Semilocal Convergence of a Three Step Fifth Order Iterative Method under Hölder Continuity Condition in Banach Spaces

Ramandeep Behl, Prashanth Maroju, S. S. Motsa

Abstract—In this paper, we study the semilocal convergence of a fifth order iterative method using recurrence relation under the assumption that first order Fréchet derivative satisfies the Hölder condition. Also, we calculate the R-order of convergence and provide some a priori error bounds. Based on this, we give existence and uniqueness region of the solution for a nonlinear Hammerstein integral equation of the second kind.

Keywords—Hölder continuity condition, Fréchet derivative, fifth order convergence, recurrence relations.

I. INTRODUCTION

WE consider the problem of solving

$$F(x) = 0 \qquad (1)$$

where $F : \Omega \subseteq X \to Y$ is a nonlinear Fréchet differentiable operator in an open convex domain Ω of a Banach space X with values in a Banach space Y. Newton's method and its variants are used to solve nonlinear equation (1). Many topics related to Newton's method still attract attentions from the researchers. We have two approaches to establish the convergence of iterative methods. Those are majorizing sequence, recurrence relation approach. Rall [1] suggested a recurrence relation approach for the convergence of iterative methods. Also, we have local and semilocal convergence analysis of iterative methods. The local convergence is based on the information around the solution. The semilocal convergence is based on the assumption at initial approximation and on domain. Using the semilocal convergence analysis, we find the existence and uniqueness regions of solution, a priori error bounds. The well known Kantorovich theorem [2] gives sufficient conditions for the semilocal convergence of Newton's method as well as the error estimates and existence-uniqueness regions of solutions. The main assumption for the semilocal convergence of iterative methods are Lipschitz/Hölder/ω-continuity conditions. The well know third order iterative methods for solving nonlinear equations are the Chebyshev method, the Halley's method and the Super-Halley's method. Many researchers [3]-[12] discussed the semilocal convergence of these several iterative methods of different orders using recurrence relation approach

Ramandeep Behl is with the School of Mathematics, Statistics and Computer Sciences, University of KwaZulu-Natal, Private Bag X01, Scottsville 3209, Pietermaritzburg, South Africa.

Prashanth Maroju is with the School of Mathematics, Statistics and Computer Sciences, University of KwaZulu-Natal, Private Bag X01, Scottsville 3209, Pietermaritzburg, South Africa (e-mail: maroju.prashanth@gmail.com).

under different continuity conditions in Banach spaces. The semilocal convergence of fifth order method in Banach spaces discussed in [15]. In recent years, the semilocal convergence of another fifth order method is discussed by [13] using recurrence relations approach. They used the assumption that the first order Fréchet derivative satisfies the Lipschitz continuity condition.

In this paper, we analyze the semilocal convergence of a fifth-order method considered in [14] under the assumption that the first order Fréchet derivative satisfies the Hölder continuity condition. We use the recurrence relation approach, where the problem in Banach space into real sequences and its properties, providing a suitable convergence domain. Finally, we apply our semilocal convergence result to a nonlinear Hammerstein integral equation of the second kind and obtain a existence and uniqueness of the solution for this type of equations.

This paper is organized in five sections. Section I is the introduction. In Section II, some preliminary results are given. Then, real sequences are generated and their properties are studied. In Section III, a convergence theorem is established for the existence and uniqueness regions along with a priori error bounds for the solution. In Section IV, numerical example is worked out to demonstrate the efficacy of our approach. Finally, conclusions form Section V.

II. PRELIMINARY RESULTS

Let $x_0 \in \Omega$ and the nonlinear operator $F : \Omega \subset X \to Y$ be continuously first order Fréchet differentiable where Ω is an open set in X and Y are Banach spaces. The fifth order iterative method for solving nonlinear equation in Banach spaces is written as

$$\left.\begin{array}{l} y_n = x_n - \Gamma_n F(x_n) \\ z_n = y_n - 5\Gamma_n F(y_n) \\ x_{n+1} = z_n - \frac{1}{5}\Gamma_n(-16F(y_n) + F(z_n)) \end{array}\right\} \qquad (2)$$

Let $F'(x_0)^{-1} = \Gamma_0 \in L(Y, X)$ exists at some $x_0 \in \Omega$, where $L(Y, X)$ is the set of bounded linear operators from Y into X. For $y_0, z_0 \in \Omega$, we assume that Kantorovich's conditions [2]

$$\left.\begin{array}{l} (i) \ \|\Gamma_0\| \leq \beta, \\ (ii) \ \|\Gamma_0 F(x_0)\| \leq \eta, \\ (iii) \ \|F'(x) - F'(y)\| \leq K\|x - y\|^p, \ \forall \ x, \\ y \in \Omega, p \in (0, 1] \end{array}\right\} \qquad (3)$$

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Let $a_0 = K\beta\eta^p$ and define the sequence $a_{n+1} = a_n f(a_n)^{p+1} g(a_n)^p$

$$f(x) = \frac{1}{1 - x(1 + h(x))^p}, \tag{4}$$

$$g(x) = \frac{x}{p+1} + h(x)(x+1) + \frac{x}{p+1} h(x)^{p+1} \tag{5}$$

and

$$h(x) = \frac{4x}{5(p+1)} + \frac{x}{5(p+1)}\left(1 + \frac{5x}{p+1}\right)^{p+1}. \tag{6}$$

We now describe the properties of the sequence $\{a_n\}$ and the real functions (4)-(6) through the following Lemmas.

Lemma 1: Let f, g and h be the functions defined in (4)-(6) respectively. Then

(i) f is an increasing function and $f(x) > 1$ for $x \in (0, t_p)$, $p \in (0, 1]$.

(ii) g and h are increasing for $x \in (0, t_p)$, $p \in (0, 1]$.

Proof: The proof is trivial and hence omitted here.

Lemma 2: Let $f(x)$, $g(x)$ be defined above and $a_0 \in (0, r_p)$, where r_p be the smallest positive zero of the polynomial $f(a_0)^{p+1}g(a_0)^p - 1 = 0$ for $p \in (0, 1]$. Then,

(i) $f(a_0)^{p+1}g(a_0)^p < 1$.

(ii) the sequence $\{a_n\}$ is decreasing and $a_n < r_p$ for $n \geq 0$

Let r_p be the smallest positive zero of the polynomial $f(a_0)^{p+1}g(a_0)^p - 1 = 0$. Using Taylor's expansion of $F(y_0)$ around x_0,

$$
\begin{aligned}
z_0 - x_0 &= y_0 - x_0 - 5\Gamma_0 F(y_0) \\
&= y_0 - x_0 - 5\Gamma_0 \\
&\quad \times \int_0^1 [F'(x_0 + t(y_0 - x_0)) - F'(x_0)] \\
&\quad (y_0 - x_0)dt.
\end{aligned}
$$

Apply norm on both sides, we get

$$\|z_0 - x_0\| \leq \|y_0 - x_0\| + \frac{5}{(p+1)}K\beta\|y_0 - x_0\|^{p+1}.$$

Also,

$$\|z_0 - y_0\| \leq \frac{5}{(p+1)}K\beta\eta^p\|y_0 - x_0\| = \frac{5}{(p+1)}a_0\|y_0 - x_0\|.$$

Again, use the Taylor's expansion of $F(z_0)$ and (2), we have

$$
\begin{aligned}
\|x_1 - x_0\| &\leq \left\| y_0 - x_0 - \frac{9}{5}\Gamma_0 \int_0^1 [F'(x_0 + t(y_0 - x_0)) \right. \\
&\quad - F'(x_0)]dt(y_0 - x_0)\Gamma_0 \int_0^1 \\
&\quad \times [F'(x_0 + t(y_0 - x_0)) - F'(x_0)]dt(y_0 - x_0) \\
&\quad - \frac{1}{5}\Gamma_0 \int_0^1 [F'(x_0 + t(z_0 - x_0)) \\
&\quad \left. - F'(x_0)]dt(z_0 - x_0) \right\| \\
&= \|y_0 - x_0\| + \frac{4K\beta}{5(p+1)}\|y_0 - x_0\|^{p+1} \\
&\quad + \frac{K\beta}{5(p+1)}\|y_0 - x_0\|^{(p+1)}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(1 + \frac{5K\beta}{(p+1)}\|y_0 - x_0\|^p\right)^{p+1} \\
&= \left(1 + \frac{4a_0}{5(p+1)} + \frac{a_0}{5(p+1)}\left(1 + \frac{5a_0}{(p+1)}\right)^{p+1}\right) \\
&\quad \times \|y_0 - x_0\| \\
&= (1 + h(a_0))\eta. \tag{7}
\end{aligned}
$$

Now, for $a_0 < r_p$ and applying assumptions (i)-(iii), we have

$$
\begin{aligned}
\|I - \Gamma_0 F'(x_1)\| &\leq \|\Gamma_0\|\|F'(x_1) - F'(x_0)\| \\
&\leq K\beta\|x_1 - x_0\|^p \leq a_0(1 + h(a_0))^p \\
&< 1. \tag{8}
\end{aligned}
$$

By the Banach Lemma, Γ_1 exists and

$$\|\Gamma_1\| \leq \frac{1}{1 - a_0(1 + h(a_0))^p}\|\Gamma_0\| = f(a_0)\|\Gamma_0\|. \tag{9}$$

For $a_0(1 + h(a_0))^p < 1$, we need $a_0 < r_p$. Now we prove the following inequalities using induction.

$$
\left.
\begin{aligned}
&(I) \;\; \|\Gamma_n\| \leq f(a_{n-1})\|\Gamma_{n-1}\|, \\
&(II) \;\; \|\Gamma_n F(x_n)\| \leq f(a_{n-1})g(a_{n-1}) \\
&\quad \|\Gamma_{n-1}F(x_{n-1})\|, \\
&(III) \;\; \|z_n - y_n\| \leq \frac{5}{(p+1)}a_0 f(a_0)^{p+2}g(a_0)^{p+1} \\
&\quad \times \|y_0 - x_0\|, \\
&(IV) \;\; K\|\Gamma_n\|\|\Gamma_n F(x_n)\|^p \leq a_n, \\
&(V) \;\; \|x_{n+1} - x_n\| \leq (1 + h(a_n))\|\Gamma_n F(x_n)\|.
\end{aligned}
\right\} \tag{10}
$$

Using mathematical induction, we prove that the above inequalities. For $n = 1$, (I) hold true from (8). To prove (II), using Taylor's formula,

$$
\begin{aligned}
F(x_1) &= F(y_0) + F'(y_0)(x_1 - y_0) \\
&\quad + \int_{y_0}^{x_1} (F'(x) - F'(y_0))dx \\
&= \int_0^1 [F'(x_0 + t(y_0 - x_0)) - F'(x_0)](y_0 - x_0)dt \\
&\quad - (F'(y_0) - F'(x_0) + F'(x_0)) \\
&\quad \times \Gamma_0\left(\frac{9}{5}F(y_0) + \frac{1}{5}F(z_0)\right) \\
&\quad - \Gamma_0\left(\frac{9}{5}F(y_0) + \frac{1}{5}F(z_0)\right) \\
&\quad \times \int_0^1 [F'(y_0 + t(x_1 - y_0)) - F'(y_0)]dt.
\end{aligned}
$$

Since,

$$\left\|\frac{9}{5}F(y_0) + \frac{1}{5}F(z_0)\right\| \leq \frac{\eta}{\beta}h(a_0).$$

Then, we get

$$
\begin{aligned}
\|F(x_1)\| &\leq \frac{K\eta^{p+1}}{p+1} + K\eta^{p+1}h(a_0) + \frac{K\eta^{p+1}}{(p+1)}h(a_0)^{p+1} \\
&\quad + \frac{\eta}{\beta}h(a_0). \tag{11}
\end{aligned}
$$

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From (9), (11), we get

$$
\begin{aligned}
\|\Gamma_1 F(x_1)\| &\leq \|\Gamma_1\|\|F(x_1)\| \\
&\leq f(a_0)\|\Gamma_0\|\|F(x_1)\| \\
&= f(a_0)\Big[\frac{a_0}{(p+1)} + (a_0+1)h(a_0) \\
&\quad + \frac{a_0}{(p+1)}h(a_0)^{p+1}\Big] \\
&= f(a_0)g(a_0)\|y_0 - x_0\|. \quad (12)
\end{aligned}
$$

Also, from (9), we get

$$
\begin{aligned}
\|z_1 - y_1\| &\leq 5\|\Gamma_1\|\|F(y_1)\| \\
&\leq \frac{5}{(p+1)}K\beta f(a_0)\|y_1 - x_1\|^{p+1} \\
&= \frac{5}{(p+1)}a_0 f(a_0)^{p+2}g(a_0)^{p+1} \\
&\quad \|y_0 - x_0\|. \quad (13)
\end{aligned}
$$

By using (I) and (II), we get

$$
\begin{aligned}
K\|\Gamma_1\|\|y_1 - x_1\|^p &= Kf(a_0)\|\Gamma_0\|f(a_0)^p g(a_0)^p \\
&\quad \times \|y_0 - x_0\|^p \\
&= K\beta\eta^p f(a_0)^{p+1}g(a_0)^p \\
&= a_0 f(a_0)^{p+1}g(a_0)^p \\
&= a_1. \quad (14)
\end{aligned}
$$

From, (7), we get (V) hold true for $n = 1$. Hence, by induction process, it can be proved that (I)-(V) hold true for $n + 1$.

III. Convergence Analysis

Theorem 1: Let X and Y be Banach spaces and $F(x)$ be a nonlinear Fréchet differentiable operator in an open convex domain Ω. Let the assumptions (i)-(iii) are satisfied. Let us denote $a_0 = K\beta\eta^p$ and $a_0 < r_p$. Then, the sequence $\{x_n\}$ defined in (2) and starting at x_0 converge to a solution x^* of (1). In that case, the solution x^* and the iterates x_n, y_n and z_n lies in $\overline{\mathcal{B}}(x_0, R\eta)$ and unique in $\mathcal{B}(x_0, \eta/a_0^{1/p})$, where, $R = \frac{5a_0}{p+1} + \frac{h(a_0)+1}{1-(f(a_0)g(a_0))^p}$.
Proof:

In order to establish the convergence of $\{x_n\}$, it is sufficient to show that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ lie in $\overline{\mathcal{B}}(x_0, R\eta)$ and a Cauchy sequence. From (10), we get

$$
\begin{aligned}
\|y_n - x_n\| &\leq f(a_{n-1})g(a_{n-1})\|y_{n-1} - x_{n-1}\| \\
&\leq \prod_{j=0}^{n-1} f(a_j)g(a_j)\|y_0 - x_0\| \\
&\leq \prod_{j=0}^{n-1} f(a_j)g(a_j)\eta, \quad (15)
\end{aligned}
$$

and

$$
\begin{aligned}
\|x_{m+n} - x_m\| &\leq \|x_{m+n} - x_{m+n-1}\| + \ldots \\
&\quad + \|x_{m+1} - x_m\| \\
&\leq (1 + h(a_{m+n-1}))\|y_{m+n-1} - x_{m+n-1}\| \\
&\quad + \ldots + (1 + h(a_m))\|y_m - x_m\| \\
&\leq (1 + h(a_m))\Big[\prod_{j=0}^{m+n-2} f(a_j)g(a_j) + \ldots \\
&\quad + \prod_{j=0}^{m-1} f(a_j)g(a_j)\Big]\eta. \quad (16)
\end{aligned}
$$

Now, for $a_0 = r_p$. We obtain $f(a_0)^{p+1}g(a_0)^p = 1$, $a_n = a_{n-1} = \ldots = a_0$. This gives

$$
\|y_n - x_n\| \leq (1 + h(a_0))(f(a_0)g(a_0))^n\|y_0 - x_0\|,
$$

and

$$
\begin{aligned}
\|x_{m+n} - x_m\| &\leq (1 + h(a_0))\|y_0 - x_0\| \\
&\quad \sum_{i=0}^{m+n-1}(f(a_0)g(a_0))^i. \quad (17)
\end{aligned}
$$

Hence, if we take $m = 0$, we get

$$
\|x_n - x_0\| \leq (1 + h(a_0))\|y_0 - x_0\|\sum_{i=0}^{n-1}(f(a_0)g(a_0))^i. \quad (18)
$$

Also,

$$
\begin{aligned}
\|y_n - x_0\| &\leq \|y_n - x_n\| + \|x_n - x_0\| \\
&\leq (1 + h(a_0))(f(a_0)g(a_0))^n\|y_0 - x_0\| \\
&\quad + (1 + h(a_0))\|y_0 - x_0\|\sum_{i=0}^{n-1}(f(a_0)g(a_0))^i \\
&= (1 + h(a_0))\Big[(f(a_0)g(a_0))^n \\
&\quad + \sum_{i=0}^{n-1}(f(a_0)g(a_0))^i\Big]\eta \\
&= (1 + h(a_0))\frac{1 - (f(a_0)g(a_0))^{n+1}}{1 - (f(a_0)g(a_0))}\eta \\
&< R\eta, \quad (19)
\end{aligned}
$$

and

$$
\begin{aligned}
\|z_n - y_n\| &\leq 5\|\Gamma_n\|\|F(y_n)\| \\
&\leq \frac{5}{p+1}K\beta f(a_0)^n\|y_n - x_n\| \\
&= \frac{5}{(p+1)}a_0(f(a_0)^{p+2}g(a_0)^{p+1})^n \\
&\quad \|y_0 - x_0\|. \quad (20)
\end{aligned}
$$

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Hence,

$$
\begin{aligned}
\|z_n - x_0\| &\leq \|z_n - y_n\| + \|y_n - x_0\| \\
&= \frac{5}{(p+1)} a_0 (f(a_0)^{p+2} g(a_0)^{p+1})^n \\
&\quad \|y_0 - x_0\| \\
&\quad +(1 + h(a_0)) \frac{1 - (f(a_0)g(a_0))^{n+1}}{1 - (f(a_0)g(a_0))} \\
&\quad \|y_0 - x_0\| \\
&< \Big(\frac{5}{(p+1)} a_0 + \\
&\quad (1 + h(a_0)) \\
&\quad \frac{1 - (f(a_0)g(a_0))^{n+1}}{1 - (f(a_0)g(a_0))} \Big) \eta \\
&< R\eta. \qquad (21)
\end{aligned}
$$

Thus, $x_n, y_n, z_n \in \overline{\mathcal{B}}(x_0, R\eta)$. Also, we can conclude that $\{x_n\}$ is a Cauchy sequence. On taking the limit as $n \to \infty$ in (18), we get $x^* \in \overline{\mathcal{B}}(x_{\alpha,0}, R\eta)$. To show that x^* is a solution of $F(x) = 0$. We have that $\|F(x_n)\| \leq \|F'(x_n)\| \|\Gamma_n F(x_n)\|$ and the sequence $\{\|F'(x_n)\|\}$ is bounded as

$$\|F'(x_n)\| \leq \|F'(x_0)\| + K\|x_n - x_0\|^p < \|F'(x_0)\| + KR\eta^p.$$

Since F is continuous, by taking limit as $n \to \infty$, we get $F(x^*) = 0$. To prove the uniqueness of the solution, if y^* be the another solution of (1) in $\mathcal{B}(x_0, \eta/a_0^{1/p}) \cap \Omega$ then we have

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))dt(y^* - x^*).$$

Clearly, $y^* = x^*$, if $\int_0^1 F'(x^* + t(y^* - x^*))dt$ is invertible. This follows from

$$
\begin{aligned}
\|\Gamma_0\| \| \int_0^1 [F'(x^* + t(y^* - x^*)) - F'(x_0)]dt\| & \qquad (22) \\
\leq K\beta \int_0^1 \|x^* + t(y^* - x^*) - x_0\|^p dt & \\
\leq K\beta \int_0^1 [(1-t)\|x^* - x_0\| + t\|y^* - x_0\|]^p dt & \\
\leq K\beta (R\eta(1-t) + (\eta/a_0^{1/p})t)^p & \\
< K\beta \int_0^1 (\eta/a_0^{1/p})^p = 1, &
\end{aligned}
$$

and by Banach Lemma. Thus, $y^* = x^*$.

IV. NUMERICAL EXAMPLES

An interesting possibility arising from the study of the convergence of iterative methods for solving equations is to obtain results of existence and uniqueness of solutions for different types of equations. In this section, we provide some results of this type for a nonlinear Hammerstein integral equation of the second kind

Example 1:

$$x(s) = 1 + \frac{1}{3} \int_0^1 G(s,t)x(t)^{3/2}dt \quad s \in [0,1] \qquad (23)$$

for $x \in X = C[a,b]$ is the space of continuous functions on $[0,1]$ with max norm $\|x\| = \max_{s \in [0,1]} |x(s)|$, where $G(s,t)$ is the kernel,

$$
G(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t, \end{cases} \qquad (24)
$$

Now, we find the First order Fréchet derivative of (23),

$$F'(x)u(s) = u(s) - \frac{1}{2} \int_0^1 G(s,t)x(t)^{1/2}u(t)dt$$

From this,

$$
\begin{aligned}
\|F'(x) - F'(y)\| &\leq \frac{1}{2}|\max_{s \in [0,1]} \int_0^1 G(s,t)|\|x - y\|^{1/2} \\
&\leq \frac{1}{16}\|x - y\|^{1/2}
\end{aligned}
$$

Here, we observe that F' does not satisfy the Lipschitz continuity condition for all $x, y \in \Omega$ but it satisfies the Hölder continuity condition. Hence, we we get, $K = 1/16$, $p = 1/2$. For a fixed $x_0(s)$, we have $\|\Gamma_0\| = \|F'(x_0)^{-1}\| \leq \frac{16}{15} = \beta$, $\|\Gamma_0 F(x_0)\| \leq \frac{4}{87} = \eta$. Using these all, we get $a_0 = K\beta\eta^p = 0.0142948 \leq r_3 = 0.442057$. Hence, we observed that the the convergence theorem satisfies all the conditions. Hence, we find that a solution of (23) exists in $\overline{\mathcal{B}}(x_0, 0.0489939) \subseteq \Omega$ and the solution is unique in the ball $\mathcal{B}(x_0, 225.00) \cap \Omega$.

Example 2:

$$x(s) = 1 + \int_0^1 G(s,t)x(t)^{5/2}dt \quad s \in [0,1] \qquad (25)$$

for $x \in X = C[a,b]$ is the space of continuous functions on $[0,1]$ with max norm $\|x\| = \max_{s \in [0,1]} |x(s)|$, where $G(s,t)$ is the kernel,

$$
G(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t, \end{cases} \qquad (26)
$$

Now, we find the First order Fréchet derivative of (25),

$$F'(x)u(s) = u(s) - \frac{5}{2} \int_0^1 G(s,t)x(t)^{3/2}u(t)dt$$

From this,

$$
\begin{aligned}
\|F'(x) - F'(y)\| &\leq \frac{5}{2}|\max_{s \in [0,1]} \int_0^1 G(s,t)|\|x - y\|^{3/2} \\
&\leq \frac{5}{16}\|x - y\|^{3/2}
\end{aligned}
$$

Here, we observe that F' does not satisfy the Lipschitz continuity condition for all $x, y \in \Omega$ but it satisfies the Hölder continuity condition. Hence, we get, $K = 5/16$, $p = 3/2$. For a fixed $x_0(s)$, we have $\|\Gamma_0\| = \|F'(x_0)^{-1}\| \leq \frac{48}{43} = \beta$, $\|\Gamma_0 F(x_0)\| \leq \frac{2}{43} = \eta$. Using these all, we get $a_0 = K\beta\eta^p = 0.00349917 \leq r_{3/2} = 0.534265$. Hence, we observed that the convergence theorem satisfies all the conditions. Hence, we find that a solution of (25) exists in $\overline{\mathcal{B}}(x_0, 0.0471155) \subseteq \Omega$ and the solution is unique in the ball $\mathcal{B}(x_0, 2.01799) \cap \Omega$.

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V. Conclusions

The semilocal convergence of fifth order iterative method for solving nonlinear equations in Banach spaces is established under the assumption that the first order Fréchet derivative satisfies the Hölder continuity condition. The existence and uniqueness region of solution for the method is obtained. Some numerical examples are worked out to demonstrate the efficiency of our convergence analysis.

References

[1] L.B.Rall, Computational Solution of Nonlinear Operator Equations, Robert E. Krieger, New York, 1969.

[2] L.V. Kantorovich, G.P. Akilov, *Functional Analysis,* Pergamon Press, Oxford, 1982.

[3] Miguel A. Hernández, Jose M.Gutierrez, Third-order iterative methods for operators with bounded second derivative,*Journal of Computational and Applied Mathematics,* 82 (1997) 171-183.

[4] P.K.Parida, D.K.Gupta, Recurrence relations for a Newton-like method in Banach spaces, *J. Comput. Appl. Math.*, 206 (2007) 873-887.

[5] S. Amat, M.A. Hernandez, and N.Romero, Semilocal convergence of a sixth order iterative method for quadratic equations, *J. Appl. Numer. Math.*, 62 (2012) 833-841.

[6] M.A. Hernandez, N. Romero, On a characterization of some Newton-like methods of R-order at least three, *J. Comput. Appl. Math.*, 183 (2005) 53-66.

[7] V.Candela, A.Marquina, Valencia, Recurrence Relation for Rational Cubic Methods I: The Halley Method, *Computing,* 44 (1990) 169-184.

[8] V.Candela, A.Marquina, Valencia, Recurrence Relation for Rational Cubic Methods II: The Chebyshev Method, *Computing,* 45 (1990) 355-367.

[9] J.A. Ezquerro, M.A. Hernandez, Recurrence relations for Chebyshev-type methods, *Appl. Math. Optim.,* 41 (2000) 227-236.

[10] J.M.Gutiérrez, M.A. Hernández, Recurrence Relations for the Super-Halley Method, *Journal of Computer Math.Applications,* 36 (1998) 1-8.

[11] M.Prashanth, D.K.Gupta, Recurrence relations for Super-Halley's Method with Hölder continuous second derivative in Banach spaces, *Kodai Math. J.,* 36 (2013), 119–136.

[12] M.Prashanth, D.K.Gupta, Semilocal convergence of a continuation method with Hölder continuous second derivative in Banach spaces, *Journal of Computational and Applied Mathematics,* 236 (2012) 3174–3185.

[13] V.Arroyo, A.Cordero, and J.R.Torregrosa, Semilocal convergence by using recurrence relations for a fifth-order method in Banach spaces, *Journal of Computational and Applied Mathematics,* 273(2015) 205-213.

[14] A.Cordero, M.A.Hernandez, N.Romero and J.R.Torregrosa, Approximation of artificial satellites preliminary orbits: the efficiency challenge, *Math. Comput. Modelling,* 54(2011) 1802-1807.

[15] Lin Zheng, Chuanqing Gu, Recurrence relations for semilocal convergence of a fifth-order method in Banach spaces, *Numer Algor,* 59(2012) 623-638.