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Algorithms for Computing of Optimization Problems with a Common Minimum-Norm Fixed Point with Applications

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Abstract—This research is aimed to study a two-step iteration process defined over a finite family of σ-asymptotically quasi-nonexpansive nonself-mappings. The strong convergence is guaranteed under the framework of Banach spaces with some additional structural properties including strict and uniform convexity, reflexivity, and smoothness assumptions. With similar projection technique for nonself-mapping in Hilbert spaces, we hereby use the generalized projection to construct a point within the corresponding domain. Moreover, we have to introduce the use of duality mapping and its inverse to overcome the unavailability of duality representation that is exploit by Hilbert space theorists. We then apply our results for σ-asymptotically quasi-nonexpansive nonself-mappings to solve for ideal efficiency of vector optimization problems composed of finitely many objective functions. We also showed that the obtained solution from our process is the closest to the origin. Moreover, we also give an illustrative numerical example to support our results.

Keywords—σ-asymptotically quasi-nonexpansive nonself-mapping, strong convergence, fixed point, uniformly convex and uniformly smooth Banach space.

I. Introduction

THE theory of fixed point is extensively studied under the nonexpansivity condition of the maps. Among the classes of generalized nonexpansive mappings, Goebel and Kirk [1] introduced the class of asymptotically nonexpansive self-mappings. Let us recall the definition in the following.

Let C be a nonempty subset of a real normed linear space E. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and $n \geq 1$ and they proved that, if C is a nonempty closed convex subset of a real uniformly convex Banach space E and T is an asymptotically nonexpansive self-mapping of C, then T has a fixed point. For more details, see, [2]–[8], [15] and references therein.

Recently, Pathak et al. [11] introduced the concept of σ-asymptotically quasi-nonexpansive mappings in Hilbert spaces and they proved some common minimum-norm fixed point theorems for σ-asymptotically quasi-nonexpansive mappings with some applications.

Let E be a real normed linear space and C be a nonempty subset of E. A mapping $T : C \to C$ is said to be *σ-asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exist two sequences of real numbers $\{k_n\}, \{c_n\}$ with $\lim_{n\to\infty} k_n = 0$ and $\sum_{n=1}^{\infty} c_n < \infty$ such that

$$\|T^n x - \hat{x}\| \leq (1 + k_n)\|x - \hat{x}\| + c_n$$

for all $x \in C$, $\hat{x} \in F(T)$ and $n \geq 1$. On the other hand, in 2006, Censor and Elfving [12] introduced the concept of a split feasibility problem in finite dimensional Hilbert space for modelling inverse problems which arise in medical image reconstruction, image restoration and radiation therapy treatment planing (see, for example, [10], [12], [13]).

Let B and C be nonempty closed convex subset of real Hilbert spaces H_1 and H_2, respectively. *The split feasibility problem* is formulated as follows: Find a point $\bar{x}$ such that

$$\bar{x} \in B \text{ and } A\bar{x} \in C, \tag{1}$$

where A is a bounded linear operator from H_1 to H_2.

Clearly, $\bar{x}$ is a solution to the split feasibility problem if and only if $\bar{x} \in B$ and $A\bar{x} - P_C A\bar{x} = 0$, where P_C is the metric projection from H_2 onto C. Set

$$\min_{x\in B} \varphi(x) := \min_{x\in B} \frac{1}{2}\|Ax - P_C Ax\|^2. \tag{2}$$

Then $\bar{x}$ is a solution of the split feasibility problem (1) if and only if $\bar{x}$ solves the *obtimization problem* (2) whics is called *the minimum-norm problem* with the minimum equal to zero.

Let C be nonempty closed convex subset of real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm

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$\|\cdot\|$ and $T : C \to C$ be a self-mapping. Recall that the *metric projection* $P_C(x)$ of x onto C is defined as:

$$P_C(x) = \min_{y\in C} \|x - y\|.$$

Some authors have studied the iterative approximations of the minimum-norm fixed points of some nonlinear mappings, for example, a nonexpansive self-mapping $T : C \to C$ and others. Especially, Yang et al. [14] introduced an explicit scheme given by

$$x_{n+1} = \beta_n T x_n + (1 - \beta_n)P_C[(1 - \alpha_n)x_n]$$

for each $n \geq 1$. They proved that, under certain conditions on $\{\alpha_n\}$ and $\{\beta_n\}$, the sequence $\{x_n\}$ converges strongly to a minimum-norm fixed point of T in real Hilbert spaces.

Let E be a real Banach space and C be a nonempty closed convex subset of E. Recently, Alber [9] introduced a generalized projection mapping Π_C in E that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, where $\phi(x, y)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle y, Jx\rangle + \|y\|^2 \tag{3}$$

for all $x, y \in E$, that is, for $x \in E$, $\Pi_C x$ is the solution to the minimization problem

$$\phi(\Pi_C x, x) = \inf_{y\in C} \phi(y, x). \tag{4}$$

Note that, in a Hilbert space H, $\Pi_C = P_C$ and, from the definition of function ϕ, it follows that

$$(\|x\| - \|y\|)^2 \leq \phi(y, x) \leq (\|x\| + \|y\|)^2 \tag{5}$$

for all $x, y \in E$;

$$\phi(x, z) = \phi(x, y) + \phi(y, z) + 2\langle x - y, Jy - Jz\rangle \tag{6}$$

for all $x, y \in E$;

$$\begin{aligned}\phi(x, y) &= \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \\ &\leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|\end{aligned} \tag{7}$$

for all $x, y \in E$;

$$\phi(x, J^{-1}(\lambda Jy+(1-\lambda)Jz)) \leq \lambda\phi(x, y)+(1-\lambda)\phi(x, z) \tag{8}$$

for all $x, y, z \in E$ and $\lambda \in (0, 1)$.

It is known that, if E is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. In a Hilbert space H, $\phi(x, y) = \|x - y\|$ for all $x, y \in H$ and Π_C is reduced to the metric projection P_C.

A mapping $T : C \to C$ is said to be *closed* if, for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Let E be a real Banach space and C be a nonempty closed convex subset of E. A mapping $T : C \to E$ is said to be *σ-asymptotically quasi-nonexpansive nonself-mapping* if there exist two sequences $\{k_n\}$, $\{c_n\}$ with $\lim_{n\to\infty} k_n = 0$ and $\sum_{n=1}^{\infty} c_n < \infty$ such that

$$\phi(T(\Pi_C T)^{n-1}x, \hat{x}) \leq (1 + k_n)\phi(x, \hat{x}) + c_n \tag{9}$$

for all $x \in C$, $\hat{x} \in F(T)$ and $n \geq 1$.

Let E be a real Banach space, C be a nonempty closed convex subset of E and $T : C \to E$ be a σ-asymptotically quasi-nonexpansive nonself-mappings with respect to $\{k_n\}$ and $\{c_n\}$. We define the iterative scheme $\{x_n\}$ as follows: for any $x_1 \in C$,

$$\begin{cases} x_1 \in C, \\ y_n = \Pi_C[(1 - \alpha_n)x_n], \\ x_{n+1} = J^{-1}\big(\beta_{n,0}Jx_n + \sum_{i=1}^{N} \beta_{n,i}JT_i(\Pi_C T_i)^{n-1}y_n\big) \end{cases} \tag{10}$$

for all $n \geq 1$, where Π_C is the generalized projection from E onto $C \subset E$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$ and $\sum_{i=0}^{N} \beta_{n,i} = 1$.

We denote the set of fixed points of T by $F(T) =: \{x \in C : Tx = x\}$.

In this paper, we use the iterative scheme (10) to study and prove some strong convergence theorems in framework real uniformly convex and uniformly smooth Banach space and give one application of the main results in this paper.

II. Some Lemmas

Lemma 1. [9] *Let E be a reflexive, smooth and strictly convex Banach space C be a nonempty closed and convex subset of E. Then the following conclusions hold:*

(1) *$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ for all $x \in C$ and $y \in E$;*

(2) *If $x \in E$ and $z \in C$, then $z = \Pi_C x$ if and only if $\langle z - y, Jx - Jz\rangle \geq 0$ for all $y \in C$;*

(3) *For all $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.*

Lemma 2. [19] *Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of E. There exits a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\Big\|\sum_{i=1}^{N}(\alpha_i x_i)\Big\|^2 \leq \sum_{i=1}^{N}(\alpha_i\|x_i\|^2) - \alpha_i\alpha_j g(x_i - x_j)$$

for all $x_1, x_2, x_3, \cdots, x_N \in B_r(0) = \{x \in E : \|x\| \leq r\}$ and $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_N \in (0, 1)$ such that $\sum_{i=1}^{N} \alpha_i = 1$.

The function $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2$$

for all $x \in E$ and $x^* \in E^*$, which was studied by Alber [9], that is, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$.

Lemma 3. [9] *Let E be a reflexive, strictly convex and smooth Banach space with E^* as its dual. Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 4. [16] *Let E be a uniformly convex and smooth Banach space and $\{x_n\}$, $\{y_n\}$ be two sequences of E. If*

$\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.

Lemma 5. [17] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following inequality:*

$$a_{n+1} \leq (1-\alpha)a_n + \alpha_n \delta_n$$

for each $n \geq n_0$, where $\{\alpha_n\} \subset (0,1)$ and $\delta \subset \mathbb{R}$ satisfy the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup \delta_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 6. [18] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} \leq a_{n_i+1}$ for all $i \in N$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all numbers for all $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $\{m_k\} = \max\{j \leq k : a_j < a_{j+1}\}$.

III. Strong Convergence Theorems

Now, we give our main results in this paper,

Theorem 1. *Let E be a real uniformly smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Let $T : C \to E$ be a closed and σ-asymptotically quasi-nonexpansive nonself-mappings with two sequences $\{k_n\}$ and $\{c_n\}$ of nonnegative real numbers with $\lim_{n\to\infty} k_n = 0$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then $F(T)$ is a closed and convex subset of C.*

Theorem 2. *Let E be a real uniformly convex and uniformly smooth Banach space, C be a nonempty closed convex subset of $E, T_i : E \to C$ be a finite family of closed and σ-asymptotically quasi-nonexpansive nonself-mappings with two sequences $\{k_{n,i}\}$, $\{c_{n,i}\}$ of nonnegative real numbers with $\lim_{n\to\infty} k_{n,i} = 0$, $\sum_{n=1}^{\infty} c_{n,i} < \infty$ for each $1 \leq i \leq N$ and $\mathcal{F} =: \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{\alpha_n\}$ and $\{\beta_{n,i}\}$ are the sequences in $(0,1)$ satisfying the following conditions:*

(i) *$\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;*

(ii) *$\sum_{i=0}^{N} \beta_{n,i} = 1$ for all $n \geq 1$ and $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ for each $1 \leq i \leq N$.*

Then the sequence $\{x_n\}$ defined by (10) *converges strongly to a common minimum-norm point of $\mathcal{F}$.*

If we assume that $N = 1$ in Theorem 2, then we have the following results:

Corollary 1. *Let E be a real uniformly convex and uniformly smooth Banach space, C be a nonempty closed convex subset of E and $T : C \to E$ be a closed and σ-asymptotically quasi-nonexpansive nonself-mapping with two sequences $\{k_n\}$ and $\{c_n\}$ of nonnegative real numbers with $\lim_{n\to\infty} k_n = 0$ and $\sum_{n=1}^{\infty} c_n < \infty$. Suppose that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_1 \in C, \\ y_n = \Pi_C[(1-\alpha_n)x_n], \\ x_{n+1} = J^{-1}\big(\beta_n J x_n + (1-\beta_n)(JT(\Pi_C T)^{n-1} y_n)\big) \end{cases} \tag{11}$$

for all $n \geq 1$, where two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) *$\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;*

(ii) *$\{\beta_n\} \subset [a,b] \subset (0,1)$ and $\liminf_{n\to\infty} \beta_n > 0$ for each $n \geq 1$.*

Then the sequence $\{x_n\}$ converges strongly to a common minimum-norm point of $F(T)$.

If we assume that each T_i is an asymptotically nonexpansive nonself-mapping and nonexpansive nonself-mapping in Theorem 2 for each $1 \leq i \leq N$, then, from Theorem 2, we have the following results:

Corollary 2. *Let E be a real uniformly convex and uniformly smooth Banach space, C be a nonempty closed convex subset of E and $T_i : C \to E$ be a finite family of asymptotically nonexpansive nonself-mappings with a sequence $\{k_{n,i}\}$ of nonnegative real numbers with $\lim_{n\to\infty} k_{n,i} = 0$ for each $1 \leq i \leq N$. Suppose that $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_1 \in C, \\ y_n = \Pi_C[(1-\alpha_n)x_n], \\ x_{n+1} = J^{-1}\big(\beta_{n,0} J x_n + \sum_{i=1}^{N} \beta_{n,i} JT_i(\Pi_C T_i)^{n-1} y_n\big) \end{cases} \tag{12}$$

for all $n \geq 1$, where two sequences $\{\alpha_n\}$ and $\{\beta_{n,i}\}$ satisfy the following conditions:

(i) *$\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;*

(ii) *$\sum_{i=0}^{N} \beta_{n,i} = 1$ for all $n \geq 1$ and $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ for each $1 \leq i \leq N$.*

Then the sequence $\{x_n\}$ converges strongly to a common minimum-norm point of $\mathcal{F}$.

Corollary 3. *Let E be a real uniformly convex and uniformly smooth Banach space, C be a nonempty closed convex subset of E and $T_i : C \to E$ be a finite family of nonexpansive nonself-mappings with a sequence $\{k_{n,i}\}$ of nonnegative real numbers with $\lim_{n\to\infty} k_{n,i} = 0$ for each $1 \leq i \leq N$. Suppose that $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_1 \in C, \\ y_n = \Pi_C[(1-\alpha_n)x_n], \\ x_{n+1} = J^{-1}\big(\beta_{n,0} J x_n + \sum_{i=1}^{N} \beta_{n,i} JT_i(\Pi_C T_i) y_n\big) \end{cases} \tag{13}$$

for all $n \geq 1$*, where two sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $\sum_{i=0}^{N} \beta_{n,i} = 1$ *for all* $n \geq 1$ *and* $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ *for each* $1 \leq i \leq N$*.*

Then, the sequence $\{x_n\}$ *converges strongly to a common minimum-norm point of* $\mathcal{F}$*.*

Corollary 4. *Let* E *be a real uniformly convex and uniformly smooth Banach space,* C *be a nonempty closed convex subset of* E *and* $T : C \to E$ *be an asymptotically nonexpansive nonself-mapping with a sequences* $\{k_n\}$ *of nonnegative real numbers with* $\lim_{n\to\infty} k_n = 0$*. Suppose that* $F(T)$ *is nonempty. Let* $\{x_n\}$ *be a sequence in* C *generated by*

$$\begin{cases} x_1 \in C, \\ y_n = \Pi_C[(1-\alpha_n)x_n], \\ x_{n+1} = J^{-1}\big(\beta_n J x_n + (1-\beta_n)(JT(\Pi_C T)^{n-1}y_n)\big) \end{cases} \tag{14}$$

for all $n \geq 1$*, which* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following conditions:*

(i) $\{\alpha_n\} \subset (0,1)$*,* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $\{\beta_n\} \subset [a,b] \subset (0,1)$ *and* $\liminf_{n\to\infty} \beta_n > 0$ *for each* $n \geq 1$*.*

Then the sequence $\{x_n\}$ *converges strongly to a common minimum-norm point of* $F(T)$*.*

Corollary 5. *Let* E *be a real uniformly convex and uniformly smooth Banach space,* C *be a nonempty closed convex subset of* E *and* $T : C \to E$ *be a nonexpansive nonself-mapping with a real sequence* $\{k_n\}$ *with* $\lim_{n\to\infty} k_n = 0$*. Suppose that* $F(T)$ *is nonempty. Let* $\{x_n\}$ *be a sequence in* C *generated by*

$$\begin{cases} x_1 \in C, \\ y_n = \Pi_C[(1-\alpha_n)x_n], \\ x_{n+1} = J^{-1}\big(\beta_n J x_n + (1-\beta_n)(JT(\Pi_C T)^{n-1}y_n)\big) \end{cases} \tag{15}$$

for all $n \geq 1$*, which two sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following conditions:*

(i) $\{\alpha_n\} \subset (0,1)$*,* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $\{\beta_n\} \subset [a,b] \subset (0,1)$ *and* $\liminf_{n\to\infty} \beta_n > 0$ *for* $n \geq 1$*.*

Then the sequence $\{x_n\}$ *converges strongly to a common minimum-norm point of* $F(T)$*.*

If $E = H$ is a real Hilbert space, then E is uniformly convex and smooth real Banach space. In this case, $J = I$ (: the identity mapping) on H and $\Pi_C = P_C$ (: the projection mapping from H onto C). Thus, we have the following result:

Corollary 6. *Let* H *be a real Hilbert space,* C *be a nonempty closed convex subset of* H *and* $T_i : E \to C$ *be a finite family of closed and* σ*-asymptotically quasi-nonexpansive nonself-mappings with two sequences* $\{k_{n,i}\}$*,* $\{c_{n,i}\}$ *of nonnegative real numbers with* $\lim_{n\to\infty} k_{n,i} = 0$*,* $\sum_{n=1}^{\infty} c_{n,i} < \infty$ *for each* $1 \leq i \leq N$*. Suppose that* $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i)$ *is nonempty. Let* $\{x_n\}$ *be a sequence in* C *generated by*

$$\begin{cases} x_1 \in C, \\ y_n = P_C[(1-\alpha_n)x_n], \\ x_{n+1} = \beta_{n,0}x_n + \sum_{i=1}^{N} \beta_{n,i} T_i (P_C T_i)^{n-1} y_n \end{cases} \tag{16}$$

for all $n \geq 1$*, where two sequences* $\{\alpha_n\}$ *and* $\{\beta_{n,i}\}$ *satisfy the following conditions:*

(i) $\{\alpha_n\} \subset (0,1)$*,* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $\{\beta_{n,i}\} \subset [a,b] \subset (0,1)$ *and* $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ *for each* $n \geq 1$ *and* $1 \leq i \leq N$*.*

Then the sequence $\{x_n\}$ *converges strongly to minimum-norm point of* $\mathcal{F}$

IV. Applications

In this section, we apply our main result to the minimum-norm in Banach spaces.

Corollary 7. *Let* E *be a real uniformly convex and uniformly smooth Banach space. Let* $A_i : C \to \mathbb{R}$ *be a continuously Fréchet differentiable convex function with* $T_i := \Pi_C(I - \mu\nabla A_i)$ *be nonexpansive nonself-mapping for some* $\mu > 0$ *and for each* $1 \leq i \leq N$*. Let* $\{x_n\}$ *be a sequence in* C *generated by*

$$\begin{cases} x_1 \in C, \\ y_n = \Pi_C[(1-\alpha_n)x_n], \\ x_{n+1} = \beta_{n,0}x_n + \sum_{i=1}^{N} \beta_{n,i}[\Pi_C(I - \mu\nabla A_i]y_n \end{cases} \tag{17}$$

for all $n \geq 1$*, where two sequences* $\{\alpha_n\}$ *and* $\{\beta_{n,i}\}$ *satisfy the following conditions:*

(i) $\{\alpha_n\} \subset (0,1)$*,* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $\{\beta_{n,i}\} \subset [a,b] \subset (0,1)$ *and* $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ *for* $n \geq 1$ *and* $1 \leq i \leq N$*,*

where ∇A_i *denotes the gradient of* A_i *at* $x \in C$*. Then the sequence* $\{x_n\}$ *converges strongly to a common minimum-norm point of* $\mathcal{F}$*.*

V. Numerical Example

Now, we give an example of a σ-asymptotically quasi-nonexpansive mapping that satisfies the conditions of Theorem 2 and some numerical experiment results to explain the conclusion of the theorem as follows:

Let $E = \mathbb{R} = H$, $C = [0,\infty)$, J be the identity mapping and $\Pi_C = P_C$ with $P_C x = x$. Assume that $T_i x = \frac{x}{9}$, $1 \leq i \leq N$ for $x \in C$. Let $k_{n,i} = \frac{1}{2^i(n^2+1)}$ and $c_{n,i} = \frac{1}{2^i n^2}$ for $n \geq 1$ and $1 \leq i \leq N$, we have

$$\begin{aligned} &\|T_i x - T_i y\| - (1-k_{i,n})\|x-y\| - c_{i,n} \\ &\quad\leq \|x-y\| - (1-k_{i,n})\|x-y\| - c_{i,n} \\ &\quad\leq 0 \end{aligned}$$

for $n \geq 1$ and $1 \leq i \leq N$ with $\lim_{n\to\infty} k_{n,i} = 0$ and $\sum_{n=1}^{\infty} c_{n,i} < \infty$, so T_i is a σ-asymptotically quasi-nonexpansive mapping. Clearly, $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) = \{0\}$. Set

$$\alpha_n = \frac{1}{n+2} \text{ and } \beta_{n,i} = \frac{1}{3^i}(\frac{1}{n+3}).$$

Thus, the conditions of Theorem 2 are fulfilled. Therefore, we can invoke Theorem 2 to demonstrate that the iterative sequence $\{x_n\}$ defined by (10) converges strongly to 0. We have the numerical analysis tabulated in Table I and shown in Fig. 1.

TABLE I
NUMERICAL EXPERIMENT

n	x_n
1	10
2	2.5617
3	0.6419
4	0.1605
5	0.0401
6	0.0100
7	0.0025
8	0.0006
9	0.0002
10	0.0000

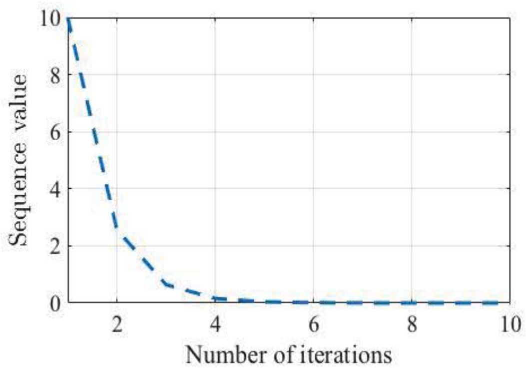

Fig. 1 The iteration chart with initial value $x_1 = 10$

VI. CONCLUSION

Our iteration can be used for proving strong convergence theorems of the proposed sequence $\{x_n\}$ in real uniformly convex and uniformly smooth Banach spaces.

APPENDIX A
PROOF OF THE THEOREM 1

Let $\{x_n\}$ be a sequence in $F(T)$ with $x_n \to v$ as $n \to \infty$. Since $Tx_n = x_n \to v$, by the closednes of T, we have $v = Tv$, that is, $v \in F(T)$. This shows that $F(T)$ is closed.

Next, we prove that $F(T)$ is convex. For any $x, y \in F(T)$ and $t \in (0,1)$, putting $u = tx + (1-t)y$. We prove that $u \in F(T)$. Let $\{v_n\}$ be a sequence generated by

$$v_1 = Tu, \;\; v_2 = T\Pi_C v_1 = T(\Pi_C T)u,$$
$$v_3 = T\Pi_C v_2 = T(\Pi_C T)^2 u, \;\; \cdots,$$
$$v_n = T\Pi_C v_{n-1} = T(\Pi_C T)^{n-1} u, \cdots$$

for each $n \geq 1$. By the definition of $\phi(x,y)$, we have

$$\begin{aligned} \phi(u, v_n) &= \|u\|^2 - 2\langle u, Jv_n\rangle + \|v_n\|^2 \\ &= \|u\|^2 - 2\langle tx + (1-t)y, Jv_n\rangle + \|v_n\|^2 \\ &= \|u\|^2 - 2t\langle x, Jv_n\rangle - 2(1-t)\langle y, Jv_n\rangle + \|v_n\|^2 \\ &= \|u\|^2 + t\phi(x, v_n) + (1-t)\phi(y, v_n) - t\|x\|^2 \\ &\quad - (1-t)\|y\|^2. \end{aligned} \tag{18}$$

Also, we have

$$\begin{aligned} &t\phi(x, v_n) + (1-t)\phi(y, v_n) \\ &\leq t[(1+k_n)\phi(x,u) + c_n] + (1-t)[(1+k_n)\phi(y,u) + c_n] \\ &= t[(1+k_n)(\|x\|^2 - 2\langle x, Ju\rangle + \|u\|^2) + c_n] \\ &\quad + (1-t)[(1+k_n)(\|y\|^2 - 2\langle y, Ju\rangle + \|u\|^2) + c_n] \\ &= t(1+k_n)\|x\|^2 + (1-t)(1+k_n)\|y\|^2 \\ &\quad - (1+k_n)\|u\|^2 + c_n \\ &= (1+k_n)[t\|x\|^2 + (1-t)\|y\|^2 - \|u\|^2] + c_n. \end{aligned} \tag{19}$$

From (18) and (19), it follows that

$$\begin{aligned} \phi(u, v_n) \leq \|u\|^2 + (1+k_n)[t\|x\|^2 + (1-t)\|y\|^2 - \|u\|^2] \\ + c_n - t\|x\|^2 - (1-t)\|y\|^2 \to 0 \end{aligned}$$

as $n \to \infty$. Thus $v_n \to u$ as $n \to \infty$, which implies that $v_{n+1} \to u$. Since T is closed and $v_{n+1} = T(\Pi_C T)^n u = T(\Pi_C T)(\Pi_C T)^{n-1} u = T\Pi_C v_n$, we have $u = T\Pi_C u$. Since $\Pi_C u = u$ for for any $u \in C$, we have $u = Tu$ and so $F(T)$ is convex. This completes the proof.

APPENDIX B
PROOF OF THE THEOREM 2

Let $\hat{x} = \Pi_{\mathcal{F}}(0) \in \mathcal{F}$, that is, $\|\hat{x}\|^2 = \phi(\hat{x}, 0) = \min_{y\in\mathcal{F}} \phi(y, 0) = \min_{y\in\mathcal{F}} \|y\|^2$ and let $k_n = \max_{1\leq i\leq N}\{k_{n,i}\}$ and $c_n = \max_{1\leq i\leq N}\{c_{n,i}\}$. It follows from (10), Lemma 1 and the property of ϕ that

$$\begin{aligned} \phi(\hat{x}, y_n) &= \phi(\hat{x}, \Pi_C((1-\alpha_n)x_n)) \\ &\leq \phi(\hat{x}, (1-\alpha_n)x_n) \\ &= \phi(\hat{x}, J^{-1}(\alpha_n J0 + (1-\alpha_n)Jx_n)) \\ &= \|\hat{x}\|^2 - 2\langle \hat{x}, \alpha_n J0 + (1-\alpha_n)Jx_n\rangle \\ &\quad + \|(1-\alpha_n)Jx_n\|^2 \\ &\leq \|\hat{x}\|^2 - 2\langle \hat{x}, \alpha_n J0\rangle - 2(1-\alpha_n)\langle \hat{x}, Jx_n\rangle \\ &\quad + \alpha_n\|J0\|^2 + \|(1-\alpha_n)x_n\|^2 \\ &= \alpha_n\phi(\hat{x}, 0) + (1-\alpha_n)\phi(\hat{x}, x_n). \end{aligned} \tag{20}$$

Then, we have

$$
\begin{aligned}
&\phi(\hat{x}, x_{n+1})\\
&= \phi(\hat{x}, J^{-1}(\beta_{n,0}Jx_n + \sum_{i=1}^{N}\beta_{n,i}JT_i(\Pi_C T_i)^{n-1}y_n))\\
&= \|\hat{x}\|^2 - 2\langle \hat{x}, \beta_{n,0}Jy_n + \sum_{i=1}^{N}\beta_{n,i}JT_i(\Pi_C T_i)^{n-1}y_n\rangle\\
&\quad + \|\beta_{n,0}Jy_n + \sum_{i=1}^{N}\beta_{n,i}JT_i(\Pi_C T_i)^{n-1}y_n\|^2\\
&\le \|\hat{x}\|^2 - 2\beta_{n,0}\langle \hat{x}, Jy_n\rangle - 2\sum_{i=1}^{N}\beta_{n,i}\langle \hat{x}, JT_i(\Pi_C T_i)^{n-1}y_n\rangle\\
&\quad + \beta_{n,0}\|Jy_n\|^2 + \sum_{i=1}^{N}\beta_{n,i}\|JT_i(\Pi_C T_i)^{n-1}y_n\|^2\\
&\quad - \beta_{n,0}\beta_{n,i}\|Jy_n - JT_i(\Pi_C T_i)^{n-1}y_n\|\\
&\le \beta_{n,0}\phi(\hat{x}, y_n) + \sum_{i=1}^{N}\beta_{n,i}\phi(\hat{x}, T_i(\Pi_C T_i)^{n-1}y_n)\\
&\quad - \beta_{n,0}\beta_{n,i}\|Jy_n - JT_i(\Pi_C T_i)^{n-1}y_n\|\\
&\le \beta_{n,0}\phi(\hat{x}, x_n) + \sum_{i=1}^{N}\beta_{n,i}\phi(\hat{x}, T_i(\Pi_C T_i)^{n-1}y_n)\\
&\le \beta_{n,0}\phi(\hat{x}, x_n) + (1-\beta_{n,0})[(1+k_n)\phi(\hat{x}, y_n) + c_n]\\
&\le \beta_{n,0}\phi(\hat{x}, x_n) + (1-\beta_{n,0})[(1+k_n)\{\alpha_n\phi(\hat{x}, 0)\\
&\quad + (1-\alpha_n)\phi(\hat{x}, x_n)\} + c_n]\\
&= \beta_{n,0}\phi(\hat{x}, x_n) + (1-\beta_{n,0})(1+k_n)\alpha_n\phi(\hat{x}, 0)\\
&\quad + (1-\beta_{n,0})(1+k_n)(1-\alpha_n)\phi(\hat{x}, x_n) + (1-\beta_{n,0})c_n\\
&= [\beta_{n,0} + (1-\beta_{n,0})(1+k_n)(1-\alpha_n)]\phi(\hat{x}, x_n)\\
&\quad + (1-\beta_{n,0})(1+k_n)\alpha_n\phi(\hat{x}, 0) + (1-\beta_{n,0})c_n\\
&= [1 - \alpha_n(1-\beta_{n,0}) + k_n(1-\beta_{n,0})\\
&\quad - \alpha_n k_n(1-\beta_{n,0})]\phi(\hat{x}, x_n)\\
&\quad + (1-\beta_{n,0})(1+k_n)\alpha_n\phi(\hat{x}, 0) + (1-\beta_{n,0})c_n\\
&= [1 - (1-\beta_{n,0})(\alpha_n - k_n + \alpha_n k_n)]\phi(\hat{x}, x_n)\\
&\quad + (1-\beta_{n,0})(1+k_n)\alpha_n\phi(\hat{x}, 0) + (1-\beta_{n,0})c_n\\
&= [1 - (1-\beta_{n,0})(\alpha_n(1+k_n) - k_n)]\phi(\hat{x}, x_n)\\
&\quad + (1-\beta_{n,0})(1+k_n)\alpha_n\phi(\hat{x}, 0) + (1-\beta_{n,0})c_n\\
&\le \Big(\prod_{i=1}^{n}\beta_{i,0}\Big)\phi(\hat{x}, x_n) + (1-\beta_{n-1,0})\phi(\hat{x}, 0) + \sum_{j=1}^{n}c_j\\
&= \Theta_1\phi(\hat{x}, x_n) + (1-\Theta_{n-1})\phi(\hat{x}, 0) + \sum_{j=1}^{n}c_j,
\end{aligned} \tag{21}
$$

where $\Theta_1 = \prod_{i=1}^{n}\beta_{i,0}$, $\Theta_{n-1} = \beta_{n-1,0}\beta_{n-2,0}\beta_{n-3,0}\cdots\beta_{1,0}$ and $\sum_{j=1}^{n}c_j = c_1 + c_2 + c_3 + \cdots + c_n$. From (8) and Lemma 3, it follows that

$$
\begin{aligned}
&\phi(\hat{x}, y_n)\\
&\le \phi(\hat{x}, (1-\alpha_n)x_n)\\
&= V(\hat{x}, J(1-\alpha_n)x_n)\\
&\le V(\hat{x}J(1-\alpha_n)x_n + \alpha_n J\hat{x})\\
&\quad - 2\langle J^{-1}J(1-\alpha_n)x_n - \hat{x}, \alpha_n J\hat{x}\rangle\\
&= \phi(\hat{x}, J^{-1}(J(1-\alpha_n)x_n + \alpha_n J\hat{x}))\\
&\quad - 2\langle (1-\alpha_n)x_n - \hat{x}, \alpha_n J\hat{x}\rangle\\
&\le (1-\alpha_n)\phi(\hat{x}, x_n) + \alpha_n\phi(\hat{x}, \hat{x})\\
&\quad - 2\alpha_n\langle (1-\alpha_n)x_n - \hat{x}, J\hat{x}\rangle\\
&= (1-\alpha_n)\phi(\hat{x}, x_n) - 2\alpha_n\langle (1-\alpha_n)x_n - \hat{x}, J\hat{x}\rangle.
\end{aligned} \tag{22}
$$

From (21) and (22), we have

$$
\begin{aligned}
&\phi(\hat{x}, x_{n+1})\\
&\le \beta_{n,0}\phi(\hat{x}, x_n) + (1-\beta_{n,0})(1+k_n)[(1-\alpha_n)\phi(\hat{x}, x_n)\\
&\quad - 2\alpha_n\langle (1-\alpha_n)x_n - \hat{x}, J\hat{x}\rangle]\\
&\quad + (1-\beta_{n,0})c_n - \beta_{n,0}\beta_{n,i}g(\|Jx_n - JT_i(\Pi_C T_i)^{n-1}y_n\|)\\
&= \Big(1 - \frac{\theta_n}{\alpha_n}\Big)\phi(\hat{x}, x_n) + \frac{\theta_n}{\alpha_n}(1+k_n)[(1-\alpha_n)\phi(\hat{x}, x_n)\\
&\quad - 2\alpha_n\langle (1-\alpha_n)x_n - \hat{x}, J\hat{x}\rangle]\\
&\quad + \frac{\theta_n}{\alpha_n}c_n - \beta_{n,0}\beta_{n,i}g(\|Jx_n - JT_i(\Pi_C T_i)^{n-1}y_n\|)\\
&= \Big(1 - \frac{\theta_n}{\alpha_n}\Big)\phi(\hat{x}, x_n) + \frac{\theta_n}{\alpha_n}(1+k_n)(1-\alpha_n)\phi(\hat{x}, x_n)\\
&\quad - 2\theta_n(1+k_n)\langle (1-\alpha_n)x_n - \hat{x}, J\hat{x}\rangle\\
&\quad + \frac{\theta_n}{\alpha_n}c_n - \beta_{n,0}\beta_{n,i}g(\|Jx_n - JT_i(\Pi_C T_i)^{n-1}y_n\|)\\
&= \Big[1 - \frac{\theta_n}{\alpha_n} + \frac{\theta_n}{\alpha_n}(1+k_n)(1-\alpha_n)\Big]\phi(\hat{x}, x_n)\\
&\quad - 2\theta_n(1+k_n)\langle (1-\alpha_n)x_n - \hat{x}, J\hat{x}\rangle\\
&\quad + \frac{\theta_n}{\alpha_n}c_n - \beta_{n,0}\beta_{n,i}g(\|Jx_n - JT_i(\Pi_C T_i)^{n-1}y_n\|)\\
&= \Big[1 - \frac{\theta_n}{\alpha_n} + \frac{\theta_n}{\alpha_n}(1+k_n) - \frac{\theta_n}{\alpha_n}(1+k_n)\alpha_n\Big]\phi(\hat{x}, x_n)\\
&\quad - 2\theta_n(1+k_n)\langle (1-\alpha_n)x_n - \hat{x}, J\hat{x}\rangle\\
&\quad + \frac{\theta_n}{\alpha_n}c_n - \beta_{n,0}\beta_{n,i}g(\|Jx_n - JT_i(\Pi_C T_i)^{n-1}y_n\|)\\
&= \Big[1 - \theta_n(1+k_n) + \frac{\theta_n}{\alpha_n}((1+k_n) - 1)\Big]\phi(\hat{x}, x_n)\\
&\quad - 2\theta_n(1+k_n)\langle (1-\alpha_n)x_n - \hat{x}, J\hat{x}\rangle\\
&\quad + \frac{\theta_n}{\alpha_n}c_n - \beta_{n,0}\beta_{n,i}g(\|Jx_n - JT_i(\Pi_C T_i)^{n-1}y_n\|)\\
&= [1 - \theta_n(1+k_n)]\phi(\hat{x}, x_n) + \frac{\theta_n}{\alpha_n}[(1+k_n) - 1]\phi(\hat{x}, x_n)\\
&\quad - 2\theta_n(1+k_n)\langle (1-\alpha_n)x_n - \hat{x}, J\hat{x}\rangle\\
&\quad + \frac{\theta_n}{\alpha_n}c_n - \beta_{n,0}\beta_{n,i}g(\|Jx_n - JT_i(\Pi_C T_i)^{n-1}y_n\|)\\
&\le (1-\theta_n)\phi(\hat{x}, x_n) - 2\theta_n\langle (1-\alpha_n)x_n - \hat{x}, J\hat{x}\rangle\\
&\quad + [(1+k_n) - 1]M + \frac{\theta_n}{\alpha_n}c_n\\
&\quad - \beta_{n,0}\beta_{n,i}g(\|Jx_n - JT_i(\Pi_C T_i)^{n-1}y_n\|)\\
&\le (1-\theta_n)\phi(\hat{x}, x_n) - 2\theta_n\langle (1-\alpha_n)x_n - \hat{x}, J\hat{x}\rangle\\
&\quad + [(1+k_n) - 1]M + \frac{\theta_n}{\alpha_n}c_n
\end{aligned} \tag{23}
$$

for some $M > 0$, where $\theta_n = \alpha_n(1-\beta_{n,0})$ for $n \ge 1$.

Now, we consider the following two cases.

Case I. Suppose that there exists $N \in \mathbb{N}$ such that $\{\phi(\hat{x}, x_n)\}$ is nonincreasing for all $n \geq N$. Then $\{\phi(\hat{x}, x_n)\}$ is convergent and so, from (23),

$$\beta_{n,0}\beta_{n,i}g(\|Jx_n - JT_i(\Pi_C T_i)^{n-1}y_n\|) \to 0$$

as $n \to \infty$. From $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$, we have

$$g(\|Jx_n - JT_i(\Pi_C T_i)^{n-1}y_n\|) \to 0$$

as $n \to \infty$. Thus, by the property of g, we have

$$\lim_{n\to\infty} \|Jx_n - JT_i(\Pi_C T_i)^{n-1}y_n\| = 0 \tag{24}$$

for each $1 \leq i \leq N$. Since J^{-1} is uniformly norm-to-norm continuous on each bounded set, we have

$$\lim_{n\to\infty} \|x_n - T_i(\Pi_C T_i)^{n-1}y_n\| = 0. \tag{25}$$

From (7), (24) and (25), we obtain

$$\lim_{n\to\infty} \phi(x_n, T_i(\Pi_C T_i)^{n-1}y_n) = 0. \tag{26}$$

Moreover, it follows from (26) that

$$\begin{aligned}
&\phi(x_n, x_{n+1}) \\
&= \phi\Big(x_n, J^{-1}\big(\beta_{n,0}Jx_n + \sum_{i=1}^{N} \beta_{n,i}JT_i(\Pi_C T_i)^{n-1}y_n)\big)\Big) \\
&\leq \beta_{n,0}\phi(x_n, x_n) + \sum_{i=1}^{N} \beta_{n,i}\phi(x_n, T_i(\Pi_C T_i)^{n-1}y_n) \\
&\leq \sum_{i=1}^{N} \beta_{n,i}\phi(x_n, T_i(\Pi_C T_i)^{n-1}y_n) \to \infty
\end{aligned} \tag{27}$$

as $n \to \infty$. Since $\lim_{n\to\infty} \alpha_n = 0$, it follows that

$$\begin{aligned}
\phi(x_n, y_n) &= \phi(x_n, \Pi_C((1-\alpha_n)x_n)) \leq \phi(x_n, (1-\alpha_n)x_n) \\
&= \phi(x_n, J^{-1}(\alpha_n J0 + (1-\alpha_n)Jx_n) \\
&\leq \alpha_n(x_n, 0) + (1-\alpha_n)\phi(x_n, x_n) \\
&= \alpha_n(x_n, 0) \to 0
\end{aligned} \tag{28}$$

as $n \to \infty$. From (26)–(28) and Lemma 4, we have

$$\begin{aligned}
&\lim_{n\to\infty} \|x_n - T_i(\Pi_C T_i)^{n-1}y_n\| = 0, \ \lim_{n\to\infty} \|x_n - y_n\| = 0, \\
&\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.
\end{aligned} \tag{29}$$

Furthermore, J is uniformly norm-to-norm continuous on each bounded set, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \|y_n - T_i(\Pi_C T_i)^{n-1}y_n\| = 0, \\
&\lim_{n\to\infty} \|Jy_n - JT_i(\Pi_C T_i)^{n-1}y_n\| = 0, \\
&\lim_{n\to\infty} \|y_n - y_{n+1}\| = 0, \quad \lim_{n\to\infty} \|y_n - y_{n+1}\| = 0.
\end{aligned}$$

From (7), it follows that

$$\lim_{n\to\infty} \phi(y_n, y_{n+1}) = 0, \ \lim_{n\to\infty} \phi(y_n, T_i(\Pi_C T_i)^{n-1}y_n) = 0 \tag{30}$$

for each $1 \leq i \leq N$ and, from (6), we have

$$\begin{aligned}
&\phi(y_n, T_i y_n) \\
&= \phi(y_n, y_{n+1}) + \phi(y_{n+1}, T_i y_n) \\
&\quad + 2\langle y_n - y_{n+1}, Jy_{n+1} - JT_i y_n\rangle \\
&= \phi(y_n, y_{n+1}) + \phi(y_{n+1}, T_i(\Pi_C T_i)^n y_{n+1}) \\
&\quad + \phi(T_i(\Pi_C T_i)^n y_{n+1}, T_i(\Pi_C T_i)^n y_n) + 2\langle T_i(\Pi_C T_i)^n y_{n+1} \\
&\quad - T_i(\Pi_C T_i)^n y_n, JT_i(\Pi_C T_i)^n y_n - JT_i y_n\rangle \\
&\quad + \langle y_{n+1} - T_i(\Pi_C T_i)^n y_{n+1}, JT_i(\Pi_C T_i)^n y_{n+1} - JT_i y_n\rangle \\
&\quad + 2\langle y_n - y_{n+1}, Jy_{n+1} - JT_i y_n\rangle \\
&\quad + \phi(T_i(\Pi_C T_i)^n y_n, T_i y_n).
\end{aligned} \tag{31}$$

Since $\phi(y_n, (\Pi_C T_i)^n y_n) \leq \phi(y_n, T_i(\Pi_C T_i)^{n-1}y_n)$, it follows from (31) that

$$\lim_{n\to\infty} \phi(y_n, (\Pi_C T_i)^n y_n) = 0.$$

By Lemma 4, we have $\lim_{n\to\infty} \|y_n - (\Pi_C T_i)^n y_n\| = 0$, that is, $\lim_{n\to\infty} \phi((\Pi_C T_i)^n y_n, y_n) = 0$. Since $\phi(T_i(\Pi_C T_i)^n y_n, T_i y_n) \leq (1+k_n)\phi((\Pi_C)^n y_n, y_n) + c_n$, we have

$$\lim_{n\to\infty} \phi(T_i(\Pi_C T_i)^n y_n, T_i y_n) = 0. \tag{32}$$

Applying (30), (32) and the definition of T_i, we have

$$\lim_{n\to\infty} \phi(y_n, T_i y_n) = 0. \tag{33}$$

By Lemma 4, we get

$$\lim_{n\to\infty} \|y_n - T_i y_n\| = 0 \tag{34}$$

for each $1 \leq i \leq N$. Let $\{x_{n_k}\}$ be a subsequence of the sequence $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (1-\alpha_n)x_n - \hat{x}, J\hat{x}\rangle = \lim_{k\to\infty} \langle (1-\alpha_n)x_{n_k} - \hat{x}, J\hat{x}\rangle$$

and $x_{n_k} \to w$. Then, from (28), it follows that $x_{n_k} \to w$. Hence, By Lemma 1 (ii), we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle (1-\alpha_n)x_n - \hat{x}, J\hat{x}\rangle &= \lim_{k\to\infty} \langle (1-\alpha_n)x_{n_k} - \hat{x}, J\hat{x}\rangle \\
&= \langle w - \hat{x}, J\hat{x}\rangle \geq 0.
\end{aligned}$$

Now, we show that $x_{n+1} \to \hat{x}$ as $n \to \infty$. Since T_i is closed, it follows from (34) that that $w \in F(T_i)$ foe each $1 \leq i \leq N$ and $w \in \bigcap_{i=1}^{N} F(T_i)$. Then, from (23), we have

$$\begin{aligned}
\phi(x_{n+1}, \hat{x}) &\leq (1-\theta_n)\phi(\hat{x}, x_n) - 2\theta_n\langle (1-\alpha_n)x_n - \hat{x}, J\hat{x}\rangle \\
&\quad + [(1+k_n) - 1]M + \frac{\theta_n}{\alpha_n}c_n.
\end{aligned}$$

Note that $\lim_{n\to\infty} \theta_n = 0$ and $\sum_{n=1}^{\infty} \theta_n = \infty$. By Lemma 5, we have $\phi(\hat{x}, x_n) \to 0$ as $n \to \infty$ and, consequently, $x_n \to \hat{x}$, $n \to \infty$.

Case II. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(\hat{x}, x_{n_i}) \leq \phi(\hat{x}, x_{n_i+1})$$

for each $i \in \mathbb{N}$. Then, by Lemma 6, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\phi(\hat{x}, x_{m_k}) \leq \phi(\hat{x}, x_{m_k+1}), \quad \phi(\hat{x}, x_k) \leq \phi(\hat{x}, x_{m_k+1})$$

for all $k \in \mathbb{N}$. Then, from (23) and $\theta_n \to 0$, it follows that

$$\begin{aligned} &\beta_{m_k,0}\beta_{m_k,i}g(\|Jx_{m_k} - JT_i(\Pi_C T_i)^{m_k-1}y_{m_k}\|) \\ &\leq (1-\theta_{m_k})\phi(\hat{x}, x_{m_k}) - \phi(x_{m_k+1}, \hat{x}) \\ &\quad - 2\theta_{m_k}\langle (1-\alpha_n)x_{m_k} - \hat{x}, J\hat{x}\rangle \\ &\quad + [(1+k_{m_k}) - 1]M + \frac{\theta_{m_k}}{\alpha_{m_k}} c_{m_k}. \end{aligned}$$

This implies that

$$g(\|Jx_{m_k} - JT_i(\Pi_C T_i)^{m_k-1}y_{m_k}\|) \to 0$$

as $n \to \infty$. Hence, following the method of Case I, we have $\|x_{m_k} - y_{m_k}\| \to 0$ as $k \to \infty$ and $\|y_{m_k} - T_i y_{m_k}\| \to 0$ as $k \to \infty$ for each $1 \leq i \leq N$. Thus there exists $w_1 \in \mathcal{F}$ such that

$$\begin{aligned} &\limsup_{n\to\infty}\langle (1-\alpha_{m_k})x_{m_k} - \hat{x}, J\hat{x}\rangle \\ &= \lim_{k\to\infty}\langle (1-\alpha_{m_k})x_{m_k} - \hat{x}, J\hat{x}\rangle \\ &= \langle w_1 - \hat{x}, J\hat{x}\rangle \\ &\geq 0. \end{aligned} \tag{35}$$

It follows from (23) that

$$\begin{aligned} &\phi(x_{m_k+1}, \hat{x}) \\ &\leq (1-\theta_{m_k})\phi(\hat{x}, x_{m_k}) - 2\theta_{m_k}\langle (1-\alpha_n)x_{m_k} - \hat{x}, J\hat{x}\rangle \\ &\quad + [(1+k_{m_k}) - 1]M + \frac{\theta_{m_k}}{\alpha_{m_k}} c_{m_k}. \end{aligned} \tag{36}$$

Since $\phi(x_{m_k}, \hat{x}) \leq \phi(x_{m_k+1}, \hat{x})$, (36) implies that

$$\begin{aligned} &\theta_{m_k}\phi(\hat{x}, x_{m_k}) \\ &\leq \phi(\hat{x}, x_{m_k}) - \phi(x_{m_k+1}, \hat{x}) - 2\theta_{m_k}\langle (1-\alpha_n)x_{m_k} - \hat{x}, J\hat{x}\rangle \\ &\quad + [(1+k_{m_k}) - 1]M + \frac{\theta_{m_k}}{\alpha_{m_k}} c_{m_k} \\ &\leq -2\theta_{m_k}\langle (1-\alpha_n)x_{m_k} - \hat{x}, J\hat{x}\rangle \\ &\quad + [(1+k_{m_k}) - 1]M + \frac{\theta_{m_k}}{\alpha_{m_k}} c_{m_k} \end{aligned} \tag{37}$$

In particular, since $\theta_{m_k} > 0$, it follows that

$$\phi(\hat{x}, x_{m_k}) \leq -2\langle (1-\alpha_n)x_{m_k} - \hat{x}, J\hat{x}\rangle + \frac{k_{m_k}}{\theta_{m_k}} M + \frac{c_{m_k}}{\alpha_{m_k}}. \tag{38}$$

Hence, from (36) and the fact that $k_{m_k} \to 0$, $k \to \infty$ and $c_{m_k} \to 0$ as $k \to \infty$, it follows that $\phi(\hat{x}, x_{m_k}) \to 0$ as $k \to \infty$, which together with (37) gives $\phi(\hat{x}, x_{m_k+1}) \to 0$ as $k \to \infty$. But, since $\phi(x_{m_k}, \hat{x}) \leq \phi(x_{m_k+1}, \hat{x})$ for all $k \in \mathbb{N}$, we obtain $x_k \to \hat{x}$ as $k \to \infty$. Therefore, from the two Cases, we can conclude that $\{x_n\}$ converges strongly to the minimum norm point of $\mathcal{F}$. This completes the proof.

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