Free Vibration Analysis of Conical Helicoidal Rods Having Elliptical Cross Sections Positioned in Different Orientation

Merve Ermis, Akif Kutlu, Nihal Eratlı, Mehmet H. Omurtag

***Abstract*—**In this study, the free vibration analysis of conical helicoidal rods with two different elliptically oriented cross sections is investigated and the results are compared by the circular cross-section keeping the net area for all cases equal to each other. Problems are solved by using the mixed finite element formulation. Element matrices based on Timoshenko beam theory are employed. The finite element matrices are derived by directly inserting the analytical expressions (arc length, curvature, and torsion) defining helix geometry into the formulation. Helicoidal rod domain is discretized by a two-noded curvilinear element. Each node of the element has 12 DOFs, namely, three translations, three rotations, two shear forces, one axial force, two bending moments and one torque. A parametric study is performed to investigate the influence of elliptical cross sectional geometry and its orientation over the natural frequencies of the conical type helicoidal rod.

***Keywords*—**Conical helix, elliptical cross section, finite element, free vibration.

I. Introduction

THE number of researches about circular or non-circular helicoidal geometry has been increased and some of these studies are about double helix-DNA [1], carbon nanotubes [2], fibers [3], polymers [4], dampers [5], and staircases [6] *etc.* Helicoidal rods are also preferred for various applications, such as, to support mechanical equipment, to transfer the forces, absorb the energy or reduce the vibration in structural system, to be used as steam generators in nuclear industries, and helical actuators used to manage thermal heating in smart material applications, etc. In the literature, the theoretical and numerical studies exist on the static/dynamic analyses of elastic helices having circular or rectangular cross-sections. In some of these studies, [7] studied the dynamic analysis of helical rods based on the exact differential equations governing static behavior of an infinitesimal element by using the finite element method (FEM). The static analysis of circular and non-circular helices having rectangular cross-section is investigated by using the mixed FEM in [8]. References [9], [10] applied to investigate the static or dynamic analysis of helices by the transfer matrix method. The free vibration analysis of a circular helicoidal bar is studied by using both the dynamic transport matrix method and the finite element method in [11]. Reference [12] employed the exact element method for the static analysis of helicoidal structures of variable cross section. The pseudospectral method is used to investigate the free vibration analysis of cylindrical helical springs with circular cross-sections in [13]. Considering the warping deformations of the cross-section, free vibration analysis of naturally curved and twisted beams are investigated by using the analytical study in [14]. The cylindrical and non-cylindrical helicoidal rods having thin-thick walled circular and non-circular cross sections, equilateral triangular, cruciform and composite cross sections are studied by using the mixed FEM in [15], [16].

In this study, the both curvatures and the arc length of helicoidal geometry are directly taken into account in the mixed FE formulation, and it is applied to dynamic analysis of viscoelastic helicoidal rods in [17], [18]. By using the Timoshenko beam theory adapted mixed FE formulation, the free vibration analysis of conical helicoidal rods having elliptical cross sections is solved. As a parametric study, the influence of the orientation of elliptical cross-sections on the natural frequencies is investigated. Also, the fundamental natural frequencies of elliptical cross sections are compared with the circular cross section keeping the cross sectional area constant for both cases. Some benchmark examples are presented for the literature.

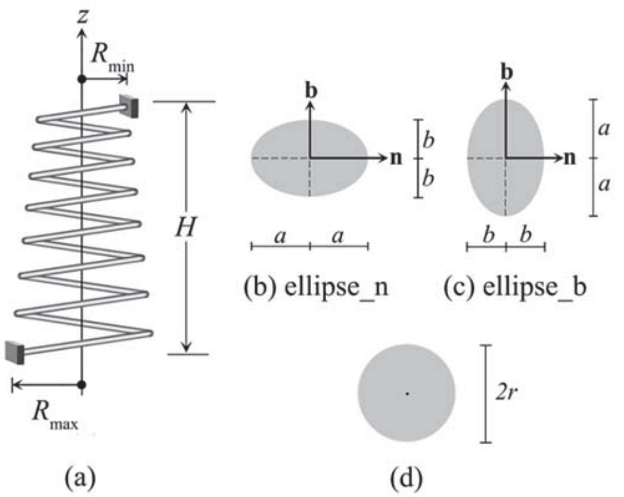

Fig. 1 Conical helix and cross-sectional geometries $a = 5\,\text{mm}$, $b = 2.5\,\text{mm}$ and $r = 3.53553\,\text{mm}$

M. Ermis, A. Kutlu, N. Eratlı, and M. H. Omurtag are with the Faculty of Civil Engineering, Istanbul Technical University, Maslak 34469, Turkey (phone: +90-212-2856551; e-mail: ermism@itu.edu.tr, e-mail: kutluak@itu.edu.tr, e-mail: eratli@itu.edu.tr, e-mail: omurtagm@itu.edu.tr).

II. Formulation

A. Helix Geometry and Functional

The parametrical representation of helical geometry can be given as: $x = R(\varphi)\cos\varphi$, $y = R(\varphi)\sin\varphi$, $z = p(\varphi)\varphi$, $p(\varphi) = R(\varphi)\tan\alpha$, horizontal angle φ, the pitch angle and the centerline radius $R(\varphi)$, the step for unit angle of the helix $p(\varphi)$. By using $c(\varphi) = \sqrt{R^2(\varphi) + p^2(\varphi)}$, the infinitesimal arc length becomes $\mathrm{d}s = c(\varphi)\,\mathrm{d}\varphi$. The radius of conical helix

$$R(\varphi) = R_{\max} + (R_{\min} - R_{\max})\varphi/2n\pi \tag{1}$$

where $R_{\max}$ and $R_{\min}$ are the bottom and top radius, respectively (see Fig. 1).

In the Frenet coordinate system for the helix geometry, based on Timoshenko beam theory the field equations, exist in [10], can be given in the form

$$\left.\begin{aligned} -\mathbf{T}_{,s} - \mathbf{q} + \rho A\ddot{\mathbf{u}} = \mathbf{0} \\ -\mathbf{M}_{,s} - \mathbf{t}\times\mathbf{T} - \mathbf{m} + \rho\mathbf{I}\ddot{\mathbf{\Omega}} = \mathbf{0} \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} \mathbf{u}_{,s} + \mathbf{t}\times\mathbf{\Omega} - \mathbf{C}_{\gamma}\mathbf{T} = \mathbf{0} \\ \mathbf{\Omega}_{,s} - \mathbf{C}_{\kappa}\mathbf{M} = \mathbf{0} \end{aligned}\right\} \tag{3}$$

where the accelerations of the displacement $\ddot{\mathbf{u}}$ and rotations $\ddot{\mathbf{\Omega}}$ vectors, the displacement vector $\mathbf{u}(u_t, u_n, u_b)$, the rotational vector $\mathbf{\Omega}(\Omega_t, \Omega_n, \Omega_b)$, the force vector $\mathbf{T}(T_t, T_n, T_b)$, the moment vector $\mathbf{M}(M_t, M_n, M_b)$, the material density ρ, the area of the cross section A, the moments of inertia $\mathbf{I}$, and compliance matrices, $\mathbf{C}_{\gamma}$ and $\mathbf{C}_{\kappa}$, the distributed external force $\mathbf{q}$ and moment $\mathbf{m}$ vectors, respectively. In the free vibration analysis, considering the harmonic motion, it is obvious that $\mathbf{q} = \mathbf{m} = \mathbf{0}$. Incorporating Gâteaux differential with potential operator concept [19] yields the functional in terms of (2), (3)

$$\begin{aligned} \mathbf{I}(\mathbf{y}) = & -\left[\mathbf{u}, \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right] + \left[\mathbf{t}\times\mathbf{\Omega}, \mathbf{T}\right] - \left[\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}s}, \mathbf{\Omega}\right] - \frac{1}{2}\left[\mathbf{C}_{\kappa}\ \mathbf{M}, \mathbf{M}\right] \\ & -\frac{1}{2}\left[\mathbf{C}_{\gamma}\ \mathbf{T}, \mathbf{T}\right] - \frac{1}{2}\rho A\omega^2\left[\mathbf{u}, \mathbf{u}\right] - \frac{1}{2}\rho\ \omega^2\left[\mathbf{I}\mathbf{\Omega}, \mathbf{\Omega}\right] \\ & -\left[\mathbf{q}, \mathbf{u}\right] - \left[\mathbf{m}, \mathbf{\Omega}\right] + \left[\left(\mathbf{T} - \hat{\mathbf{T}}\right), \mathbf{u}\right]_{\sigma} + \left[\left(\mathbf{M} - \hat{\mathbf{M}}\right), \mathbf{\Omega}\right]_{\sigma} \\ & + \left[\hat{\mathbf{u}}, \mathbf{T}\right]_{\varepsilon} + \left[\hat{\mathbf{\Omega}}, \mathbf{M}\right]_{\varepsilon} \end{aligned} \tag{4}$$

where square brackets indicate the inner product, the terms with hats are known values on the boundary and the subscripts ε and σ represent the geometric and dynamic boundary conditions, respectively.

B. The Mixed FE Method and Free Vibration Analysis

Linear shape functions, $\phi_i = (\varphi_j - \varphi)/\Delta\varphi$ and $\phi_j = (\varphi - \varphi_i)/\Delta\varphi$ are employed, where $\Delta\varphi = (\varphi_j - \varphi_i)$. A two-noded curvilinear element is used to discretize the helicoidal rod domain, where the subscripts *i* and *j* shows the node numbers of the element. The degree of freedom (DOF) of the curved element is 24, where each node of the element has 12 DOFs. The variable vectors are $\mathbf{u}, \mathbf{\Omega}, \mathbf{T}, \mathbf{M}$ for per node. The curvatures and arc length of the helix are directly taken into consideration through the mixed FE formulation [11], [12].

The problem of determining the natural frequencies of a structural system reduces to the solution of a standard eigenvalue problem $\left([\mathbf{K}] - \omega^2[\mathbf{M}]\right)\{\mathbf{u}\} = \{\mathbf{0}\}$, where the system matrix $[\mathbf{K}]$, the mass matrix $[\mathbf{M}]$, the eigenvector (mode shape) $\mathbf{u}$ and the natural angular frequency ω of the system. In the mixed formulation, the explicit form of standard eigenvalue problem can be given as

$$\left(\begin{bmatrix} [\mathbf{K}_{11}] & [\mathbf{K}_{12}] \\ [\mathbf{K}_{22}] & [\mathbf{K}_{22}] \end{bmatrix} - \omega^2 \begin{bmatrix} [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{M}] \end{bmatrix}\right) \begin{Bmatrix} \{\mathbf{F}\} \\ \{\mathbf{U}\} \end{Bmatrix} = \begin{Bmatrix} \{\mathbf{0}\} \\ \{\mathbf{0}\} \end{Bmatrix} \tag{5}$$

where the nodal force and the moment vectors $\{\mathbf{F}\}$,and, the nodal displacement and rotation vectors $\{\mathbf{U}\} = \{\mathbf{u}\ \mathbf{\Omega}\}^T$. In order to attain consistency between (5) and $\left([\mathbf{K}] - \omega^2[\mathbf{M}]\right)\{\mathbf{u}\} = \{\mathbf{0}\}$, the $\{\mathbf{F}\}$ is eliminated in (5), which yields to the condensed system matrix $[\mathbf{K}^*] = [\mathbf{K_{22}}] - [\mathbf{K_{12}}]^{\mathrm{T}}[\mathbf{K_{11}}]^{-1}[\mathbf{K_{12}}]$. In the mixed formulation, the eigenvalue problem becomes $\left([\mathbf{K}^*] - \omega^2[\mathbf{M}]\right)\{\mathbf{U}\} = \{\mathbf{0}\}$.

III. Numerical Example

This is a parametric study of a conical helix, which has three different cross-sections (circular and two different elliptical orientations), three different number of active turns ($n = 2, 4, 6$), and three different pitch angles. The objective of this study is to investigate the influence of the above cited geometric properties on the natural frequency of the conical helix. Helicoidal rod is clamped at both ends. The orientations of the two different elliptical cross sections are as shown in (see Figs. 1 (b), (c)). The abbreviations "ellipse_n" and "ellipse_b" are used elliptical cross sections with long side oriented horizontal and vertical direction, respectively. The natural frequencies of the conical helix having the elliptical cross sections are compared with the results of the circular cross section. The net areas of the all three cross-sectional geometries are equal to each other.

The material and geometric properties of the helix are as follows: the modulus of elasticity is $E = 206\,\mathrm{GPa}$, Poisson's ratio is $\upsilon = 0.3$, density of material is $\rho = 7850\,\mathrm{kg/m^3}$, the taper ratio $R_{\min}/R_{\max}$ 0.5 where $R_{\max} = 100\,\mathrm{mm}$. The three different number of active turns and the height of the helix are

$n = 2, 4, 6$, and $H = 200, 400, 600\,\text{mm}$, respectively. The cross-sectional dimensions of the circular and elliptical cross sections are $a = 5\,\text{mm}$ and $b = 2.5\,\text{mm}$; $r = 3.53553\,\text{mm}$ (see Figs. 1 (b)-(d)). The torsional moment of inertia [14] for an elliptical cross section is given as

$$I_t = \pi \frac{a^3 b^3}{a^2 + b^2} \tag{6}$$

The natural frequencies of the conical helix having circular cross section are obtained via the mixed finite element and these results are compared for first six natural frequencies with the commercial programSAP2000 and verified. The convergence by the present study and the SAP2000 is given for the fundamental natural frequency in Fig. 2. In this study 200 mixed finite element results are compared by the 1000 displacement type SAP2000 elements. The normalized percent differences between these two finite elements models are calculated and the results are tabulated in Tables I-III, where the absolute maximum percent difference is 0.38%. The natural frequencies of elliptical cross sections ("ellipse_n", "ellipse_b") and the percent differences normalized in case of "ellipse_n" with respect to "ellipse_b" are also given in Tables IV-VI.

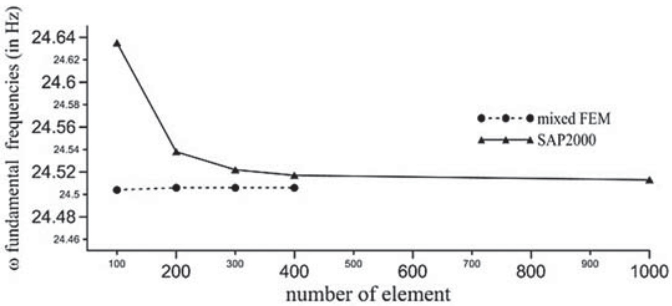

Fig. 2 The convergence analysis for ω fundamental frequency of the conical helix having circular cross section for the number of active turn n=4 and the height H=400 mm (see Table II)

The fundamental natural frequencies decrease by increasing the number of active turns n for each value of $H = 200, 400, 600\,\text{mm}$ (see Tables I-VI). The fundamental natural frequency values of $n = 4, 6$ are normalized with respect to $n = 2$ that correspond $H = 200, 400, 600\,\text{mm}$ and the decrease of the percent differences are given in Table VII for all type of cross-sections. The fundamental natural frequencies decrease by increasing the height H for each value of $n = 2, 4, 6$ (see Tables I-VI). The fundamental natural frequency values of $H = 400, 600\,\text{mm}$ are normalized with respect to $H = 200\,\text{mm}$ that correspond $n = 2, 4, 6$ and the decrease of the percent differences are tabulated in Table VIII for all type of cross-sections.

The fundamental natural frequencies of the conical helix are normalized with respect to the results of circular cross-section and the normalized percent differences are given in Table IX. It is observed that the fundamental natural frequencies of the elliptical cross sections ("ellipse_n", "ellipse_b" see Fig. 1 (b), (c)) decreased with respect to the circular cross section ("circle") in the range of 9.6%~15.4% and 11.0%~ 31.7%, respectively. When the fundamental frequencies of the elliptical cross-section ("ellipse_b") are compared with the respective results of the elliptical cross-section ("ellipse_n") to investigate the effect of the orientation of the long axis of the ellipse, the reductions by increasing the helix height are observed. These reductions are approximately 14.7%~15.2%, 7.4%~12.2% and -3.7%~-2.3%, respectively (see Tables IV-VI).

TABLE I
THE NATURAL FREQUENCIES OF CONICAL HELIX HAVING CIRCULAR CROSS SECTION FOR THE DIFFERENT NUMBER OF ACTIVE TURNS ($H = 200\,\text{mm}$)

n		ω (in Hz)					
		1	2	3	4	5	6
2	this study	46.9	52.8	63.3	80.9	89.5	120.3
	SAP2000	47.0	52.8	63.3	80.8	89.6	119.9
	diff.%	-0.21	0.00	0.00	0.12	-0.11	0.33
4	this study	26.6	28.4	34.5	37.0	46.6	49.2
	SAP2000	26.7	28.4	34.6	37.0	46.6	49.3
	diff.%	-0.38	0.00	-0.29	0.00	0.00	-0.20
6	this study	18.3	19.5	23.8	24.8	33.6	35.1
	SAP2000	18.3	19.5	23.8	24.8	33.6	35.2
	diff.%	0.00	0.00	0.00	0.00	0.00	-0.28

diff. % = (This study-SAP2000) ×100/This study)

TABLE II
THE NATURAL FREQUENCIES OF CONICAL HELIX HAVING CIRCULAR CROSS SECTION FOR THE DIFFERENT NUMBER OF ACTIVE TURNS ($H = 400\,\text{mm}$)

n		ω (in Hz)					
		1	2	3	4	5	6
2	this study	39.4	44.9	55.8	68.3	100.7	115.8
	SAP2000	39.4	44.9	55.8	68.1	100.5	115.4
	diff.%	0.00	0.00	0.00	0.29	0.20	0.35
4	this study	24.5	25.8	29.3	30.0	44.7	49.5
	SAP2000	24.5	25.8	29.3	30.0	44.8	49.4
	diff.%	0.00	0.00	0.00	0.00	-0.22	0.20
6	this study	17.3	18.0	19.6	19.9	33.4	35.1
	SAP2000	17.3	18.0	19.6	19.9	33.4	35.1
	diff.%	0.00	0.00	0.00	0.00	0.00	0.00

diff. % = (This study-SAP2000) ×100/This study)

TABLE III
THE NATURAL FREQUENCIES OF CONICAL HELIX HAVING CIRCULAR CROSS SECTION FOR THE DIFFERENT NUMBER OF ACTIVE TURNS ($H = 600\,\text{mm}$)

n		ω (in Hz)					
		1	2	3	4	5	6
2	this study	31.5	34.4	47.3	60.0	84.0	116.4
	SAP2000	31.6	34.4	47.3	59.9	83.7	116.2
	diff.%	-0.32	0.00	0.00	0.17	0.36	0.17
4	this study	20.1	20.4	26.4	28.8	40.4	43.1
	SAP2000	20.1	20.4	26.4	28.8	40.4	43.1
	diff.%	0.00	0.00	0.00	0.00	0.00	0.00
6	this study	14.1	14.2	18.3	19.5	30.6	30.9
	SAP2000	14.1	14.2	18.3	19.5	30.6	30.9
	diff.%	0.00	0.00	0.00	0.00	0.00	0.00

diff. % = (This study-SAP2000) ×100/This study)

TABLE IV
THE NATURAL FREQUENCIES OF CONICAL HELIX HAVING ELLIPTICAL CROSS SECTIONS FOR THE DIFFERENT NUMBER OF ACTIVE TURNS ($H = 200\text{mm}$)

n		ω (in Hz)					
		1	2	3	4	5	6
2	ellipse_n	42.8	61.7	71.3	92.0	115.5	124.1
	ellipse_b	36.3	45.3	49.2	63.4	73.1	108.9
	diff.%	15.19	26.58	31.00	31.09	36.71	12.25
4	ellipse_n	23.8	35.2	37.2	40.6	44.0	53.1
	ellipse_b	20.2	24.1	26.5	28.9	38.8	41.1
	diff.%	15.13	31.53	28.76	28.82	11.82	22.6
6	ellipse_n	16.3	24.9	25.5	27.9	30.7	37.3
	ellipse_b	13.9	16.4	18.4	19.3	27.2	29.8
	diff.%	14.72	34.14	27.84	30.82	11.40	20.11

diff. % = (ellipse_n - ellipse_b)×100/ ellipse_n)

TABLE V
THE NATURAL FREQUENCIES OF CONICAL HELIX HAVING ELLIPTICAL CROSS SECTIONS FOR THE DIFFERENT NUMBER OF ACTIVE TURNS ($H = 400\text{mm}$)

n		ω (in Hz)					
		1	2	3	4	5	6
2	ellipse_n	35.2	43.3	64.3	75.8	115.2	137.9
	ellipse_b	32.6	38.2	45.3	56.1	79.3	93.2
	diff.%	7.39	11.78	29.55	25.99	31.16	32.41
4	ellipse_n	22.0	24.5	27.4	39.6	40.8	47.8
	ellipse_b	19.6	21.8	23.5	26.0	36.5	41.6
	diff.%	10.91	11.02	14.23	34.34	10.54	12.97
6	ellipse_n	15.6	17.0	18.0	27.5	29.8	34.8
	ellipse_b	13.7	15.2	16.0	17.1	26.5	29.7
	diff.%	12.18	10.59	11.11	37.82	11.07	14.66

diff. % = (ellipse_n- ellipse_b)×100/ ellipse_n)

TABLE VI
THE NATURAL FREQUENCIES OF CONICAL HELIX HAVING ELLIPTICAL CROSS SECTIONS FOR THE DIFFERENT NUMBER OF ACTIVE TURNS ($H = 600\text{mm}$)

n		ω (in Hz)					
		1	2	3	4	5	6
2	ellipse_n	27.3	31.1	47.8	72.9	86.5	138.2
	ellipse_b	28.3	30.4	38.6	50.1	68.4	94.4
	diff.%	-3.66	2.25	19.25	31.28	20.92	31.69
4	ellipse_n	17.6	17.9	23.9	35.6	38.3	41.0
	ellipse_b	18.0	18.4	20.5	24.7	34.0	37.7
	diff.%	-2.27	-2.79	14.23	30.62	11.23	8.05
6	ellipse_n	12.3	12.4	16.4	26.9	27.4	28.8
	ellipse_b	12.7	12.9	14.0	16.6	25.4	27.0
	diff.%	-3.25	-4.03	14.63	38.29	7.30	6.25

diff. % = (ellipse_n - ellipse_b) ×100/ ellipse_n)

TABLE VII
THE PERCENT DECREASE OF THE FUNDAMENTAL NATURAL FREQUENCIES OF CONICAL HELIX HAVING CIRCULAR AND ELLIPTICAL CROSS SECTIONS IN THE CASE OF $n = 4,6$ WITH RESPECT TO $n = 2$

H(mm)	n	circle	ellipse_n	ellipse_b
200	4	43.3%	44.4%	44.4%
	6	61.0%	61.9%	61.9%
400	4	37.8%	37.5%	37.5%
	6	56.1%	55.7%	55.7%
600	4	36.2%	35.5%	35.5%
	6	55.2%	54.9%	54.9%

TABLE VIII
THE PERCENT DECREASE OF THE FUNDAMENTAL NATURAL FREQUENCIES OF CONICAL HELIX HAVING CIRCULAR AND ELLIPTICAL CROSS SECTIONS IN THE CASE OF $H = 400,600\text{mm}$ WITH RESPECT TO $H = 200\text{mm}$

n	H(mm)	circle	ellipse_n	ellipse_b
2	400	16.0%	17.8%	10.2%
	600	32.8%	36.2%	22.0%
4	400	7.9%	7.6%	3.0%
	600	24.4%	26.1%	10.9%
6	400	5.5%	4.3%	1.4%
	600	23.0%	24.5%	8.6%

TABLE IX
THE PERCENT DECREASE OF THE FUNDAMENTAL NATURAL FREQUENCIES OF CONICAL HELIX HAVING ELLIPTICAL CROSS SECTIONS WITH RESPECT TO CIRCULAR CROSS SECTION

H(mm)	n	ellipse_n	ellipse_b
200	2	9.6%	29.2%
	4	11.8%	31.7%
	6	12.3%	31.7%
400	2	11.9%	20.9%
	4	11.4%	25.0%
	6	10.9%	26.3%
600	2	15.4%	11.3%
	4	14.2%	11.7%
	6	14.6%	11.0%

IV. CONCLUSIONS

The free vibration analysis of conical helicoidal rod circular and elliptical cross sections is investigated using the mixed finite element formulation. Curved finite element is derived based on Timoshenko beam theory. The analytical functions of both curvatures and arc length defining the helix geometry are directly inserted into the mixed finite element formulation. The results of this formulation are compared with the commercial program SAP2000 for circular cross section. Some parametric studies are performed to observe the effects of the number of active turns, the height of helix and the orientation of the elliptical cross-section on the natural frequencies of the conical helix. The increase in the number of active turns n and the helix vertical height decreased the natural frequencies of the conical helix.

REFERENCES

[1] A. A Lucas and P. Lambin, "Diffraction by DNA, carbon nanotubes and other helical nanostructures", *Rep. Prog. Phys.*, vol. 68, pp. 1181-1249, 2005.

[2] J. Cherusseri, R. Scharma and K.K. Kar, "Helically coiled carbon nanotube electrodes for flexible supercapacitors", *Carbon*, vol. 105, pp.113-125, 2016.

[3] D. Tentori, A. Garcia-Wadner and J.A. Rodriguez-Garcia, "Use of fiber helical coils to obtain polarization insensitive fiber devices", *Opt. Fiber Technol.*, vol. 31, pp.13-19, 2016.

[4] E.H. Egelman, "Three-dimensional reconstruction of helical polymers", *Arch. Biochem. Biophys.*, vol. 581, 54-58, 2015.

[5] N. Barbieri, R. Barbieri, R.A. de Silva, M.J. Mannala and L.D.S.V. Barbieri "Nonlinear dynamic analysis of wire-rope isolator and Stockbridge damper", *Nonlinear Dyn.*, doi: 10.1007/s11071-016-2903-1, 2016.

[6] A. Baratta and I. Corbi, "Equilibrium models for helicoidal laterally supported staircases", *Comput. Struct.*, vol. 124, pp. 21-28, 2013.

[7] J.E. Mottershead, "Finite elements for dynamical analysis of helical rods", *Int. J. Mech. Sci.*, vol. 22, pp. 267–283, 1980.

[8] M.H. Omurtag and A.Y. Aköz, "The mixed finite element solution of helical beams with variable cross-section under arbitrary loading", *Comput. Struct.*, vol. 43, pp. 325–331, 1992.

[9] V. Haktanır and E. Kıral, "Statical analysis of elastically and continuously supported helicoidal structures by the transfer and stiffness matrix methods", *Comput. Struct.*, vol. 49, pp. 663–677, 1993.

[10] V. Yıldırım, "Investigation of parameters affecting free vibration frequency of helical springs", *Int. J. Numer. Methods Eng.*, vol. 39, pp. 99–114, 1996.

[11] S.A. Alghamdi, M.A. Mohiuddin and H.N. Al-Ghamedy, "Analysis of free vibrations of helicoidal beams", *Eng. Comput.*, vol. 15, pp. 89–102, 1998.

[12] W. Busool and M. Eisenberger, "Exact static analysis of helicoidal structures of arbitrary shape and variable cross section", *J. Struct. Eng.*, vol. 127, pp. 1266–1275, 2001.

[13] J. Lee, "Free vibration analysis of cylindrical helical springs by the pseudospectral method", *J. Sound Vib.*, vol. 302, pp. 185–196, 2007.

[14] A.M. Yu, C.J. Yang and G.H. Nie, "Analytical formulation and evaluation for free vibration of naturally curved and twisted beams", *J. Sound Vib.*, vol. 329, pp. 1376–1389, 2010.

[15] N. Eratlı, M. Ermis and M.H. Omurtag, "Free vibration analysis of helicoidal bars with thin-walled circular tube cross-section via mixed finite element method", *Sigma Journal of Engineering and Natural Sciences*, vol. 33(2), pp. 200-218, 2015.

[16] N. Eratlı, M. Yılmaz, K. Darılmaz and M.H. Omurtag, "Dynamic analysis of helicoidal bars with non-circular cross-sections via mixed FEM", *Struct. Eng. Mech.*, vol. 57(2), pp. 221-238, 2016.

[17] M. Ermis, *The dynamic analysis of non-cylindrical viscoelastic helical bars using mixed finite element*, Msc Thesis, Istanbul Technical University, Istanbul, 2015.

[18] M. Ermis, N. Eratlı, Argeso H., A. Kutlu and M.H. Omurtag, "Parametric Analysis of Viscoelastic Hyperboloidal Helical Rod", *Adv. Struct. Eng.*, doi: 10.1177/1369433216643584,2016.

[19] J.T. Oden and J.N. Reddy, *Variational Method in Theoretical Mechanics*, Springer-Verlag, Berlin, 1976.

[20] S. Timoshenko and J.N. Goodier, *Theory of elasticity*, McGraw-Hill, New York, 1951.