)$ is the shape function matrix, and a_i $(i = 1, \dots, 9)$ is the additional degree of freedom.

The programming scales using different elements are present in Table I. The number of variables, as well as the number of constraints is reduced largely by using an 8-node non-conforming element.

The application of static shakedown approach on a real composite structure or structural element usually leads to a large-scale nonlinear optimization problem. There are many optimization algorithms and corresponding software packages, like LANCELOT [7], which is based on an augmented Lagrangian method, and IPDCA (Interior Point with DC regularization Algorithm), which is based on interior-point method and especially designed for shakedown problems [8].

The efficiency of IPDCA was proved in [9], but it was not suitable for composite materials. In this work, AMPL+IPOPT are adopted as the numerical solver. AMPL, an algebraic modeling language for mathematical programming, may deal with the linear and nonlinear optimization problem with discrete or continuous variables. IPOPT, short for "Interior Point OPTimizer", is an open software library for large-scale nonlinear optimization of continuous systems.

TABLE I
COMPARISON OF PROGRAMMING SCALES FOR A SIMPLE NUMERICAL EXAMPLE WITH 100 SOLID ELEMENTS

	8-node non-conforming	20-node second order
Element		
Nr. Elem	100	100
Nr. Var.	**4801**	**162001**
Nr. Eq.	1626	2403
Nr. InEq.	800	2700

To make the application on real industrial structure possible, a numerical platform, integrating with Finite Element Software and Optimization Solver, was developed under the framework of MATLAB, as shown in Fig. 4.

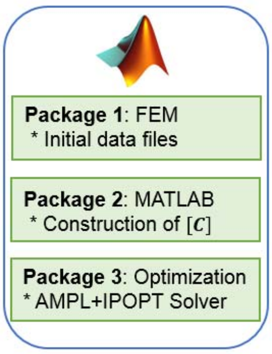


Fig. 4 Illustration of Numerical Platform

IV. FAILURE CRITERION PREDICTION

Based on the homogenization theory, three states of the composites during the failure are defined:

– Onset of plasticity

$$\boldsymbol{\Sigma}_{EL} = \alpha_{EL}\langle\boldsymbol{\sigma}^{e}\rangle \tag{14}$$

– Shakedown state

$$\boldsymbol{\Sigma}_{SD} = \alpha_{SD}\langle\boldsymbol{\sigma}^{e}\rangle \tag{15}$$

– Limit state

$$\boldsymbol{\Sigma}_{LM} = \alpha_{LM}\langle\boldsymbol{\sigma}^{e}\rangle \tag{16}$$

As the flowchart Fig. 5 shows, under the assumption that each phase of the composites is elastic-perfectly plastic, the obtained homogenized limit domain can predict the yield strength, while the predicted strength from shakedown domain is the fatigue limit, which means the material will never fail below that cyclic load. In this work, the yield strength was discussed firstly [5].

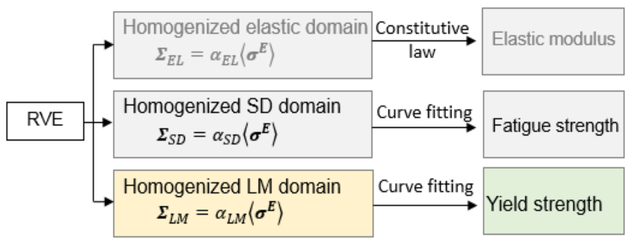


Fig. 5 Flowchart for prediction of homogenized properties

There are two ways to derive the yield criterion: to find the best fitted mathematical formulation or to identify the limit domain using existing criteria. Here, we take a continuously unidirectional fiber reinforced composites as a numerical example, which can be treated as transversely homogeneous material. Therefore, the Hill's yield criterion can be adopted to fit the limit domain.

From (16), the homogenized limit stress components Σ_{ij} depend on the orientation of the coordinate system. Nevertheless, there are certain invariants, i.e. the principle stresses, associated with every tensor. They can be determined through the characteristic equation:

$$\boldsymbol{\sigma}^3 - I_1\boldsymbol{\sigma}^2 + I_2\boldsymbol{\sigma} + I_3 = 0 \quad (17)$$

I_1, I_2 and I_3 are the first, second and third stress invariants, respectively. The Hill's yield criterion is written as:

$$F(\sigma_2 - \sigma_3)^2 + G(\sigma_3 - \sigma_1)^2 + H(\sigma_1 - \sigma_2)^2 = 1 \quad (18)$$

with $F = \frac{1}{2}\left(\frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2}\right)$, $G = \frac{1}{2}\left(\frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2}\right)$ and $H = \frac{1}{2}\left(\frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}\right)$. X, Y and Z are the axial strength of the orthotropic material. For transversely homogeneous material, with the assumption $Y = Z$, i.e. $G = H$. The line $\sigma_1 = \sigma_2 = \sigma_3$ is defined as the hydrostatic axis, the plane $\sigma_1 + \sigma_2 + \sigma_3 = 0$ is named as π −plane. Hill's yield surface in the principle stress coordinate is represented by an ellipse column around the hydrostatic axis.

Let $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$ are the original principle and transformed coordinate systems, respectively, as shown in Fig. 6, z_2 is the hydrostatic axis.

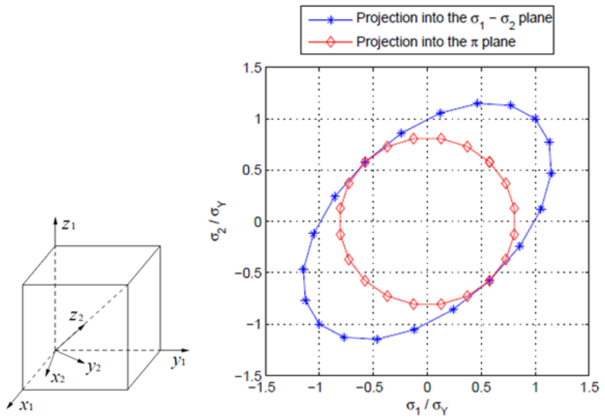


Fig. 6 Illustration of two coordinate systems and an example of the projection for von Mises yield criterion into π −plane

The obtained principle stresses $\{\sigma_1, \sigma_2, \sigma_3\}$ in $\{x_1, y_1, z_1\}$ coordinate system can be transformed to $\{\gamma_1, \gamma_2, \gamma_3\}$ in $\{x_2, y_2, z_2\}$ system under:

$$\boldsymbol{\sigma} = \mathbf{T}\boldsymbol{\gamma} \quad (19)$$

T is the rotation matrix [5]. Finally, the projection of Hill's criterion into π-plane is an ellipse:

$$(F + 1.866H + 0.134G)\,\gamma_1^2 + (F + 0.134H + 1.866G)\,\gamma_2^2 + (G - 2F + H)\,\gamma_1\gamma_2 = 1 \quad (20)$$

For transversely homogeneous material, (20) can be written as:

$$(F + 2H)\,\gamma_1^2 + (F + 2H)\,\gamma_2^2 + (2H - 2F)\,\gamma_1\gamma_2 = 1 \quad (21)$$

There are two parameters to identify here in (21).

An ellipse with the coordinate origin as the center can be expressed parametrically in the trace of a point $\{x(t),\ y(t)\}$:

$$x(t) = a\cos(t)\cos(\psi) - b\sin(t)\sin(\psi) \quad (22)$$
$$y(t) = a\cos(t)\cos(\psi) + b\sin(t)\sin(\psi)$$

ψ is the angle between the x-axix and the major axis of the ellipse, here $\psi = \pi/4$. Parameter t varies in $[0, 2\pi]$, a and b represent the major and minor radii, respectively.

After a series deduction, the two parameters to identify in (21) are reformulated in variables a and b:

$$H = 1/3a^2 \quad (23)$$
$$F = 1/2b^2 - 1/6a^2$$

For the general orthotropic material, there are three parameters to identify.

V. Numerical Illustration

In this paper, a simple square patterned unidirectional fiber reinforced periodic metal matrix composites is used to verify the accuracy of the proposed approach. The RVE is shown in Fig. 7, with perfect interface.

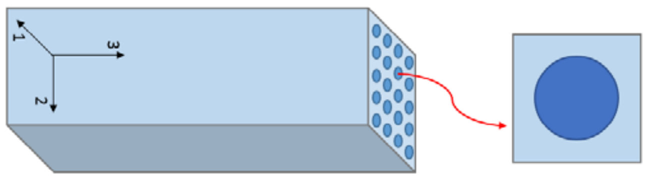


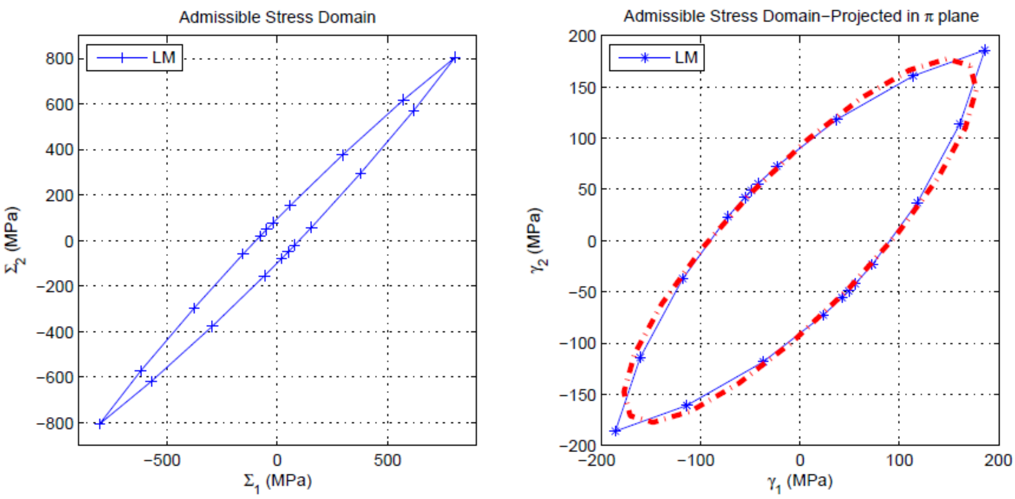
Fig. 7 Illustration of Periodic Composites and the RVE

The fiber ratio is 40%. The material properties of each phase are given in Table I, elastic-perfectly plastic, with the assumption that each phase is isotropic.

TABLE II
Units for Magnetic Properties

Properties	Matrix (Al)	Fiber (Al_2O_3)
E (GPa)	70	370
v	0.3	0.3
σ_Y (MPa)	80	2000
σ_U (MPa)	120	

The homogenized limit principle stresses ($\boldsymbol{\Sigma}_1$-$\boldsymbol{\Sigma}_2$), as shown in Fig. 8 (a), is projected into π −plane. Based on the least square fitting method, the curve fitted using Hill's yield criterion was plotted in dash line in Fig. 8 (b). The value of the major axis a and minor axis b will be obtained. Follow (23), the homogenized axial strength is 296.18 MPa, equally $3.70\sigma_Y^{\mathrm{m}}$, while the homogenized transverse strength is 98.5 MPa, equally $1.23\sigma_Y^{\mathrm{m}}$, in which σ_Y^{m} is the yield strength of the matrix.



(a) (b)

Fig. 8 Yield criterion fitting (a) Homogenized limit principle stresses ($\boldsymbol{\Sigma}_1$-$\boldsymbol{\Sigma}_2$) (b) Hill's yield criterion fitting in π −plane

The numerical results match the physical phenomenon understanding, the axial direction is reinforced largely by the unidirectional fiber, yet the transverse direction is slightly reinforced.

VI. Conclusion

The work in this paper addresses to predict the homogenized parameters of heterogeneous materials that compose smart structures. Making good use of the advantage, which is that direct approach has no requirement for loading evolution information, and thus, over RVE, under the assumption of elastic-perfectly material model for each phase, the shakedown load can be regarded as the fatigue strength while the limit load is treated as the yield strength of the global material. A simple periodic composite material is tested numerically, which proves the feasibility of the proposed model for design reference; however, the engineering accuracy still needs further experimental confirmation.

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