Skew Cyclic Codes over $F_q+uF_q+\ldots+u^{k-1}F_q$

Jing Li, Xiuli Li

***Abstract*—**This paper studies a special class of linear codes, called skew cyclic codes, over the ring $R= F_q+uF_q+\ldots+u^{k-1}F_q$, where q is a prime power. A Gray map ϕ from R to F_q and a Gray map ϕ' from R^n to F^n_q are defined, as well as an automorphism Θ over R. It is proved that the images of skew cyclic codes over R under map ϕ' and Θ are cyclic codes over F_q, and they still keep the dual relation.

***Keywords*—**Skew cyclic code, gray map, automophism, cyclic code.

I. Introduction

In recent years, the study of coding theory on finite chain has attracted the attention of many scholars. Reference [1] shows cyclic codes of odd length and self-dual codes over ring F_2+uF_2. The structure and weight of the cyclic code of arbitrary length over Z_2+uZ_2 and $Z_2+uZ_2+u^2Z_2$ has been given in [2]. Reference [3] shows skew codes over F_4+vF_4 ($v^2=v$), and shows the relationship between the cyclic codes and the cyclic codes over the ring F_2+vF_2 and F_4, by defining the Gray map.

As a finite ring in more general sense, the research of the structure of cyclic codes, cyclic codes and quasi cyclic codes over the ring $R=F_q+uF_q+\cdots+u^{k-1}F_q$ has aroused the interest of many people. Reference [4] provides the structure and ideal over the ring $F_q+uF_q+\cdots+u^{k-1}F_q$ length $p^s n$ where p,n are coprime, and obtains the direct sum and spectral representation (MS polynomial) of the cyclic codes over the ring by using the discrete Fourier transform and inverse isomorphism. According to [5], the structure and the number of codewords of all $(u\lambda-1)$ - cyclic codes with length p^e over finite chain ring $F_q+uF_q+\cdots+u^{k-1}F_q$ are generated by finite ring theory. Reference [6] studies the Gray image of constacyclic codes over finite chain rings; it is proved that the Gray image of arbitrary cyclic codes over finite chain rings is equivalent to quasi cyclic codes over finite fields. Reference [7] shows quasi cyclic codes over the ring $F_p+uF_p+\cdots+u^{k-1}F_p$, and establishes the relation between cyclic codes over $F_p+uF_p+\cdots+u^{k-1}F_p$ and quasi cyclic codes over F_p. By using the torsion codes of arbitrary $(1+\lambda u)$-length constacyclic codes over $R=F_{p^m}[u]/<u^k>$, the bound of homogeneous distance of these constacyclic codes is obtained in [8], and a new Gray map is defined to establish the relation between the constacyclic codes over R and the linear codes over F_{p^m}, then some optimal linear codes are constructed.

This paper will study the properties of skew cyclic codes over $R=F_q+uF_q+\cdots+u^{k-1}F_q$ where q is prime power.

II. Large Basic Knowledge

$R=F_q+uF_q+\cdots+u^{k-1}F_q$ is a finite ring where $q=p^m$, p is arbitrary prime, and m is positive integer. Any element c in the ring R can be represented uniquely by $c=r_0(c)+ur_1(c)+\cdots+u^{k-1}r_{k-1}(c)$ where $r_i(c)\in F_q$, $0\le i\le k-1$.

A subset C of the ring R is called a code over R, in which the element is called a codeword. And a linear cyclic code length n over R can be considered as a R-submodule of R^n.

There are two forms to express these elements in C the first one is $c=(c_0,c_1,\cdots,c_{n-1})\in C$ in vector form, another one is $f(x)=c_0+c_1x+\cdots+c_{n-1}x^{n-1}\in C$ in functional form.

Define the new Grey map ϕ as follows:

$$\phi: R\to F_q^k$$

$$\phi(r_0+ur_1+\cdots+u^{k-1}r_{k-1})=(r_0,r_0+r_1,r_0+r_1+r_2,\cdots,r_0+r_1+\cdots+r_{k-1})$$

Thus, there is another Grey map ϕ' which is derived as:

$$\phi': R^n\to F_q^{kn}$$

$$\begin{aligned}\phi'(c_0,c_1,\cdots,c_{n-1})&=(\phi(c_0),\phi(c_1),\cdots,\phi(c_{n-1}))\\&=(r_{0,0},r_{0,0}+r_{1,0},r_{0,0}+r_{1,0}+r_{2,0},\cdots,r_{0,0}+r_{1,0}+\cdots+r_{k-1,0},\\&\quad r_{0,1},r_{0,1}+r_{1,1},r_{0,1}+r_{1,1}+r_{2,1},\cdots,r_{0,1}+r_{1,1}+\cdots+r_{k-1,1},\\&\quad\cdots\cdots\\&\quad r_{0,n-1},r_{0,n-1}+r_{1,n-1},r_{0,n-1}+r_{1,n-1}+r_{2,n-1},\cdots,r_{0,n-1}+r_{1,n-1}+\cdots+r_{k-1,n-1})\end{aligned}$$

The Hamming weight of codeword $c=(c_0,c_1,\cdots,c_{n-1})$ in R is defined as $w_H(c)=\sum_{i=0}^{n-1}w_H(c_i)$, where

$$w_H(c_i)=\begin{cases}1, c_i\ne 0\\0, c_i=0\end{cases},\ 0\le i\le n-1.$$

The Hamming distance of code C is defined as

Jing Li is with the School of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao, China (corresponding author, phone: 18765279986; e-mail: 672092400@qq.com).

Xiuli Li is with Qingdao University of Science and Technology, Qingdao, China (e-mail:lixiuli2004@tom.com).

$$d_H(C) = \min d_H(c, c'),$$

where $\forall c, c' \in C$, $c \neq c'$, $d_H(c, c') = w_H(c - c')$.

We define the Lee weight of codeword $c = (c_0, c_1, \cdots, c_{n-1})$ in R as $w_L(c) = \sum_{i=0}^{n-1} w_H(\phi(c_i))$, where $w_H(\phi(c_i))$ is Hamming weight of $\phi(c_i)$. We also define the Lee distance between c and c' as $d_L(C) = \min d_L(c, c')$, where $\forall c, c' \in C$, $c \neq c'$, $d_L(c, c') = w_L(c - c')$.

Obviously, the Gray map ϕ' is an isometric mapping from R^n (Lee distance) to F_q^{kn} (Hamming distance).

Theorem 1. If C is $[n, M]$ linear code over R and $d_L(C) = d$, then $\phi'(C)$ is $[nk, M]$ linear code over F_q and $d_H(\phi'(C)) = d$.

Proof. $d_L(C) = d_H(\phi'(C))$ is known. It can be seen easily that the length of $\phi'(C)$ is nk. Next, it needs to prove that ϕ' keeps linear operation.

Let $c = (c_0, c_1, \cdots, c_{n-1})$, $e = (e_0, e_1, \cdots, e_{n-1}) \in R^n$, when $0 \le i < n-1$, there are

$$c_i = r_0(c_i) + ur_1(c_i) + \cdots + u^{k-1} r_{k-1}(c_i)$$

$$e_i = r_0(e_i) + ur_1(e_i) + \cdots + u^{k-1} r_{k-1}(e_i)$$

Thus,

$$\begin{aligned}
&\phi'(c+e) \\
&= (\phi'(c_0+e_0), \phi'(c_1+e_1), \cdots, \phi'(c_{n-1}+e_{n-1})) \\
&= (r_0(c_0+e_0), r_0(c_0+e_0) + r_1(c_0+e_0), \cdots, \\
&\quad r_0(c_0+e_0) + r_1(c_0+e_0) + \cdots + r_{k-1}(c_0+e_0), \cdots, \\
&\quad r_0(c_{n-1}+e_{n-1}), r_0(c_{n-1}+e_{n-1}) + r_1(c_{n-1}+e_{n-1}), \cdots, \\
&\quad r_0(c_{n-1}+e_{n-1}) + r_1(c_{n-1}+e_{n-1}) + \cdots + r_{k-1}(c_{n-1}+e_{n-1})) \\
&= (r_0(c_0), r_0(c_0) + r_1(c_0), \cdots, r_0(c_0) + r_1(c_0) + \cdots + r_k(c_0), \cdots\cdots \\
&\quad r_0(c_{n-1}), r_0(c_{n-1}) + r_1(c_{n-1}), \cdots, r_0(c_{n-1}) + r_1(c_{n-1}) + \cdots + r_k(c_{n-1})) \\
&\quad + (r_0(e_0), r_0(e_0) + r_1(e_0), \cdots, r_0(e_0) + r_1(e_0) + \cdots + r_k(e_0), \cdots\cdots \\
&\quad r_0(e_{n-1}), r_0(e_{n-1}) + r_1(e_{n-1}), \cdots, r_0(e_{n-1}) + r_1(e_{n-1}) + \cdots + r_k(e_{n-1})) \\
&= \phi'(c_0, c_1, \cdots, c_{n-1}) + \phi'(e_0, e_1, \cdots, e_{n-1}) \\
&= \phi'(c) + \phi'(e)
\end{aligned}$$

If $\lambda \in F_q$, $c \in R$, then

$$\phi'(\lambda c) = \phi'(\lambda c_0, \lambda c_1, \cdots, \lambda c_{n-1}) = \lambda \phi'(c_0, c_1, \cdots, c_{n-1}) = \lambda \phi'(c)$$

So, ϕ' keeps linear operation and ϕ' is a bijection. Thus, the number of codewords in C and $\phi'(C)$ is the same. This gives the proof.

Now, define a ring automorphism θ as follows

$$\begin{aligned}
\theta(c) &= \theta(r_0 + ur_1 + \cdots + u^{k-1} r_{k-1}) \\
&= r_0 + u^{k-1} r_1 + u^{k-2} r_2 + \cdots + u^2 r_{k-2} + ur_{k-1}
\end{aligned}$$

for all $c = r_0(c) + ur_1(c) + \cdots + u^{k-1} r_{k-1}(c)$ in R. One can verify that θ is an automorphism and $\theta^2(a) = a$ for any $a \in R$. This implies that θ is an automorphism with order 2.

A ring like

$$R[x, \theta] = \{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} : a_i \in R, 0 \le i \le n-1, n \in N\}$$

is called skew polynomial ring. For a given automorphism θ of R, the set $R[x, \theta]$ of formal polynomials forms a ring under usual addition of polynomial and where multiplication is defined using the rule $(ax^i) * (bx^j) = a\theta^i(b) x^{i+j}$.

Let $f(x) = \sum_{i=0}^{s} f_i x^i$, $g(x) = \sum_{i=0}^{t} g_i x^i$, where f_i and g_i are units of R, then there exist unique polynomials $u(x)$ and $v(x)$ of $R[x, \theta]$ which make $g(x) = u(x) * f(x) + v(x)$ establish where $v(x) = 0$ or $\deg(v(x)) < \deg(f(x))$. When $v(x) = 0$, $f(x)$ is called the right divisor of $g(x)$; that is, $f(x)$ right divides $g(x)$ exactly.

Let $R_n = R[x, \theta] / (x^n - 1)$, define multiplication from left as

$$r(x) * (f(x) + (x^n - 1)) = r(x) * f(x) + (x^n - 1),$$

where $f(x) + (x^n - 1)$ is element of R_n, and $r(x) \in R[x, \theta]$.

For any $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$ in R^n, the inner product is defined as $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$. Let C be linear code over R, the dual code of C is $C^\perp = \{x \in R^n \mid \langle x, c \rangle = 0, \forall c \in C\}$. A code C is called self-dual code if $C = C^\perp$.

Definition 1. A subset C of R^n is called a quasi-cyclic code of length N ($N = ns$) if C satisfies the following conditions:

(1) C is a R-submodule of R^n;

(2) If

$$\begin{aligned}
c = (&c_{0,0}, c_{0,1}, \cdots, c_{0,n-1}, | c_{1,0}, c_{1,1}, \cdots, c_{1,n-1}, \cdots, | \\
&c_{s-1,0}, c_{s-1,1}, \cdots, c_{s-1,n-1}) \in C
\end{aligned},$$

then

$$\varphi_n(c) = \left(c_{s-1,0}, c_{s-1,1}, \cdots, c_{s-1,n-1}, \left|c_{0,0}, c_{0,1}, \cdots, c_{0,n-1}, \right| \cdots, \left|c_{s-2,0}, c_{s-2,1}, \cdots, c_{s-2,n-1}\right) \in C\right.$$

Particularly, C is cyclic code when $n=1$.

Definition 2. A subset C of R^n is called a skew cyclic code of length n if C satisfies the following conditions:

(1) C is a R^n-submodule of R^n;

(2) If $c=(c_0, c_1, \cdots, c_{n-1}) \in C$, then

$$\varphi_\theta(c) = \left(\theta(c_{n-1}), \theta(c_0), \cdots, \theta(c_{n-2})\right) \in C.$$

III. Construction

Theorem 1. The center of $R[x,\theta]$ is $F_q\left[x^2\right]$.

Proof. The subring of the elements of R that are fixed by θ is F_q. Since θ is an automorphism with order 2, for any $a \in R$, there is $\left(x^{2i}\right) * a = \theta^{2i}(a) x^{2i} = \left(\theta^2\right)^i (a) x^{2i} = a x^{2i}$. Thus x^{2i} is in the center of $R[x,\theta]$. This implies that any $f(x) = \varepsilon_0 + \varepsilon_1 x^2 + \varepsilon_2 x^4 + \cdots + \varepsilon_s x^{2s}$ is a center element with $\varepsilon_i \in F_q$, $0 \le i \le s$.

Conversely, let $Z\left(R[x,\theta]\right)$ be the center of R, so $f(x) * a = a * f(x)$ for any $f(x) \in Z\left(R[x,\theta]\right)$ and any $a \in R$. Since $f(x) = \varepsilon_0 + \varepsilon_1 x + \varepsilon_2 x^2 + \cdots + \varepsilon_n x^n$ for $\varepsilon_i \in F_q$, $0 \le i \le n$, there are

$$\begin{gathered} f(x) * a = a * f(x), \\ \left(\varepsilon_0 + \varepsilon_1 x + \varepsilon_2 x^2 + \cdots + \varepsilon_n x^n\right) * a = a * \left(\varepsilon_0 + \varepsilon_1 x + \varepsilon_2 x^2 + \cdots + \varepsilon_n x^n\right), \\ a\varepsilon_0 + \varepsilon_1 \theta(a) x + \varepsilon_2 \theta^2(a) x^2 + \cdots + \varepsilon_n \theta^n(a) x^n \\ = a\varepsilon_0 + a\varepsilon_1 x + a\varepsilon_2 x^2 + \cdots + a\varepsilon_n x^n \end{gathered}.$$

It is known that $\left|\langle\theta\rangle\right| = 2$, so there are $\varepsilon_i x^i * a = a\varepsilon_i x^i$ when i is even, and $\varepsilon_i x^i * a \ne a\varepsilon_i x^i$ when i is odd. Hence, any $f(x) = \varepsilon_0 + \varepsilon_1 x + \varepsilon_2 x^2 + \cdots + \varepsilon_n x^n$ of $Z\left(R[x,\theta]\right)$ only exists even power term of x, that is $f(x) = \varepsilon_0 + \varepsilon_2 x^2 + \varepsilon_4 x^4 + \cdots + \varepsilon_{2s} x^{2s}$. Thus, any element of center is in $F_q\left[x^2\right]$. This gives the proof.

Theorem 2. Let $R_n = R[x,\theta] / (x^n - 1)$, a code C in R_n is a skew cyclic code if and only if C is a left $R[x,\theta]$-submodule of the left $R[x,\theta]$ module R_n.

Proof. Suppose C is θ-cyclic code, so $\left(\theta(c_{n-1}), \theta(c_0), \cdots, \theta(c_{n-2})\right) \in C$ for $c = (c_0, c_1, \cdots, c_{n-1}) \in C$, that is for any $f(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} \in C$, there is $x * f(x) \in C$. Next, $g(x) * f(x) \in C$ for any $g(x) \in R[x,\theta]$ from linear property, then C is a left $R[x,\theta]$-submodule of the left $R[x,\theta]$ module R_n.

Now suppose that C is a left $R[x,\theta]$-submodule of the left $R[x,\theta]$ module R_n, so

$$x * f(x) = \left(\theta(c_{n-1}), \theta(c_0), \cdots, \theta(c_{n-2})\right) \in C$$

for any $f(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} \in C$, this implies that

$$\left(\theta(c_{n-1}), \theta(c_0), \cdots, \theta(c_{n-2})\right) \in C$$

for any $c = (c_0, c_1, \cdots, c_{n-1}) \in C$. Thus, C is θ-cyclic code. This gives the proof.

Theorem 3. Let C be a θ-cyclic code in $R_n = R[x,\theta] / (x^n - 1)$ and let $f(x)$ be a polynomial in C of minimal degree. If $f(x)$ is monic polynomial, then $C = \langle f(x) \rangle$ where $f(x)$ is a right division of $x^n - 1$.

Proof. Suppose $g(x) = u(x) * f(x) + v(x)$ for any $g(x) \in C$ where $v(x) = 0$ or $\deg(v(x)) < \deg(f(x))$. Since $f(x) \in C$, then $v(x) = g(x) - u(x) * f(x) \in C$. Also since $f(x)$ is polynomial in C of minimal degree, we have $v(x) = 0$, this implies that $C = \langle f(x) \rangle$.

Since the θ-cyclic codes over R_n and its left $R[x,\theta]$-submodule are corresponding one by one, thus $f(x)$ is a right division of $x^n - 1$. This gives the proof.

Theorem 4. Let n be even. If codes C over R are θ-cyclic codes, so is its dual codes $C^\perp$.

Proof. Let $c = (c_0, c_1, \cdots, c_{n-1}) \in C^\perp$, $a = (a_0, a_1, \cdots, a_{n-1}) \in C$, so $\langle c, a \rangle = 0$ for any c and a. Since C is θ-cyclic codes, then $\left(\theta(a_{n-1}), \theta(a_0), \cdots, \theta(a_{n-2})\right) \in C$. Thus,

$$\left(\theta^{n-1}(a_1), \theta^{n-1}(a_2), \cdots, \theta^{n-1}(a_0)\right) \in C.$$

Therefore,

$$\begin{gathered} c_0 \theta^{n-1}(a_1) + c_1 \theta^{n-1}(a_2) + \cdots + c_{n-1} \theta^{n-1}(a_0) = 0 \\ \theta(c_0) \theta^n(a_1) + \theta(c_1) \theta^n(a_2) + \cdots + \theta(c_{n-1}) \theta^n(a_0) = 0 \end{gathered}$$

It is known n is even, then have $\theta^n(a_j) = a_j$ for $a_j \in R$. Hence,

$$a_0 \theta(c_{n-1}) + a_1 \theta(c_0) + \cdots + a_{n-1} \theta(c_{n-2}) = 0$$

by transforming formulas. Thus

$(\theta(c_{n-1}),\theta(c_0),\cdots,\theta(c_{n-2}))\in C^{\perp}$ and $C^{\perp}$ is θ-cyclic codes. This gives the proof.

Theorem 5. Let n be even, then the θ-cyclic codes C generated by a monic right divisior $g(x)$ over R are cyclic codes if and only if the coefficients of $g(x)$ are elements of F_q.

Proof. Let $g(x)=x^m+\sum_{i=0}^{m-1}g_ix^i$ where $g_i\in F_q$. So, $\theta(g_i)=g_i$, $x*g(x)=g(x)*x$ from definition of θ, thus the θ-cyclic codes C generated by a monic right divisior $g(x)$ over R are cyclic codes.

Let the θ-cyclic codes C generated by $g(x)$ over R be cyclic codes, then $x*g(x)\in C$, $g(x)*x\in C$. Hence,

$$\begin{aligned}u(x)&=x*g(x)-g(x)*x\\&=(\theta(g_0)-g_0)x+(\theta(g_1)-g_1)x^2+\cdots+(\theta(g_{m-1})-g_{m-1})x^m\in C\end{aligned}$$

Since $g(x)$ is the right divisor of $u(x)$, there exists $u(x)=t*g(x)=tx^m+tg_{m-1}x^{m-1}+\cdots+tg_1x+g_0$ where t is a constant. Comparing two formulas of $u(x)$, then

$$\begin{gathered}\theta(g_{m-1})-g_{m-1}=t,\\\theta(g_{m-2})-g_{m-2}=tg_{m-1},\\\vdots\\\theta(g_1)-g_1=tg_2,\\\theta(g_0)-g_0=tg_1,\\tg_0=0.\end{gathered}$$

If $t=0$, then $u(x)=0$, this theorem is proved. If $t\neq 0$, $g_0=0$, it shows that $g_i=0$, $1\le i\le m-1$, hence $g(x)=x^m$, $\theta(g_i)=g_i$, $0\le i\le m$. Thus, the coefficients of $g(x)$ are elements of F_q. This gives the proof.

Theorem 6. Let n be odd and C be a skew cyclic code of length n over R. Then, C is equivalent to cyclic code of length n over R.

Proof. Since n is odd, $\gcd(2,n)=1$. Hence, there exist integers b,c such that $2b+cn=1$. Thus, $2b=1-cn=1+zn$ where $z>0$.

Let $a(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}\in C$, we have

$$\begin{aligned}&x^{2b}*a(x)\\&=\theta^{2b}(a_0)x^{1+zn}+\theta^{2b}(a_1)x^{2+zn}+\cdots+\theta^{2b}(a_{n-1})x^{n+zn}\\&=a_{n-1}+a_0x+a_1x^2+\cdots+a_{n-2}x^{n-1}\in C\end{aligned}$$

Thus, C is cyclic code of length n over R. This gives the proof.

Corollary 1. If C is a skew cyclic code of length n over R, then the Gray image $\phi'(C)$ of C is equivalent to quasi-cyclic code of length nk over F_q.

Proof. Let $(c_0,c_1,\cdots,c_{n-1})\in C$, each element c in C can be expressed as $c=r_0(c)+ur_1(c)+\cdots+u^{k-1}r_{k-1}(c)$. It is known that $\varphi_\theta(c)=(\theta(c_{n-1}),\theta(c_0),\cdots,\theta(c_{n-2}))\in C$, that is $\varphi_\theta(C)=C$. For ϕ', $\phi'(\varphi_\theta(C))=\phi'(C)$. From Theorem 1 in Section II, $\phi'(C)$ is linear code over F_q and ϕ' keeps linear operation, so

$$\begin{aligned}&\phi'(\varphi_\theta(c_0,c_1,\cdots,c_{n-1}))\\&=\phi'(\theta(c_{n-1}),\theta(c_0),\cdots,\theta(c_{n-2}))\\&=\big(\phi(r_{n-1,0}+u^{k-1}r_{n-1,1}+u^{k-2}r_{n-1,2}+\cdots+ur_{n-1,k-1}),\\&\quad\phi(r_{0,0}+u^{k-1}r_{0,1}+u^{k-2}r_{0,2}+\cdots+ur_{0,k-1}),\cdots\cdots,\\&\quad\phi(r_{n-2,0}+u^{k-1}r_{n-2,1}+u^{k-2}r_{n-1,2}+\cdots+ur_{n-2,k-1})\big)\\&=\big(r_{n-1,0},r_{n-1,0}+r_{n-1,1},r_{n-1,0}+r_{n-1,1}+r_{n-1,2},\cdots,\\&\quad r_{n-1,0}+r_{n-1,1}+r_{n-1,2}+\cdots+r_{n-1,k-1}\big|r_{0,0},r_{0,0}+r_{0,1},r_{0,0}+r_{0,1}+r_{0,2},\\&\quad\cdots,r_{0,0}+r_{0,1}+r_{0,2}+\cdots+r_{0,k-1}\big|,\cdots\cdots,\big|r_{n-2,0},r_{n-2,0}+r_{n-2,1},\\&\quad r_{n-2,0}+r_{n-2,1}+r_{n-2,2},\cdots,r_{n-2,0}+r_{n-2,1}+r_{n-2,2}+\cdots+r_{n-2,k-1}\big)\end{aligned}$$

Now, each section of right side of equation is a cyclic code of length nk. Thus, $\phi'(C)$ is quasi-cyclic code of length nk over F_q. This gives the proof.

REFERENCES

[1] Bonnecaze A, Udaya P. Cyclic codes and self-dual codes over F_2+uF_2 (J). IEEE Trans Inform Theory, 1999, 45(4):1250-1255.

[2] Abualr T, Siap T. Cyclic codes over the rings Z_2+uZ_2 and $Z_2+uZ_2+u^2Z_2$ (J). Designs Codes and Cryptography, 2007, 42(3):273-287.

[3] Xu Xian-qi, Zhu Shi-xin. Skew cyclic codes over the ring F_4+vF_4 (J). Journal of Hefei University of Technology, 2011, 34(9):1429-1432.

[4] Han Mu, Ye You-pei, Zhu Shi-xin. Cyclic codes over $F_p+uF_p+\cdots+u^{k-1}F_p$ with length p^sn (J). Information Sciences, 2011, 181(4):926-934.

[5] Zhu Shi-xin, Li Ping, Wu Bo. A Class of Repeated-root Constacyclic Codes over the Ring $F_q+uF_q+\cdots+u^{k-1}F_q$ (J). Journal of Electronics & Information Technology, 2008, 30(6):1394-1396.

[6] Qian Jian-fa, Ma Wen-ping. Constacyclic and cyclic codes over finite chain rings(J). Science Direct, 2009, 16(3):122-125.

[7] Li Fu-lin, Zhu Shi-xin. Quasi-cyclic codes over $F_p+uF_p+\cdots+u^kF_p$ (J). Journal of Hefei University of Technology, 2009, 32(11):1766-1768.

[8] Ding Jian, Li Hong-ju. Construction of optimal codes with Homogenuous distance(J). Journal of University of Science and Technology of China, 2015, 45(7): 588-593.