

# Group Invariant Solutions of Nonlinear Time-Fractional Hyperbolic Partial Differential Equation

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**Abstract**—In this paper, we have investigated the nonlinear time-fractional hyperbolic partial differential equation (PDE) for its symmetries and invariance properties. With the application of this method, we have tried to reduce it to time-fractional ordinary differential equation (ODE) which has been further studied for exact solutions.

**Keywords**—Nonlinear time-fractional hyperbolic PDE, Lie Classical method, exact solutions.

## I. PRELIMINARIES

**P**ROBLEMS of physical interest are often translated in terms of differential equations that may turn out to be linear or nonlinear, ordinary or partial. Exact solutions of these resulting equations are of much interest both from mathematical and application points of view. On account of their applications in the disciplines of mathematics, chemistry, engineering and in almost all branches of theoretical physics, differential equations and their symmetries have retained their central role. One reason for the overall prominence of the concept of symmetry is its nativeness and its simplicity. Intuitively speaking, a symmetry is a transformation of an object leaving this object invariant. Lie symmetry analysis of differential equations provides a powerful and fundamental framework to the exploitation of systematic procedures leading to the integration by quadrature of ordinary differential equations, to the determination of invariant solutions of initial and boundary value problems, to the derivation of conservation laws, to the construction of links between different differential equations that turn out to be equivalent.

During the past three decades, Fractional Differential Equations (FDEs) have gained considerable popularity and importance and to solve such equations, the subject of Fractional Calculus [1], [2] provides several potentially useful tools. With the application of this subject, the earlier differential equations of integer order can be generalized to FDEs of non-integer order and such type of FDEs are more beneficial in explanation of nonlinear phenomenon. In spite of this, FDEs are evolved in several scientific fields such as Biology, Finance, Probability and Statistics, Chemical Physics, Optics, Rheology, Viscoelasticity etc., so to sharpen the concepts of such fields, it becomes quite important to

solve such equations. In this direction, several numerical and analytical methods have been proposed to solve such type of FDEs. Among the various available different methods, the Lie group method [3]-[5], which is based upon the study of the invariance under one-parameter Lie group of point transformations is one of the most powerful methods to determine solutions of PDEs. In the recent past there have been considerable developments in symmetry methods for differential equations as is evident by the number of research papers [13]-[17], books [3]-[5] but there is only few development of this method for FDEs is available in literature [6]-[12]. In this paper, our aim is to enhance this Lie symmetry method by applying it to Nonlinear time-fractional Hyperbolic PDE [7]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial x} \left( u(x, t) \frac{\partial u}{\partial x} \right), t > 0, x \in \mathbb{R}, 1 < \alpha \leq 2 \quad (1)$$

In [7], Sumudu decomposition method has been developed to solve this equation. We have studied (1) for the exact solutions using combination of Lie symmetry method and several other methods.

The layout of paper is as follows. In Section II, we have provided some definitions and basics of Fractional Calculus which are required to obtain symmetries of (1). Section III is devoted to the outline of Lie classical method to generate various symmetries of (1) which in turn reduces (1) to fractional ODE. Section IV contains the solution of reduced ODE. Some concluding remarks are given in Section V.

## II. DEFINITIONS

In this section, we are giving the definitions of the Riemann-Liouville fractional integrals and fractional derivatives on a finite interval of the real line (as available in several books of Fractional Calculus [1], [2]).

Let  $\Omega = [a, b]$  be a finite interval on the real axis  $\mathbb{R}$ . The Riemann-Liouville fractional integrals  $I_{a+}^\alpha f$  and  $I_{b-}^\alpha f$  of order  $\alpha \in \mathbb{C}$  are defined by

$$(I_{a+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t) dt}{(x-t)^{1-\alpha}}, (x > a; R(\alpha) > 0) \quad (2)$$

and

$$(I_{b-}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t) dt}{(t-x)^{1-\alpha}}, (x < b; R(\alpha) > 0) \quad (3)$$

where  $\Gamma(\alpha)$  is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals.

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The Riemann-Liouville fractional derivatives  $D_{a+}^\alpha y$  and  $D_{b-}^\alpha y$  of order  $\alpha \in \mathbb{C}$  are defined by

$$\left( D_{a+}^\alpha(y) \right) (x) = \left( \frac{d}{dx} \right)^n \left( I_{a+}^{n-\alpha} y \right) (x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^n \int_a^x \frac{y(t) dt}{(x-t)^{\alpha-n+1}} \quad (4)$$

and

$$\left( D_{b-}^\alpha(y) \right) (x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{-d}{dx} \right)^n \int_x^b \frac{y(t) dt}{(t-x)^{\alpha-n+1}} \quad (5)$$

### III. SYMMETRY ANALYSIS

In this section, we first determine the Lie point symmetries of (1) and then use them to reduce the equation to lower dimension equations. A Lie point symmetry of a differential equation is an invertible transformation of the dependent and independent variables that leave the equation unchanged. Determining all the symmetries of a differential equation is a formidable task. However, the Norwegian mathematician Sophus Lie realized that if we restrict ourselves to symmetries that depend continuously on a small parameter and that form a group (continuous one-parameter group of transformations), one can linearize the symmetry conditions and end up with an algorithm for calculating continuous symmetries. For a detailed study of Lie group theory the interested reader is referred to the well-known books [2]-[4]. Let us consider the Lie group of point transformations as

$$\begin{aligned} u^* &= u + \epsilon \eta(x, t, u) + O(\epsilon^2), \\ x^* &= x + \epsilon \xi(x, t, u) + O(\epsilon^2), \\ t^* &= t + \epsilon \tau(x, t, u) + O(\epsilon^2), \end{aligned} \quad (6)$$

which leaves the system (1) invariant. In other words, the transformations are such that if  $u$  is a solution of (1), then  $u^*$  is also a solution. The method for determining the symmetry group of (1) consists of finding the infinitesimals  $\eta, \xi$  and  $\tau$ , which are functions of  $x, t, u$ . Assuming that the system (1) is invariant under the transformations (6), we get the following relation from the coefficients of the first order of  $\epsilon$ :

$$\eta_\alpha^0 - 2u_x \eta^x - \eta^{xx} u - \eta u_{xx} = 0, \quad (7)$$

where  $\eta_\alpha^0, \eta^x, \eta^{xx}$  are extended (prolonged) infinitesimals acting on an enlarged space that includes all derivatives of the dependent variables and are given as:

$$\begin{aligned} \eta^x &= \eta_x + u_x \eta_u - (\xi_x + u_x \xi_u) u_x - (\tau_x + u_x \tau_u) u_t, \\ \eta^{xx} &= \eta_{xx} + u_x \eta_{xx} - (\xi_{xx} + u_x \xi_{xu}) u_x \\ &\quad - (\tau_{xx} + u_x \tau_{xu}) u_t - 2(\tau_x + u_x \tau_u) u_{xt} \\ &\quad + (\eta_{xu} + u_x \eta_{uu} - (\xi_{xu} + u_x \xi_{uu}) u_x - (\tau_{xu} + u_x \tau_{uu}) u_t) u_x \\ &\quad + u_{xx} (\eta_u - u_x \xi_u - u_t \tau_u) - 2(\xi_x + u_x \xi_u) u_{xx}, \\ \eta_\alpha^0 &= \eta_t^\alpha + (\eta_u - \alpha(\tau_t + u_t \tau_u)) u_t^\alpha - u(\eta_u)_t^\alpha \\ &\quad - \sum_{n=1}^\infty \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x) \\ &\quad + \sum_{n=1}^\infty \binom{\alpha}{n} \partial_t^n \eta_u - \binom{\alpha}{n+1} D_t^{n+1}(\tau) D_t^{\alpha-n}(u) + \dots \end{aligned} \quad (8)$$

Now using the above prolonged generalised vector fields in (7) and by equating the coefficients of various derivative terms we get a system of determining equations and on solving those equations, we get

$$\begin{aligned} \xi &= C_1 x + C_2, \\ \tau &= \frac{2C_1}{\alpha} t, \\ \eta &= 0, \end{aligned} \quad (9)$$

where  $C_1, C_2$  are arbitrary parameters. Using the characteristic equations

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}, \quad (10)$$

we get the similarity transform and similarity variable as

$$u(x, t) = f(z), \quad z = xt^{\frac{-\alpha}{2}}, \quad (11)$$

which reduces (1) to the following ODE of fractional order as:

$$\left( P_{\frac{\alpha}{2}}^{1-\alpha, \alpha} f \right) (z) - \left( \frac{df}{dz} \right)^2 - f(z) \frac{d^2 f(z)}{dz^2} = 0, \quad (12)$$

where  $\left( P_{\frac{\alpha}{2}}^{1-\alpha, \alpha} f \right) (z)$  is the Erdélyi-Kober fractional differential operator given in [18].

### IV. SOME EXACT SOLUTIONS OF (1)

In spite of symmetry analysis, we have also tried to furnish exact solutions of (1) by certain transformations. For this, if we consider

$$u(x, t) = f(z), \quad z = x + \frac{t^\alpha}{\Gamma(1+\alpha)}, \quad (13)$$

we get

$$D_t^\alpha u = \frac{df}{dz}, \quad \frac{\partial u}{\partial x} = \frac{df}{dz}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{d^2 f}{dz^2}, \quad (14)$$

On substituting (13) and (14) in (1), we get

$$\frac{df}{dz} - \left( \frac{df}{dz} \right)^2 - f(z) \frac{d^2 f}{dz^2} = 0, \quad (15)$$

On solving above equation, we get the following solutions:

$$\begin{aligned} f(z) &= z + C_1, \\ f(z) &= -C_1 \left( LambertW \left( \frac{-e^{-1}}{C_1 e^{C_1} e^{\frac{z}{C_1}}} \right) + 1 \right), \end{aligned} \quad (16)$$

where  $C_1, C_2$  are arbitrary constants. Hence, corresponding to the solutions (16), we get the following solutions of (1):

$$\begin{aligned} u(x, t) &= x + \frac{t^\alpha}{\Gamma(1+\alpha)} + C_1, \\ u(x, t) &= -C_1 \left( LambertW \left( \frac{-e^{-1}}{C_1 e^{C_1} e^{\frac{x + \frac{t^\alpha}{\Gamma(1+\alpha)}}{C_1}}} \right) + 1 \right) \end{aligned} \quad (17)$$

#### A. Analysis of Solutions

We have obtained such solutions so that one can choose arbitrary constants  $C_1, C_2, \alpha$  in suitable manner, to simulate the physical solutions governed by (1). We have also plotted the graphs of solution surfaces to understand solutions. As shown in Figs. 1-4, certain kinky solutions are obtained through (17). The behaviour of the solution has been shown for different values of  $\alpha$  in Figs. 5 and 6 that helps us to understand more the physical situations of (1).

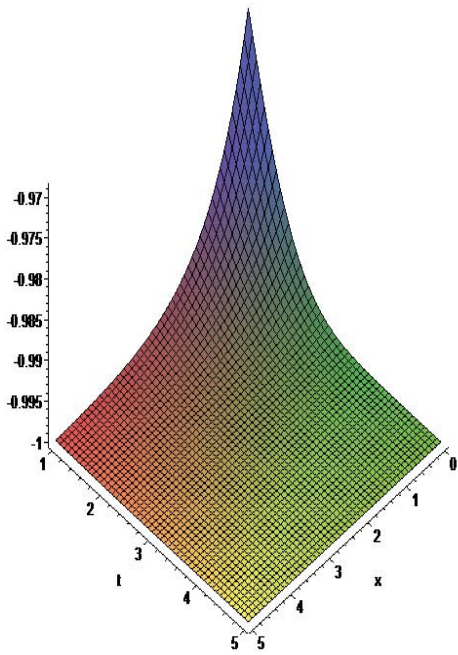


Fig. 1 Solutions (17)(ii) when  $C_1 = 1, C_2 = 2, \alpha = 2$

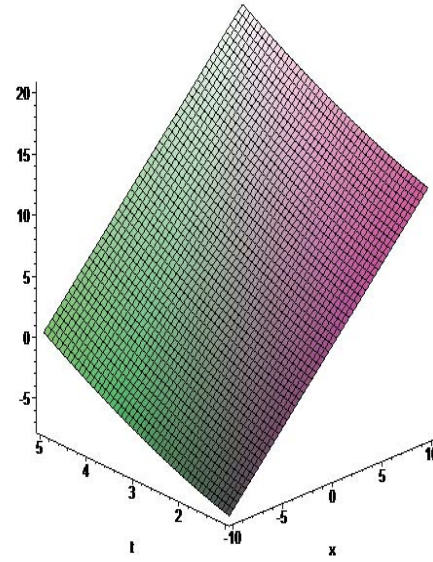


Fig. 3 Solutions (17)(i) when  $C_1 = 2, \alpha = \frac{3}{2}$

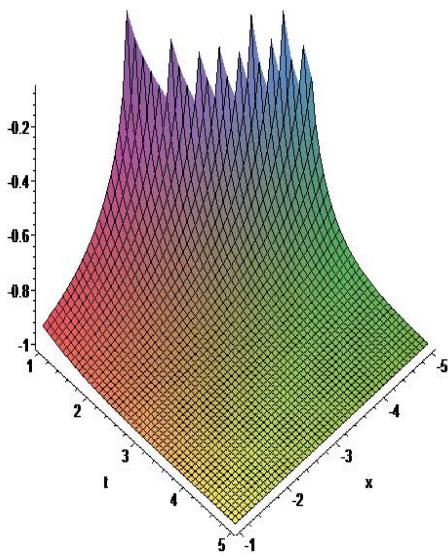


Fig. 2 Solutions (17)(ii) when  $C_1 = 1, C_2 = 2, \alpha = \frac{3}{2}$

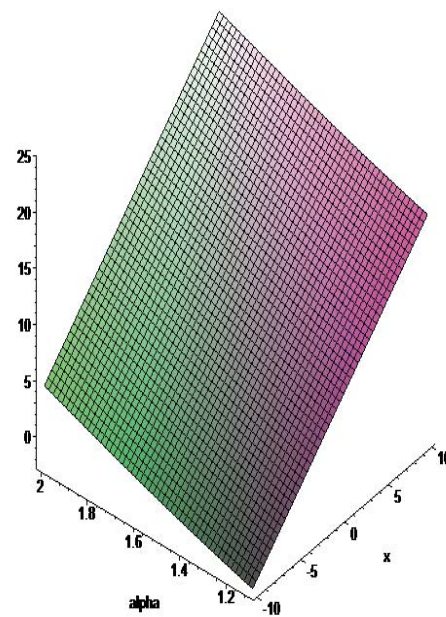


Fig. 4 Solutions (17)(i) when  $C_1 = 2, t = 5$

## V. CONCLUSION AND DISCUSSIONS

In this paper, we have attempted to apply Lie Symmetry method to fractional order nonlinear hyperbolic PDE. On achieving the symmetries, equation (1) has been reduced to ODE of fractional order. Finally, we furnished some exact solutions of undergone equation including kinky solutions.

- Equation (1) has been reduced to ODE (12) by means of the classical Lie group method using symmetries (9).
- The availability of mathematical computer software like Maple facilitates the tedious algebraic calculations. It is worth to mention here that the correctness of the solutions has been checked with the aid of software Maple.

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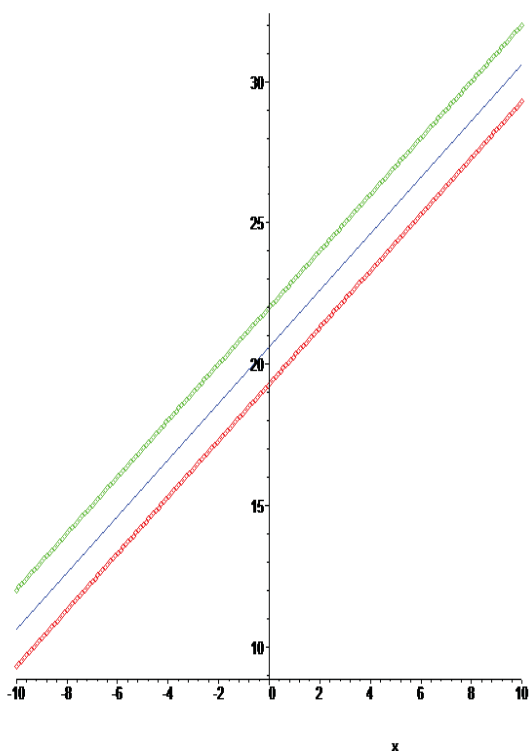


Fig. 5 Solutions (17)(i) when  $C_1 = 7, t = 3, \alpha = 1.6$  (red),  $\alpha = 1.7$ (blue) and  $\alpha = 1.8$ (green)

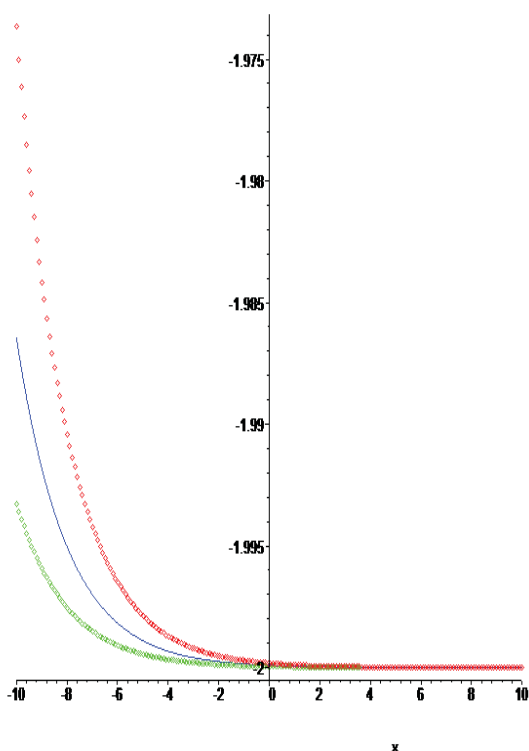


Fig. 6 Solutions (17)(ii) when  $C_1 = 2, C_2 = 3, t = 6, \alpha = 1.6$  (red),  $\alpha = 1.7$ (blue) and  $\alpha = 1.8$ (green)