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A Partially Accelerated Life Test Planning with Competing Risks and Linear Degradation Path under Tampered Failure Rate Model

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Abstract**—**In this paper, we propose a method to model the relationship between failure time and degradation for a simple step stress test where underlying degradation path is linear and different causes of failure are possible. It is assumed that the intensity function depends only on the degradation value. No assumptions are made about the distribution of the failure times. A simple step-stress test is used to shorten failure time of products and a tampered failure rate (TFR) model is proposed to describe the effect of the changing stress on the intensities. We assume that some of the products that fail during the test have a cause of failure that is only known to belong to a certain subset of all possible failures. This case is known as masking. In the presence of masking, the maximum likelihood estimates (MLEs) of the model parameters are obtained through an expectation-maximization (EM) algorithm by treating the causes of failure as missing values. The effect of incomplete information on the estimation of parameters is studied through a Monte-Carlo simulation. Finally, a real example is analyzed to illustrate the application of the proposed methods.

Keywords**—**Expectation-maximization (EM) algorithm, cause of failure, intensity, linear degradation path, masked data, reliability function.

I. Introduction

TODAY'S products are designed to work without failures for years. Hence, traditional life testing is not an appropriate method to collect information on the failure of such products. In such situations, an accelerated life testing (ALT) is needed to shorten failure times. For more details on ALTs and Bayesian analysis of ALTs, the reader is referred to [8]-[10]. However, as it is mentioned in [8], for highly reliable products, little information about reliability is provided by life testings. In certain situations, it is possible to measure the product degradation along with failure time and even find a degradation model. In the context of joint modeling of failure times and degradation under ALT and competing risks, [7] proposed a modeling approach based on the Cumulative Exposure (CE) model. They derived MLEs of the model parameters and studied their asymptotic properties.

In practice, products are exposed to different causes of failure. The duration to the first failure is defined to be the failure time of the product. When a product fails, we generally observe the cause of failure. However, in certain situations, the cause of failure for some products is not observed while their failure times are observed. This is known as masking. The problem of estimation of model parameters in the presence of masked causes of failure using EM algorithm was considered by [4], [5], [11]. They treated the cause of failure as missing information. The statistical inference for masked data in step-stress ALT has been considered by [6], while the works regarding the Bayesian analyses of this problem have been studied by [12], [13].

In this work, we propose an approach to model failure times and linear degradation data under simple step-stress test and different causes of failure, where masked cause of failure is possible. Our model is based on the tampered failure rate (TFR) model proposed by [3]. We estimate the model parameters using EM algorithm based on a complete-data likelihood function by treating the cause of failure as missing values. We also study the effect of masked causes of failure on the estimation of parameters through a Monte-Carlo simulation.

The rest of the paper is organized as follows. In Section I.A, we present the assumptions and notation. The model is described in Section I.B. In Section I.C, complete-data likelihood function is derived and EM algorithm is applied to obtain the MLEs. Numerical results are presented in Section II.

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A. Assumptions and Notation

We make the following assumptions:

Assumption 1: A simple step-stress life test with two stress levels S_0 and $S_1 (S_0 < S_1)$ is considered. The test is conducted as follows. The test units are initially placed under normal stress S_0, and the stress level remains at S_0 until the changing point of stress τ. Then, the stress is increased to higher stress level S_1 and the test continues until all remaining units fail.

Assumption 2: The degradation path of the test unit, Z, is an increasing function of time t, and follows a linear model $Z(t) = \frac{t}{A}$, where A is a random vector with distribution function π, dependent on the nature of the unit.

Assumption 3: Multiple causes of failure are possible. The causes of failure are indexed by the integers 1 to s. For each test unit, we observe the failure time $T = \min(T^1, ..., T^s)$, where $T^j, j = 1, ..., s$ is latent failure time for j-th cause of failure. We assume that $T^1, ..., T^s$ are statistically independent. Moreover, it is assumed that the intensity function associated with the j-th cause of failure denoted by $\lambda^j(.)$, depends only on degradation. This is a practical assumption when the failure occurs due to wear, fatigue or mechanical damages. The reader is referred to [1], [2], [7].

Assumption 4: It is possible that the cause of failure is not observed for some units on test. That is, for a failed unit one of the following cases occurs: (1) the cause of failure is observed, (2) the cause of failure is not observed and it is only known that the cause of failure belongs to a subset of causes of failure $g \subset \{1, \ldots, s\}$, called masking group. Let $M = 2^s - 1$ be the number of masking groups. The corresponding masking groups are denoted by $g_1, \ldots, g_M$. Note that the masking group with a single member represents the exact cause of failure.

Assumption 5: A TFR model is proposed to relate the intensities at high level of stress to intensities at lower level of stress.

Consider that n units are on test and n_1 units fail under S_0 and n_2 units fail under S_1. For the units failed under S_0, the failure time (t), the degradation level at the moment of failure (z) and associated masking group are recorded. For these units, the value of a is obtained by $a = \frac{t}{z}$. Before putting the unfailed units under S_1, the degradation levels of units are recorded at time τ, denoted by z^τ. Then, the value of a for these units could be obtained by $a = \frac{\tau}{z^\tau}$. Next, unfailed units are subjected to S_1, and the test continuous till the failure time and associated masking groups for remaining units are observed.

Similar to [4], [5], we represent the observation for the i-th unit on test by complete data set

$$(t_i, a_i, \gamma_{ig_1}, ..., \gamma_{ig_M}, \delta_{i1}, ..., \delta_{is}), \tag{1}$$

where $\gamma_{ig}, g \in \{g_1, \ldots, g_M\}$, is the indicator that the cause of failure of the i-th unit is masked to group g. The $\delta_{ij}, j = 1, ..., s$, is the indicator that the actual cause of failure of the i-th unit is j. If the i-th unit is masked, then all $\delta_{ij}, j = 1, ..., s$, are unknown.
Given the cause of failure j, the probability that the cause of failure is masked to group g, is denoted as $P_{g|j}$ and is given by

$$\begin{aligned} P_{g|j} = P(&\text{cause of failure is masked to group } g \mid \\ &\text{actual cause of failure is } j\,), \quad j \in g, \end{aligned}$$

where $\sum_{g:j\in g} P_{g|j} = 1$.

B. Model

The proposed model is based on the TFR model that describes the effect of the changing stress on the intensity in a simple step-stress test. From the assumptions 3 and 5, we define

$$\lambda^j(z(t)) = \begin{cases} \lambda_0^j(\frac{t}{a}), & t \le \tau, \\ \lambda_1^j(\frac{t}{a}) = \alpha_j \lambda_0^j(\frac{t}{a}), & t > \tau. \end{cases} \tag{2}$$

where λ_l^j, is the intensity function corresponding to the j-th cause of failure, $j = 1, ..., s$, under the l-th, $l = 0, 1$, level of stress, and α_j is accelerated factor that depends on the cause of failure.
Let $R_l^j(.|A = a)$, be the conditional reliability function corresponding to the j-th cause of failure at the l-th level of stress given the value of A. From the assumption that the model is a TFR model, the conditional reliability function of a test unit in the presence of multiple causes of failure under simple step-stress test is expressed as

$$R^j(t|A=a) = \begin{cases} R_0^j(t|A=a) & 0 \le t < \tau; \\ R_1^j(t|A=a) = \left[R_0^j(\tau|A=a)\right]^{1-\alpha_j} \left[R_0^j(t|A=a)\right]^{\alpha_j} & \tau \le t < \infty. \end{cases}$$

C. Maximum Likelihood Function and EM Algorithm

In the proposed model, no assumptions are made about the distribution of failure times. However, we suppose that $\lambda^j(z)$ belongs to a parametric class $\lambda^j(z, \boldsymbol{\eta}_j)$, where $\boldsymbol{\eta}_j = (\eta_{1j}, ..., \eta_{mj})$ is a vector of m parameters.
Based on the complete data set in (1), the complete-data likelihood function is represented as follows:

$$\begin{aligned} L_C(\Theta) \quad &\propto \quad \left[\prod_{i=1}^{n}\prod_{j=1}^{s}\left\{\lambda_0^j(z_i, \boldsymbol{\eta}_j)\right\}^{\delta_{ij}}\right]\left[\prod_{i=1}^{n_1}\prod_{j=1}^{s} R_0^j(t_i|a_i, \boldsymbol{\eta}_j)\right] \\ &\times \quad \left[\prod_{i=1}^{n_2}\prod_{j=1}^{s} {\alpha_j}^{\delta_{ij}} \left\{R_0^j(\tau|a_i, \boldsymbol{\eta}_j)\right\}^{1-\alpha_j} \left\{R_0^j(t_i|a_i, \boldsymbol{\eta}_j)\right\}^{\alpha_j}\right] \\ &\times \quad \prod_{i=1}^{n}\prod_{j=1}^{s}\left((1 - \sum_{g\in M_j^*} P_{g|j})^{1-\sum_{g\in M_j^*}\gamma_{ig}} \prod_{g\in M_j^*} P_{g|j}^{\gamma_{ig}}\right)^{\delta_{ij}}, \end{aligned}$$

where Θ is the vector of parameters $\boldsymbol{\eta}_j$, α_j and $P_{g|j}$ and M_j^* is the set of masking groups that contain j, except $\{j\}$ for $j = 1, \ldots, s$.

Similar to [4], [5], [11], we treat the δ_{ij} as missing data and apply the EM algorithm which is a general iterative algorithm for ML estimation in incomplete data problems. The EM algorithm consists of two steps: E step and M step. In E step, the conditional expectation of the log-likelihood function given the observed data (OBS), and current estimated parameters $\Theta^{(h)}$, is obtained as follow:

$$\begin{aligned} \mathcal{Q}(\Theta|\Theta^{(h)}) \quad &= \quad E_{\Theta^{(h)}}\left[l_C(\Theta)|OBS\right] \\ &\propto \quad \sum_{i=1}^{n}\sum_{j=1}^{s} E_{\Theta^{(h)}}(\delta_{ij}|OBS)\log\left(\lambda_0^j(z_i, \boldsymbol{\eta}_j)\right) \\ &+ \quad \sum_{i=1}^{n_1}\sum_{j=1}^{s} \log\left(R_0^j(t_i|a_i, \boldsymbol{\eta}_j)\right) \\ &+ \quad \sum_{i=1}^{n_2}\sum_{j=1}^{s}\Big\{E_{\Theta^{(h)}}(\delta_{ij}|OBS)\log(\alpha_j) \\ &+ \quad (1-\alpha_j)\log\left(R_0^j(\tau|a_i, \boldsymbol{\eta}_j)\right) \\ &+ \quad \alpha_j\log\left(R_0^j(t_i|a_i, \boldsymbol{\eta}_j)\right)\Big\} \\ &+ \quad \sum_{i=1}^{n}\sum_{j=1}^{s} E_{\Theta^{(h)}}(\delta_{ij}|OBS) \\ &\quad\quad \Big\{\Big(1 - \sum_{g\in M_j^*}\gamma_{ig}\Big)\log\Big(1 - \sum_{g\in M_j^*} P_{g|j}\Big) \\ &+ \quad \sum_{g\in M_j^*}\gamma_{ig}\log P_{g|j}\Big\}. \end{aligned} \tag{3}$$

where

$$E_{\Theta^{(h)}}(\delta_{ij}|OBS) = \begin{cases} \dfrac{\lambda_l^j(z)P_{g|j}}{\sum_{j'\in g_i}\lambda_l^{j'}(z)P_{g|j'}} & j \in g \\ 0 & j \notin g, \end{cases} \tag{4}$$

and the intensity function is considered to be

$$\lambda^j(z,\theta,\nu) = (\frac{z}{\theta_j})^{\nu_j}, \quad j = 1, ..., s. \tag{5}$$

In M-step of EM algorithm, $\Theta^{(h+1)}$ is derived by maximizing the expected log likelihood function such that:

$$\mathcal{Q}(\Theta^{(h+1)}|\Theta^{(h)}) \geq \mathcal{Q}(\Theta|\Theta^{(h)}), \qquad \text{for all } \Theta.$$

Let $\mu_{ij}^{(h)} = E_{\Theta^{(h)}}(\delta_{ij}|OBS)$. Maximizing $\mathcal{Q}(\Theta|\Theta^{(h)})$ with respect to Θ yields

$$P_{g|j}^{(h+1)} = \frac{\sum_{i=1}^n \mu_{ij}^{(h)} \gamma_{ig}}{\sum_{i=1}^n \mu_{ij}^{(h)}},$$

$$(\eta_{1j}^{(h+1)}, \ldots, \eta_{mj}^{(h+1)}, \alpha_j^{(h+1)}) = \underset{(\eta_{1j},\ldots,\eta_{mj},\alpha_j)}{\text{argmax}} \mathcal{Q}(\Theta|\Theta^{(h)}),$$

for $j = 1, \ldots, s$. The reliability function from linear degradation and failure time data with power intensity (5), under normal stress S_0, could be estimated as

$$\hat{R}(t) = \int_0^\infty \exp\left(- a \sum_{j=1}^s \hat{\Lambda}_0^j(\frac{t}{a})\right) dF_A(a), \tag{6}$$

where $\hat{\Lambda}_0^j(t) = \int_0^t \lambda_0^j(u,\hat{\theta},\hat{\nu})du$. For more details the reader is referred to Haghighi and Nikulin (2010).

II. Simulation

We conducted a simulation study to assess the performance of the estimation method discussed in previous sections. We supposed that two causes of failure with the intensity function $\lambda_0^j(z,\theta,\nu) = (\frac{z}{\theta_j})^{\nu_j}, \quad j = 1,2$ are possible. The degradation path is assumed to be linear and π follows a Weibull distribution with the shape parameter a and scale parameter b. A simple step-stress test with $\tau = 67, 70, 73$ is considered. Given $a = 10.6$, $b = 4.77$, $\nu_1 = 6.883$, $\theta_1 = 23.256$, $\alpha_1 = 3.88$, $\nu_2 = 10.116$, $\theta_2 = 20$, $\alpha_2 = 2.26$, and $n = 100, 200$, the algorithm proposed by Han and Kundu (2015), has been adapted to our model in order to generate data under step-stress test with two causes of failures. We generated $B = 1000$ samples and masked the obtained data based on the different masking probabilities (MP) $(P_{g|1}, P_{g|2}) = (0,0), (0.5,0.3), (0.6,0.7)$. Then, we estimated the parameters of the model. The parameter estimates as well as the corresponding MSE are reported in Tables I and II. From the results, it is observed that the MSEs become larger as the masking probabilities increase. We also found that as τ increases, the MSE of the estimators for α_1 and α_2 increases in most cases, while the MSE of the other estimators decreases. The reason for this is that when τ increases, large numbers of failures occur before τ and small numbers of failures occur after τ, resulting in higher variability in the estimation of α_1 and α_2 and lower variability in the estimation of the other parameters. As we expected , for fixed value of τ and masking probability, when n increases, the MSEs become smaller.

III. Conclusion

In this paper, we have proposed a method to estimate the reliability of highly reliable products which are exposed to two causes of failure and have a linear degradation path. We have assumed that the latent failure times are independent while no assumptions have been made on the failure time distribution. A step stress TFR model has been proposed and the parameters of the model have been estimated through an EM algorithm. Based on the estimated parameters of intensity functions, the reliability of product has been estimated. The proposed method is easier when compared to that of [7], and leads to closed form MLEs for most parameters. The problem of masked causes of failure has been considered in the

TABLE I
Summary of Estimation Results for Model Parameters and Mean Squared Errors (in parentheses) Based on Simulated Masked Data, when $a = 10.6$, $b = 4.77$, $\nu_1 = 6.883$, $\theta_1 = 23.256$, $\alpha_1 = 3.88$, $\nu_2 = 10.116$, $\theta_2 = 20$, $\alpha_2 = 2.26$, $n = 100$

MP	Par	$\tau_1 = 67$	$\tau_2 = 70$	$\tau_3 = 73$
$(0,0)$	ν_1	7.02(1.28)	7.02(1.22)	7.05(1.22)
	θ_1	23.57(6.54)	23.50(5.25)	23.41(4.40)
	α_1	4.10(1.70)	4.08(1.60)	4.08(1.96)
	ν_2	10.43(2.50)	10.47(2.37)	10.37(1.62)
	θ_2	20.02(1.41)	19.98(1.22)	19.98(0.95)
	α_2	2.41(0.67)	2.34(0.68)	2.39(0.75)
	$P_{g\|1}$	0.01(0.001)	0.01(0.001)	0.01(0.001)
	$P_{g\|2}$	0.01(0.002)	0.01(0.002)	0.01(0.001)
$(0.5,0.3)$	ν_1	6.85(2.02)	6.85(1.81)	6.94(1.73)
	θ_1	24.87(17.07)	24.67(12.49)	24.30(9.55)
	α_1	4.39(3.83)	4.41(3.41)	4.30(3.71)
	ν_2	10.15(2.81)	10.16(2.79)	10.16(2.72)
	θ_2	19.92(1.70)	19.93(1.61)	19.95(1.40)
	α_2	2.56(0.97)	2.49(0.98)	2.51(1.12)
	$P_{g\|1}$	0.42(0.01)	0.42(0.01)	0.42(0.01)
	$P_{g\|2}$	0.42(0.02)	0.40(0.01)	0.39(0.01)
$(0.6,0.7)$	ν_1	7.04(2.57)	7.16(2.32)	7.18(2.24)
	θ_1	24.25(15.21)	23.64(9.42)	23.48(8.01)
	α_1	4.39(4.56)	4.12(3.09)	4.16(4.33)
	ν_2	10.43(5.52)	10.39(5.01)	10.32(4.68)
	θ_2	20.20(4.07)	20.22(3.41)	20.29(3.77)
	α_2	2.58(1.81)	2.55(1.86)	2.56(2.28)
	$P_{g\|1}$	0.58(0.004)	0.59(0.004)	0.59(0.004)
	$P_{g\|2}$	0.72(0.005)	0.71(0.004)	0.70(0.004)

TABLE II
Summary of Estimation Results for Model Parameters and Mean Squared Errors (in Parentheses) Based on Simulated Masked Data, when $a = 10.6$, $b = 4.77$, $\nu_1 = 6.883$, $\theta_1 = 23.256$, $\alpha_1 = 3.88$, $\nu_2 = 10.116$, $\theta_2 = 20$, $\alpha_2 = 2.26$, $n = 200$

MP	Par	$\tau_1 = 67$	$\tau_2 = 70$	$\tau_3 = 73$
$(0,0)$	ν_1	6.91(0.61)	6.92(0.53)	6.94(0.55)
	θ_1	23.47(2.52)	23.42(2.22)	23.36(2.10)
	α_1	4.01(0.73)	3.99(0.75)	3.94(0.82)
	ν_2	10.31(1.23)	10.29(1.07)	10.23(0.94)
	θ_2	19.96(0.68)	19.99(0.54)	20.00(0.46)
	α_2	2.30(0.31)	2.33(0.33)	2.34(0.37)
	$P_{g\|1}$	0.004(0.000)	0.004(0.001)	0.005(0.000)
	$P_{g\|2}$	0.006(0.000)	0.005(0.000)	0.0055(0.000)
$(0.5,0.3)$	ν_1	6.75(0.90)	6.76(0.80)	6.75(0.73)
	θ_1	24.52(6.77)	24.38(5.59)	24.31(4.69)
	α_1	4.18(1.35)	4.19(1.44)	4.26(1.64)
	ν_2	10.04(1.31)	10.05(1.11)	10.04(1.15)
	θ_2	19.87(0.88)	19.85(0.59)	19.88(0.56)
	α_2	2.43(0.43)	2.40(0.41)	2.38(0.41)
	$P_{g\|1}$	0.41(0.009)	0.42(0.008)	0.42(0.008)
	$P_{g\|2}$	0.42(0.01)	0.40(0.01)	0.39(0.009)
$(0.6,0.7)$	ν_1	6.96(1.10)	6.95(0.94)	6.95(0.91)
	θ_1	23.74(4.85)	23.62(3.73)	23.48(3.46)
	α_1	4.10(1.47)	4.07(1.43)	4.03(1.66)
	ν_2	10.12(2.20)	10.17(1.91)	10.25(2.04)
	θ_2	20.10(1.69)	20.08(1.23)	20.07(1.05)
	α_2	2.42(0.66)	2.41(0.77)	2.42(0.89)
	$P_{g\|1}$	0.58(0.002)	0.59(0.002)	0.59(0.002)
	$P_{g\|2}$	0.72(0.002)	0.71(0.002)	0.70(0.002)

model and simulation results show that the masking probabilities have an effect on the parameter estimates.

REFERENCES

[1] V. Bagdonavicius, A. Bikelis and V. Kazakevicius, " Statistical analysis of linear degradation and failure time data with multiple failure modes," *Lifetime data analysis,* vol. 1, no. 10, pp. 65-81, 2004.

[2] V. Bagdonavicius, F. Haghighi and M. Nikulin, " Statistical analysis of general degradation path model and failure time data with multiple failure modes," *Communications in statistics-Theory and methods,* vol. 34, no. 8, pp. 1771-1791, 2005.

[3] G. Bhattacharyya and Z. Soejoeti, " A tampered failure rate model for step-stress accelerated life test," *Communications in statistics- Theory and methods,* vol. 18, no. 5, pp. 1627-1643, 1989.

[4] R. V. Craiu and T. Duchesne," Inference based on the EM algorithm for the competing risk model with masked causes of failure," *Biometrika,* vol. 91, no. 3, pp. 543558, 2004.

[5] R. V. Craiu and B. Reiser," Inference for the dependent competing risks model with masked causes of failure," *Lifetime data analysis,* vol. 12, no. 1, pp. 2133, 2006.

[6] R. V. Fan and B. Wang," Accelerated Life Tests for Weibull Series Systems With Masked Data," *IEEE Transactions on Reliability,* vol. 60, no. 3, pp. 557569, 2011.

[7] F. Haghighi and S. J. Bae, " Reliability estimation from linear degradation and failure time data with competing risks under a step-stress accelerated degradation test," *IEEE Trans. Reliability,* vol.64, no. 3, pp.960-971 , 2015.

[8] M. Hamada, " Bayesian analysis of step-stress accelerated life tests and its use in planning," *Quality Engineering,* vol. 27, no. 3, pp. 276-282, 2015.

[9] W. Q. Meeker and L. A. Escobar, *Statistical methods for reliability data*. John Wiley and Sons, 1998.

[10] W. B. Nelson, *Accelerated testing: statistical models, test plans, and data analyses*. John Wiley and Sons, 2004.

[11] C. Park, " Parameter estimation of incomplete data in competing risks using the EM algorithm , " *IEEE Transactions on reliability,* vol. 54, no. 2, pp. 282-290, 2005.

[12] A. Xu, S. Basu, and Y.A. Tang, A Full Bayesian Approach for Masked Data in Step-Stress Accelerated Life Testing, *IEEE Transactions on Reliability*, vol. 63, no. 3, pp. 798–806, 2014.

[13] A. Xu, Y. Tang, and Q. Guan, Bayesian Analysis of Masked Data in Step-Stress Accelerated Life Testing, *Communications in Statistics-Simulation and Computation*, vol. 43, no. 8, pp. 2016–2030, 2014.

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