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Measurement of CES Production Functions Considering Energy as an Input

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***Abstract*—**Because of its flexibility, CES attracts much interest in economic growth and programming models, and the macroeconomics or micro-macro models. This paper focuses on the development, estimating methods of CES production function considering energy as an input. We leave for future research work of relaxing the assumption of constant returns to scale, the introduction of potential input factors, and the generalization method of the optimal nested form of multi-factor production functions.

***Keywords*—**Bias of technical change, CES production function, elasticity of substitution, energy input.

I. Introduction

THE first of 'Kaldor's stylized facts about economic growth' has been widely accepted by researchers for a long time. This means that the ratio of shares of input factors, like capital and labor stays constant over time. However, the income shares of capital and labor do not always remain stable, but are actually quite volatile [1]-[3]. Solow [4] explained that a Constant Elasticity of Substitution (CES) production function with an elasticity larger than one more easily explains sustained economic growth within the neoclassical growth model. This is mainly because capital can partly substitute for labor (increasing labor costs) and the value of the marginal product of capital in the long term is always greater than zero. References [5] and [6] conclude several reasons for the increasing interest of the CES technology, especially its potential importance for the analysis of the short-run [7]. In June 2008, the Journal of Macroeconomics devoted a special issue to CES production function to inspire new and exciting research using the family of CES production functions.

II. The Form of CES Production Function

A. The Development of CES with Two and More Input Factors

The concept of CES was introduced into economics by [8] to model 'a more general kind of national-income function'. Later, [4] proposed the endogenous growth theory as well as a new production function with constant elasticity of substitution, the CES-type production function, as shown in Table I. ρ is an indispensable parameter of CES, from which we can get the elasticity of substitution. Reference [9] improved Solow's model by defining α as the additional value of capital and adding the additional value of labor parameter β. Reference [10] laid a solid theoretical and mathematical foundation by making a generalized derivation process of CES, which has been recognized as the first rigid derivation of the CES. Reference [11] presented one form of CES production function with the capital-augmenting technical progress parameter B (the level of efficiency of the conventional inputs of capital) and labor-augmenting technical progress parameter C (the level of efficiency of the conventional inputs of labor), which provides the mathematical foundation of biased technology progress. As shown in Table I, although the forms of the above-mentioned production function are different, they can be deduced from the definition of elasticity of substitution and can be transformed into each other under given assumptions. When all is added up, the forms of CES production function with two inputs are substantially improved and each parameter has its economic interpretation and theoretical basis.

In the literature on integrated assessment models of the energy and climate, energy seems to be an indispensable production factor under separability aspects [12]-[16]. One of the recurring themes has been the nested way of various input factors and the associated extent of substitution of elasticities among energy and non-energy inputs [17].

TABLE I
Variants of CES Production Function with Two Input Factors

Authors	Model
Reference [4]	$Y=(\alpha K^{\rho}+L^{\rho})^{\frac{1}{\rho}}$
Reference [9]	$Y=(\alpha K^{-\rho}+\beta \mathrm{L}^{-\rho})^{-(\frac{1}{\rho})}$
Reference [10]	$Y=A[\delta K^{-\rho}+(1-\delta)L^{-\rho}]^{-(\frac{1}{\rho})}$
Reference [11]	$Y=[(BK)^{-\rho}+(CL)^{-\rho}]^{-(\frac{1}{\rho})}$

TABLE II
The Major Differences between The Bayes Estimation And the Traditional Estimation Methods

	Bayes estimation	Traditional estimation
Form of parameter	a random variable with a specific distribution	a fixed value
Information	prior and sample information	sample information
Distribution of error term	specific distribution	no requirements
Solving criterion	minimize the loss function	minimize the sum of squared residuals

Reference [18] found that most researchers thought (KE) L is the appropriate form of CES production function. Other input factors such as material, land, ordinary capital,

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innovation capital, unskilled labor and skilled labor are often included as an input in the production function of a final good [19]-[21].

B. The Normalized Form of CES Production Function

To examine the variation of the effects of substitution, the concept of normalization was introduced by [22], [23]. It suggests choosing the appropriate baseline values for factor income shares. The significance of normalization has been emphasized on the parameters of aggregate CES production function [24]. Later, [6] made detailed analysis about the intrinsic links between production, factor substitution and normalization. And this normalization has been successfully applied in a series of theoretical papers discussing a wide variety of topics [25], [26]. Due to the normalization of CES functions, all the parameters of the derived aggregate production function can be provided with a sound interpretation [6], [27].

Reference [23] presented the normalized method and [24] proposed the corresponding normalized system-supply method which matches the normalized CES. The main point of the normalized CES production function is to maintain other parameters constant in economic terms while only the elasticity of substitution can be variant. It means the parameters other than the elasticity have the same baseline. Thus, we need to choose the benchmark base for the output, the inputs, the income shares of each factor and the factor-augmenting technical progress, in order to ensure that variation depends only on the elasticity of substitution [6]. The typical CES with element-augmenting technical progress is given by:

$$Y_t = \{[(A_t K_t)^{-\rho_1} + (B_t L_t)^{-\rho_1}]^{\rho/\rho_1} + (C_t E_t)^{-\rho}\}^{-1/\rho} \tag{1}$$

Suppose K_0, L_0 and E_0 are the initial inputs of capital, labor and energy respectively, and r_0, w_0, p_0 are the corresponding prices for each factor at the baseline point. For (1), to satisfy the requirements of manufacturers' profits maximization ($\left.\frac{\partial Y_t}{\partial K_t}\right|_{t=0} = r_0$, $\left.\frac{\partial Y_t}{\partial L_t}\right|_{t=0} = w_0$, $\left.\frac{\partial Y_t}{\partial E_t}\right|_{t=0} = p_0$), the original element-augmenting technical progress A_0, B_0, C_0 can be defined as:

$$A_0 = (\alpha_0)^{-1/\rho_1} (\beta_0)^{-1/\rho} \frac{Y_0}{K_0} \tag{2}$$

$$B_0 = (1-\alpha_0)^{-1/\rho_1} (\beta_0)^{-1/\rho} \frac{Y_0}{L_0} \tag{3}$$

$$C_0 = (1-\beta_0)^{-1/\rho} \frac{Y_0}{E_0} \tag{4}$$

We assume that the expression of the element-augmenting technical progress takes the following form,

$$A_t = A_0 e^{\gamma_t},\ B_t = B_0 e^{\mu_t},\ C_t = C_0 e^{\nu_t} \tag{5}$$

The element-augmenting technical progress represents exponential growth as $\gamma_t = \mu_t = \nu_t = rt$. We take the average value of the initial output and inputs considering the non-linearization of CES, testing $Y_0 = \delta \overline{Y}$ (where δ is the scale factor), $K_0 = \overline{K}$, $L_0 = \overline{L}$, $E_0 = \overline{E}$. Substituting (2)-(5) into (1), the normalization of CES production function can be derived:

$$Y_t = Y_0 \{\beta_0 [\alpha_0 (e^{\gamma_t} \frac{K_t}{K_0})^{-\rho_1} + (1-\alpha_0)(e^{\gamma_t} \frac{L_t}{L_0})^{-\rho_1}]^{\rho/\rho_1} + (1-\beta_0)(e^{\gamma_t} \frac{E_t}{E_0})^{-\rho}\}^{-1/\rho} \tag{6}$$

III. Estimation of CES Production Function

In general, the nonlinear equation can be estimated by the logarithmic and linearized treatments. Here we introduce and compare four widely used methods.

A. The Kmenta Approximation

The Kmenta approximation was proposed in 1967 by [28]. To estimate the Kmenta, we expand the production function using a Taylor series expansion to remove the second-and-above-order items. Then the production function is transformed into the linear form. There are two ways to treat the linearization, one is OLS (ordinary least squares), and the other is ridge regression. When the model is collinear or lacks data, the ridge regression has more priorities than OLS. By decreasing the fitness of the model to increase the significance of the regression variables, this method usually attains lower value of R-squared and larger t-Statistics compared with OLS.

Because of its simplicity, the Kmenta approximation is widely used. However, it has two limitations. One is that technical change should be Hicks neutral. The other is that the removed items would enlarge the error in a multi-factor production function. Therefore, the Kmenta approximation is not appropriate for estimating a CES production function with more than two factors [29].

B. The Bayes Estimation

The Bayes estimation of CES production function is based on Bayes theorem, as discussed by [30]. The posterior distribution is in proportion to the product of prior distribution and sample information. Based on Bayes theorem, the integration of the general information of prior distribution with the sample information would make the posterior distribution more consistent with reality. The mathematical expression is as follows:

$$p(\mu|\mathrm{Q,K,L,E}) \propto p(\mu) \times \phi(\mu|\mathrm{Q}) \tag{7}$$

$u=(\alpha,\beta,\rho,\rho_1)$ is the set of unknown parameters, and $p(\mu)$ is the collection of the prior distribution of parameters which mainly refer to the previous studies. $\phi(\mu|Q)$ is the likelihood function obtained from the sample information. The estimated parameters of marginal distribution density function can be obtained by skew integral operation.

The Bayes estimation requires both the sample information and the prior information, while only the sample information is needed by the traditional estimation methods (such as the Kmenta approximation, the system-supply method and the first-order conditional estimation method, discussed later). These two estimation methods are fundamentally different in terms of four aspects, as shown in Table II. When the prior distributions of the parameters are unclear or weak, the Bayes estimation is not recommended. Alternatively, it would get more reliable and convergent results.

C. The System-Supply Method

The above-mentioned two methods belong to the single equation estimation. But a single equation cannot investigate the linkage between parameters, which may cause the results to deviate from reality and have systematic bias [24]. The system-supply method could address the profit maximization problem. It integrates the conditions with the initial production function to establish simultaneous equations. Hence it can be a good description of the relationship among the parameters, which would lead to more robust results. Suppose the production function is $Y_t=F(K,L,E)$, the manufacturer's conditions of profit maximization can now be written as:

$$\begin{cases} F_K=(1+u)r_t \\ F_L=(1+u)w_t \\ F_E=(1+u)p_t \end{cases} \tag{8}$$

F_K, F_L and F_E are the partial derivatives of the production function. μ refers to the price-augmenting index of the factor which is determined by the market conditions. $u=0$ reflects a perfectly competitive market and $u>0$ implies imperfect competition. Its value is decided by the price elasticity of factors. Here we assume that the manufacturers are in a state of perfect competition, which means $u=0$, and w_t, p_t and r_t are the prices of labor, energy and capital. Add the assumption $u=0$ to (8), which delivers

$$\begin{cases} \dfrac{K_t r_r}{Y_t}=\dfrac{K_t}{Y_t}F_K \\ \dfrac{L_t w_r}{Y_t}=\dfrac{L_t}{Y_t}F_L \\ \dfrac{E_t p_r}{Y_t}=\dfrac{E_t}{Y_t}F_E \end{cases} \tag{9}$$

The original production function and (9) imply the simultaneous relation:

$$\begin{cases} \ln(\dfrac{K_t r_r}{Y_t})=\ln(\dfrac{K_t}{Y_t}F_K) \\ \ln(\dfrac{L_t w_r}{Y_t})=\ln(\dfrac{L_t}{Y_t}F_L) \\ \ln(\dfrac{E_t p_r}{Y_t})=\ln(\dfrac{E_t}{Y_t}F_E) \\ \ln(Y_t)=\ln F(K,L,E) \end{cases} \tag{10}$$

D. The First-Order Conditional Estimation Method

The first-order conditional estimation method was described in detail by [31]. Here we present this approach in the nested form (KL)E of CES production function as an example:

$$Y_t=\{\alpha[\beta(A_tK_t)^{-\rho_1}+(1-\beta)(B_tL_t)^{-\rho_1}]^{\rho/\rho_1}+(1-\alpha)(C_tE_t)^{-\rho}\}^{-1/\rho} \tag{11}$$

To facilitate the calculations, we decompose (11) into two simple CES production functions:

$$Y_t=\{\alpha[Z_t]^{\frac{\sigma_{(KL)E}-1}{\sigma_{(KL)E}}}+(1-\alpha)(C_tE_t)^{\frac{\sigma_{(KL)E}-1}{\sigma_{(KL)E}}}\}^{\frac{\sigma_{(KL)E}}{\sigma_{(KL)E}-1}} \tag{12}$$

$$Z_t=[\beta(A_tK_t)^{\frac{\sigma_{KL}-1}{\sigma_{KL}}}+(1-\beta)(B_tL_t)^{\frac{\sigma_{KL}-1}{\sigma_{KL}}}]^{\frac{\sigma_{KL}}{\sigma_{KL}-1}} \tag{13}$$

When we accept the exogenous assumption of price in [32], the cost minimization of manufacturers can be decomposed into a two-stage problem: (1) In the case of given price and technology, to get the optimal use of K and L per unit of Z; (2) With Z obtained from (1), to determine the optimal demand for E and Z. And we are able to conclude a system of three equations:

$$e-y=(\sigma_{(KL)E}-1)c+\sigma_{(KL)E}(p_Y-p_E) \tag{14}$$

$$\tilde{G}_{KZ}=(\sigma_{KL}-1)a_K+\frac{\sigma_{KL}-1}{1-\sigma_{(KL)E}}\tilde{G}_{ZY}+(1-\sigma_{KL})(p_K-p_Y) \tag{15}$$

$$\tilde{G}_{LZ}=(\sigma_{KL}-1)a_L+\frac{\sigma_{KL}-1}{1-\sigma_{(KL)E}}\tilde{G}_{ZY}+(1-\sigma_{KL})(p_L-p_Y) \tag{16}$$

where $\ln I_t-\ln I_{t-1}=i$, $\ln P_I^t-\ln P_I^{t-1}=p_I$, $I=K,L,E,Z,Y$, $\ln U_t-\ln U_{t-1}=u$ where $U=A,B,C$, $\tilde{G}_{ij}=p_I+i-(p_J+j)$, $I=K,L,E$, $J=Z,Y$, representing the change of share of input I that is to derive output J. The first-order conditional estimation method substitutes the change in the combination of the factors for the change of price and quantity of intermediate inputs, in other words $p_Z+z=d\ln(P_KK+P_LL)$: (14)-(16) above can be transformed into linear form:

$$\begin{cases} u_1=\alpha_1+\beta_1v_1 \\ u_2=\alpha_2+\beta_{21}v_{21}+\beta_{22}v_{22} \\ u_3=\alpha_3+\beta_{31}v_{31}+\beta_{32}v_{32} \end{cases} \tag{17}$$

where $\beta_{22} = \beta_{32}$, $\beta_{21} = \beta_{31} = \frac{\beta_{22}}{\beta_1 - 1}$.

(1) when $\beta_{21} = \beta_{31} = 1$, the form of CES production function is KLE;

(2) when $\beta_1 = 1$, $\beta_{22} = \beta_{32} = 0$, CES production function is transformed into C-D production function;

(3) when $\frac{\alpha_2}{\beta_{22}} = \frac{\alpha_3}{\beta_{32}} = 0$, CES production function is characterized by Hicks neutral, $A = B = C = 1$.

With the estimated parameters from the linear equations, we are able to derive the elasticity of substitution as well as the rate of technical change. And it is not necessary to get the shares of input factors. Hence, this method is more flexible.

From the above discussion, we prefer the system-supply method, but it has some limitations on the range of parameters, especially for the normalized CES production function. When only the elasticity of substitution and the rate of technical change are to be estimated, the first-order condition estimation method can be applied. The Kmenta approximation might be preferable in the CES production function with two factors. The Bayes estimation is only recommended when the parameters have very strong prior information.

IV. Conclusion

Challenges to the study of the CES production function with multi-factors have brought the nested approach of input, normalization, and estimation methods to the fore. Due to the divergences of elasticities between inputs, special care should be taken to obtain nested structure as well as precise estimates of these parameter values. The normalized CES production function is increasingly applied. And the estimation methods differ according to the situation. We find that by relaxing the assumption of constant returns to scale and the introduction of potential input factors, the generalization method of the optimal nested form of multi-factor production function may enhance future research of CES.

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