An Improved Variable Tolerance RSM with a Proportion Threshold

Chen Wu, Youquan Xu, Dandan Li, Ronghua Yang, Lijuan Wang

Abstract**—**In rough set models, tolerance relation, similarity relation and limited tolerance relation solve different situation problems for incomplete information systems in which there exists a phenomenon of missing value. If two objects have the same few known attributes and more unknown attributes, they cannot distinguish them well. In order to solve this problem, we presented two improved limited and variable precision rough set models. One is symmetric, the other one is non-symmetric. They all use more stringent condition to separate two small probability equivalent objects into different classes. The two models are needed to engage further study in detail. In the present paper, we newly form object classes with a different respect comparing to the first suggested model. We overcome disadvantages of non-symmetry regarding to the second suggested model. We discuss relationships between or among several models and also make rule generation. The obtained results by applying the second model are more accurate and reasonable.

Keywords**—**Incomplete information system, rough set, symmetry, variable precision.

I. Introduction

ROUGH set theory, proposed by Z. Pawlak in 1980s [1], [2], has been found to be a very useful mathematics tool for studying inexact, uncertain or vague information systems. Indiscernibility relation (reflexive, symmetric and transitive) is the basis of Z. Pawlak's rough set theory which is primarily applied to complete information system. In real world, due to the data measuring error or the limited ability in comprehending or acquiring data, we have to confront incomplete information systems (IIS) in knowledge discovery. Because of existing null values in incomplete information systems, such an indiscernibility relation as a kind of equivalence relation in Z. Pawlak's rough set theory, it is hard to construct due to the comparison between null value and real value is impossible. So, it is impossible for us to immediately cope with incomplete information with such kinds of indiscernibility relations.

Two approaches have been employed in rough set theory to deal with incomplete information systems. One is to transfer incomplete information table into complete information table by substituting null values with frequent attribute values, called indirect way. Another is to extend Z. Pawlak's rough set theory to incomplete information table, called direct way.

The direct approach attracts much more attention from scientists. For example, Kryszkiewicz, Stefanowski, Guoyin Wang respectively suggested tolerance relation [3], similarity relation [4], and limited tolerance relation [5], which are three popular models. Discernability of tolerance relation is very limited, since the equivalent probability of two objects with only few equal known attributes and much more unknown attributes is very small. Discernability of similarity relation is a little bit strong for it restricts that the second object's non-null value attribute number cannot be greater than the first one. Discernable ability of limited tolerance relation is also finite since its loose requirement of common non-null value attribute number. Based on the above discussion, we presented a variable precision rough set model [6], [7] by setting a proportion threshold for two objects in common non-null value attribute number to determine whether they belong to the same class or not. This variable precision classification relation is of only reflexivity, representing a generalized form of tolerance relation and similarity relations. Probabilistic rough set approximations are discussed in [8]. Other study ways are also can be seen in some other materials, reflecting that the research about rough set is energetic. For instance, on inconsistent incomplete decision tables approximation reduction method is explored in [9]. Variable precision rough set based decision tree classifier is researched in [10]. We also suggested another variable precision relation for rough set model [11], which remains symmetric, keeps advantages and overcomes some shortcomings of limited tolerance relation. It can be used to dispose incomplete information system to get satisfied result according to the requirements of specific data by setting appropriate precision value. On two universes and rough entropy, probabilistic rough set is researched in [12].

In the present paper, our suggested two improved limited and variable precision rough set models [6], [7], [11] are further studied. Our first suggested model builds its classes of a given object in a different way from the before. Our second suggested model completes the definition of our limited variable precision relation. It discusses relationships between or among classes, upper and lower approximations by tolerance relation, similarity relation and limited tolerance relation and our suggested two relations. Through generating determine and possible rule by applying our second suggested model on an example. It shows that the number of obtained rules is more and the accuracy is high.

II. Basic Concepts

An Incomplete Information System (IIS) can be denoted as

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$S=(U,AT,V,f)$. Here, U, a non-empty set of finite objects, is called the universe of discourse. AT is a nonempty set of finite attributes. V_a is the domain of attribute a. Set $V=\cup_{a\in AT}V_a$. f_a is information function, for $\forall a\in AT$, $\forall x\in U$, $f_a(x)=f(x,a)\in V_a$. If it contains at least one attribute, say a, its domain is V_a, the value of an object at attribute a is * (usually "*" is used to represent unknown attribute), then we say the information system S is incomplete, otherwise complete.

Definition 1. Let $S=(U,AT,V,f)$ be an incomplete information system [1]. $A\subseteq AT$ is any attribute subset. The tolerance relation referring to A is defined as

$$T_A=\{(x,y)\in U^2:\forall a\in A, f_a(x)=f_a(y)\vee f_a(x)=*\vee f_a(y)=*\} \tag{1}$$

where $f_a(x)$ represents the value of object x at attribute a.

For $\forall x\in U$, the tolerance class of x is denoted by

$$T_A(x)=\{y\in U:(x,y)\in T_A\} \tag{2}$$

Definition 2. Let S be an incomplete information system. $A\subseteq AT$. Then, for $\forall X\subseteq U$, the upper approximation and lower approximation of X in terms of T_A are expressed by $\overline{T_A}(X)$ and $\underline{T_A}(X)$, respectively, where,

$$\overline{T_A}(X)=\{x\in U:\ T_A(x)\cap X\neq\varnothing\} \tag{3}$$

$$\underline{T_A}(X)=\{x\in U:\ T_A(x)\subseteq X\} \tag{4}$$

Definition 3. Let S be an incomplete information system. $A\subseteq AT$. The similarity relation [2] referring to A is denoted as

$$S_A=\{(x,y)\in U^2:\ \forall a\in A, f_a(x)=f_a(y)\vee f_a(x)=*\} \tag{5}$$

We can clearly see that S_A is reflexive and transitive, but not necessarily symmetric. According to the definition of similarity relation, we can then define two sets for any object x: The set of objects similar to x, denoted by $S_A(x)$, the set of objects to which x is similar, denoted by $S_A^{-1}(x)$ respectively, where

$$S_A(x)=\{y\in U:(y,x)\in S_A\} \tag{6}$$

$$S_A^{-1}(x)=\{y\in U:\ (x,y)\in S_A\} \tag{7}$$

Definition 4. Let S be an incomplete information system. $A\subseteq AT$. Then, for $\forall X\subseteq U$, the upper approximation and lower approximation of X in terms of the similarity relation S_A are denoted by $\overline{S_A}(X)$ and $\underline{S_A}(X)$ respectively, where,

$$\overline{S_A}(X)=\cup_{x\in X}S_A(x) \tag{8}$$

$$\underline{S_A}(X)=\{x\in U:\ S_A^{-1}(x)\subseteq X\} \tag{9}$$

Through further study on relationships between tolerance and similarity relation, Guoyin Wang recognized of that the needing conditions of tolerance relation are too loose, and it is subject to grouping two objects, which do not have any same attribute value, into an indistinguishable block. On the contrary, the needing conditions of similarity relation are too strict, and this is subject to dividing two objects which are very similar but with only a slight bit of incomplete information into different blocks. This results in two extreme conclusions. Regarding the above two facts, he proposed limited tolerance relation [5].

Definition 5. Let S be an incomplete information system. $A\subseteq AT$. The limited tolerance relation [5] in terms of A, denoted by L_A, is defined by

$$\begin{aligned}L_A=\{(\mathrm{x},\mathrm{y})\in U^2:\forall a\in A(\ f_a(x)=f_a(y)\\ =*)\vee((P_A(x)\cap P_A(y)\neq\varnothing)\\ \wedge\forall a\in A\ (f_a(x)\neq*)\wedge(f_a(y)\neq*)\rightarrow f_a(x)=f_a(y)))\ \}\end{aligned} \tag{10}$$

where

$$P_A(x)=\{a\in A:\ f_a(x)\neq\varnothing\}.$$

The block grouped by limited tolerance relations is between that by tolerance relation and that by similarity relation. It excludes the weakness of loose requirement in tolerance relation by the needing of that they should have the same value when two objects are all not empty at an attribute. At the same time, it deletes the requirement in similarity relation that y could not be more incomplete than x. That is to say, it relaxes the needing conditions of similarity relation, and enhanced the needing conditions on tolerance relation.

III. Two Kinds of VPRST Models

In the limited tolerance relation, when the values of two different objects on all attributes are empty, this only illustrates they have indiscernible possibility, but this possibility is often relatively small. Another situation is that the values of two objects are only the same on one attribute, and the remaining values are not comparable and they are still regarded as in a class or block. When the attribute is large, this condition is obviously still too loose.

A. An Improved Limited and VPRS Model

Realizing that the needing condition of limited tolerance relation is still not restrictive, we suggested a limited and variable precision classification model [6] as:

Definition 6. Let S be an incomplete information system. $A \subseteq AT$. The variable precision classification relation [6] in terms of A is denoted by V_A^α where,

$$V_A^\alpha = \{(x,y) \in U^2 : \forall a \in P_A(x) \cap P_A(y) \\ (f_a(x) = f_a(y)) \wedge |P_A(x) \cap P_A(y)| / |P_A(x)| \geq \alpha\} \cup I_U \quad (11)$$

where $\alpha \in [0,1]$, $|\cdot|$ represents the cardinality of the set, and $I_U = \{(x,x) : x \in U\}$.

It is easy to see that V_A^α is of only reflexivity, but not necessarily of symmetry and transitivity. In the limited tolerance relation, x ={*,1,*, 2,3,*,1,*} and y ={1,*,0,*, *,*, 1,*} are recognized to be belonging to the same class. However, x and y have the same value at only one attribute of the eight ones. Therefore, we have the reason of believing that their belonging to the same class is not possible and putting them into a class becomes very farfetched. If we set $\alpha = 0.1$, then $(x,y) \notin V_A^\alpha$ and $(y,x) \notin V_A^\alpha$. That is, we can separate them into two categories by using variable precision relation. By this, we can see that variable precision limited tolerance relation is actually a modified form and is more realistic.

Because V_A^α is not always symmetric, $\{y \in U : (y,x) \in V_A^\alpha\}$ may be not the same as $\{y \in U : (x,y) \in V_A^\alpha\}$. Like Definition 3 and 4 to similar relation and dislike the related definition in [6], the following two definitions are given.

Definition 7. Let S be an incomplete information system. $A \subseteq AT$. Then, for $\forall x \in U$, the set of objects limitedly tolerant to x with variable precision α, denoted by $V_A^\alpha(x)$, and the set of objects to which x is limitedly tolerant with variable precision α, denoted by $V_A^{-1,\alpha}(x)$, are respectively defined by:

$$V_A^\alpha(x) = \{y \in U : (y,x) \in V_A^\alpha\} \quad (12)$$

$$\underline{V_A^\alpha}(X) = \{x \in U : V_A^{-1,\alpha}(x) \subseteq X\} \quad (13)$$

Definition 8. Let S be an incomplete information system. $A \subseteq AT$. Then, for $\forall X \subseteq U$, the upper approximation and lower approximation of X in terms of V_A^α are denoted by $\overline{V_A^\alpha}(X)$ and $\underline{V_A^\alpha}(X)$ respectively, where

$$\overline{V_A^\alpha}(X) = \cup_{x \in X} V_A^\alpha(x) \quad (14)$$

$$\underline{V_A^\alpha}(X) = \{x \in U : V_A^{-1,\alpha}(x) \subseteq X\} \quad (15)$$

Theorem 1. Let S be an incomplete information system. For $\forall A \subseteq AT$, $\forall x \in U$, $\forall X \subseteq U$, we have

i. $S_A^{-1}(x) \subseteq V_A^{-1,\alpha}(x) \subseteq T_A(x), S_A(x) \subseteq V_A^\alpha(x) \subseteq T_A(x)$ (16)

ii. $\underline{T_A}(X) \subseteq \underline{V_A^\alpha}(X) \subseteq \underline{S_A}(X)$ (17)

iii. $\overline{S_A}(X) \subseteq \overline{V_A^\alpha}(X) \subseteq \overline{T_A}(X)$ (18)

Proof.

i. For any $y \in S_A^{-1}(x)$, we have $(x,y) \in S_A$

$$\Rightarrow \forall a \in A(f_a(x) = f_a(y) \vee f_a(x) = *) \\ \Rightarrow \forall a \in (P_A(x) \cap P_A(y))(f_a(x) = f_a(y)) \wedge (P_A(x) \subseteq P_A(y)) \\ \Rightarrow \forall a \in (P_A(x) \cap P_A(y))(f_a(x) = f_a(y)) \\ \wedge (|P_A(x) \cap P_A(y)| / |P_A(x)| = 1 \geq \alpha) \Rightarrow (x,y) \in V_A^\alpha \\ \Rightarrow y \in V_A^{-1,\alpha}(x)$$

So

$$S_A^{-1}(x) \subseteq V_A^{-1,\alpha}(x).$$

For any $y \in V_A^{-1,\alpha}(x)$, we have

$$(x,y) \in V_A^\alpha \Rightarrow \forall a \in (P_A(x) \cap P_A(y))(f_a(x) = f_a(y)) \\ \wedge (|P_A(x) \cap P_A(y)| / |P_A(x)| \geq \alpha) \\ \Rightarrow \forall a \in (P_A(x) \cap P_A(y))(f_a(x) = f_a(y)) \Rightarrow (x,y) \in T_A$$

Thus, $y \in T_A(x)$. So $V_A^{-1,\alpha}(x) \subseteq T_A(x)$.

For any $y \in S_A(x)$, we have

$$(y,x) \in S_A \Rightarrow \forall a \in A(f(y,a) = f_a(x) \vee f_a(y) = *) \\ \Rightarrow \forall a \in (P_A(y) \cap P_A(x))(f_a(y) = f_a(x)) \wedge (P_A(y) \subseteq P_A(x)) \\ \Rightarrow \forall a \in (P_A(y) \cap P_A(x))(f_a(y) = f_a(x)) \\ \wedge (|P_A(x) \cap P_A(y)| / |P_A(y)| = 1 \geq \alpha) \Rightarrow (y,x) \in V_A^\alpha \\ \Rightarrow y \in V_A^\alpha(x)$$

So

$$S_A(x) \subseteq V_A^\alpha(x).$$

For any $y \in V_A^\alpha(x)$, we have

$$(y,x) \in V_A^\alpha \Rightarrow \forall a \in (P_A(y) \cap P_A(x))(f_a(y) = f_a(x)) \\ \wedge (|P_A(y) \cap P_A(x)| / |P_A(y)| \geq \alpha) \\ \Rightarrow \forall a \in (P_A(y) \cap P_A(x))(f_a(y) = f_a(x)) \Rightarrow (y,x) \in T_A.$$

Thus, $y \in T_A(x)$. So $V_A^\alpha(x) \subseteq T_A(x)$.

From $V_A^{-1,\alpha}(x) \subseteq T_A(x), V_A^\alpha(x) \subseteq T_A(x)$ in the above, we can infer that

$$V_A^{-1,\alpha}(x) \cup V_A^\alpha(x) \subseteq T_A(x).$$

ii. By $S_A^{-1}(x) \subseteq V_A^{-1,\alpha}(x) \subseteq T_A(x)$, for any $y \in \underline{T_A}(X)$, we have $T_A(y) \subseteq X$. For $V_A^{-1,\alpha}(y) \subseteq T_A(y)$ thus we have $V_A^{-1,\alpha}(y) \subseteq X$. Therefore, $y \in \underline{V_A^\alpha}(X)$, and then $\underline{T_A}(X) \subseteq \underline{V_A^\alpha}(X)$. From $S_A^{-1}(y) \subseteq V_A^{-1,\alpha}(y)$, we also can get $V_A^{-1,\alpha}(y) \subseteq X$, $S_A^{-1}(y) \subseteq X$ and then $y \in \underline{S_A}(X)$ from $y \in \underline{V_A^\alpha}(X)$. So $\underline{V_A^\alpha}(X) \subseteq \underline{S_A}(X)$.

iii. By definitions, $\overline{S_A}(X) = \cup_{x\in X} S_A(x)$, $\overline{V_A^\alpha}(X) = \cup_{x\in X} V_A^\alpha(x)$, $\overline{T_A}(X) = \{y\in U: T_A(x) \cap X \neq \varnothing\}$. For any $y \in \overline{S_A}(X)$, we have that for some $x \in X$, $y \in S_A(x)$ and then $(y,x) \in S_A$. That is,

$$\begin{gathered}\forall a \in A(f_a(y) = f_a(x) \vee f_a(y) = *)\\ \Rightarrow \forall a \in (P_A(y) \cap P_A(x))(f_a(y) = f_a(x)) \wedge (P_A(y) \subseteq P_A(x))\\ \Rightarrow \forall a \in (P_A(y) \cap P_A(x))(f_a(y) = f_a(x))\\ \wedge(|P_A(y) \cap P_A(x)| / |P_A(y)| = 1 \geq \alpha). \Rightarrow (y,x) \in V_A^\alpha\\ \Rightarrow y \in V_A^\alpha(x) \subseteq \cup_{x\in X} V_A^\alpha(x) = \overline{V_A^\alpha}(X)\end{gathered}$$

So $\overline{S_A}(X) \subseteq \overline{V_A^\alpha}(X)$. For any $y \in \overline{V_A^\alpha}(X) = \cup_{x\in X} V_A^\alpha(x)$, we have that for some $x \in X$, $(y,x) \in V_A^\alpha$.

$$\begin{gathered}\forall a \in (P_A(y) \cap P_A(x))(f_a(y) = f_a(x))\\ \wedge(|P_A(y) \cap P_A(x)| / |P_A(y)| \geq \alpha)\\ \Rightarrow \forall a \in (P_A(y) \cap P_A(x))(f_a(y) = f_a(x))\\ \Rightarrow (y,x) \in T_A \Rightarrow y \in T_A(x), x \in T_A(y)\end{gathered}$$

$\Rightarrow T_A(y) \cap X \supseteq \{x\}$. i.e. $T_A(y) \cap X \neq \varnothing$, and then $y \in \overline{T_A}(X)$.

So $\overline{V_A^\alpha}(X) \subseteq \overline{T_A}(X)$.

Lemma 1. Let S be an incomplete information system. For $\forall A \subseteq AT$, we have

i. $$T_A = \{(x,y) \in U^2 : \forall a \in P_A(x) \cap P_A(y) \ (f_a(x) = f_a(y))\} \tag{19}$$

ii. $$S_A = \{(x,y) \in U^2 : \forall a \in P_A(x) \cap P_A(y) \ (f_a(x) = f_a(y)) \wedge (P_A(x) \subseteq P_A(y))\} \tag{20}$$

Proof.

i. Because $\forall a \in P_A(x) \cap P_A(y)$ $(f_a(x) = f_a(y))$ is logically equivalent to $\forall a \in A(f_a(x) = f_a(y) \vee f_a(x) = * \vee f_a(y) = *)$, it is hold.

ii. $\forall a \in P_A(x) \cap P_A(y)(f_a(x) = f_a(y)) \wedge (P_A(x) \subseteq P_A(y))$ is also an equivalent expression of $\forall a \in A(f_a(x) = f_a(y) \vee f_a(x) = *)$, which excludes the cases of $f_a(x) \neq * \wedge f_a(y) = *$ and $* \neq f_a(x)$, so it also holds.

B. An Improved Symmetric Limited and VPRS Model

For objects $x = \{1, *, *, *, *, *, *, *, *, *\}$ and $y = \{1, *, *, *, *, *, *, *, *, *\}$, they cannot be discriminated by tolerance, similarity and limited tolerance relation. By Definition 6 and Definition 7, no matter which value α takes, as long as we take the attribute subset A containing the first attribute, they are still regarded as the same class. But the fact that these two objects belong to the same class is hardly possible since they have only one attribute value identical and have many null values. In order to solve this problem, an improved limited and variable precision rough set model with symmetry is proposed [7], in which, object relation is of symmetry. It extends the tolerance rough set model.

Definition 9. Let $S = (U, AT, V, f)$ be an incomplete information system. $\forall x, y \in U, \forall A \subseteq AT$, we define

$$\mu(x, y) = \begin{cases} \dfrac{|P_A(x) \cap P_A(y)|}{\min\{|P_A(x)|, |P_A(y)|\}}, \\ \quad if \min\{|P_A(x)|, |P_A(y)|\} \neq 0, \\ 0, if \min\{|P_A(x)|, |P_A(y)|\} = 0. \end{cases} \tag{21}$$

Definition 10. Let $S = (U, AT, V, f)$ be an incomplete information system. For $\forall x, y \in U$, $\forall A \subseteq AT$, $0 \leq \alpha \leq 1$, a binary relation is called an improved symmetric limited and variable precision tolerance relation, where

$$\begin{gathered}NL_A^\alpha = \{(x, y) \in U^2 : \forall a \in P_A(x) \cap P_A(y)\\ (f_a(x) = f_a(y)) \wedge \mu(x, y) \geq \alpha\} \cup I_U\end{gathered} \tag{22}$$

From this definition, it is clear that the proportion of the number of non null common attributes to the total non null attributes of objects x, y should greater or equal to α, reaching at above some threshold. Definition 10 is different with Definition 1 in [9] for that here it is ensured to be reflexive by union with I_U. It is also different with Definition 6; symmetry is satisfied here but not there. We can see that this symmetric limited and variable precision tolerance relation is reflexive and symmetric but maybe not transitive. So NL_A^α is a tolerance relation consistent or a compatible relation in discrete mathematics with any value α in [0, 1]. α maybe is set to the filling factor of the system.

Theorem 3. Let S be an incomplete information system. $\forall A \subseteq AT$. Then, we can get NL_A^α and $NL_A^0 = T_A$, and $NL_A^1 = R'_A$, where

i. $T_A = \{(x, y) \in U^2 : \forall a \in P_A(x) \cap P_A(y)(f_a(x) = f_a(y))\}$ (23)

ii. $R'_A = \{(x, y) \in U^2 : \forall a \in P_A(x) \cap P_A(y)$

$$(f_a(x)=f_a(y))\wedge((P_A(x)\subseteq P_A(y)\vee P_A(y)\subseteq P_A(x))\wedge P_A(x)\neq\varnothing\wedge P_A(y)\neq\varnothing\}\cup I_U \tag{24}$$

Proof.

i. Because $NL_A^0=\{(x,y)\in U^2:$

$$\forall a\in P_A(x)\cap P_A(y)(f_a(x)=f_a(y))\wedge\mu(x,y)\geq 0\}\cup I_U$$
$$=\{(x,y)\in U^2:\forall a\in P_A(x)\cap P_A(y)(f_a(x)=f_a(y))\}=T_A$$

So it is right.

ii. Because $\mu(x,y)=1$ if and only if $\min\{|P_A(x)|,|P_A(y)|\}\neq 0$ and $\frac{|P_A(x)\cap P_A(y)|}{\min\{|P_A(x)|,|P_A(y)|\}}=1$. Thus, $|P_A(x)\cap P_A(y)|=\min\{|P_A(x)|,|P_A(y)|\}$.

$$P_A(x)\subseteq P_A(y) \text{ or } P_A(y)\subseteq P_A(x),\ P_A(x)=\varnothing\wedge P_A(y)\neq\varnothing.$$

So $NL_A^1=\{(x,y)\in U^2:\forall a\in P_A(x)\cap P_A(y)$

$$(f_a(x)=f_a(y))\wedge\mu(x,y)=1\}\cup I_U$$
$$=\{(x,y)\in U^2:\forall a\in P_A(x)\cap P_A(y)(f_a(x)=f_a(y))$$
$$\wedge((P_A(x)\subseteq P_A(y)\vee P_A(y)\subseteq P_A(x))$$
$$\wedge P_A(x)\neq\varnothing\wedge P_A(y)\neq\varnothing\}\cup I_U$$

i.e. $NL_A^1=R'_A$. R'_A is really a relation with reflexivity, symmetry.

Compared with Definition 6 and Definition 7, the improved limited and variable precision relation is a tolerance relation.

Definition 11. Let S be an incomplete information system. $\forall A\subseteq AT$. Then, for $\forall x\in U$, the tolerance class of x , denoted by $L_A^\alpha(x)$, is defined by

$$NL_A^\alpha(x)=\{y\in U:(x,y)\in NL_A^\alpha\} \tag{25}$$

Definition 12. Let S be an incomplete information system. For $\forall A\subseteq AT$ and $\forall X\subseteq U$, the upper approximation and lower approximation of X , denoted by $\overline{NL_A^\alpha}(X)$ and $\underline{NL_A^\alpha}(X)$ respectively, are defined by

$$\overline{NL_A^\alpha}(X)=\{x\in U:\ NL_A^\alpha(x)\cap X\neq\varnothing\} \tag{26}$$

$$\underline{NL_A^\alpha}(X)=\{x\in U:\ NL_A^\alpha(x)\subseteq X\} \tag{27}$$

Theorem 4. Let S be an incomplete information system. $\forall A\subseteq AT$. If $0\leq\alpha_1\leq\alpha_2\leq 1$, then for $\forall x\in U$, $\forall X\subseteq U$, we have

i. $NL_A^{\alpha_2}(x)\subseteq NL_A^{\alpha_1}(x)$ (28)

ii. $\overline{NL_A^{\alpha_2}}(X)\subseteq\overline{NL_A^{\alpha_1}}(X)$ (29)

iii. $\underline{NL_A^{\alpha_1}}(X)\subseteq\underline{NL_A^{\alpha_2}}(X)$ (30)

Proof.

i. For $\forall y\in NL_A^{\alpha_2}(x)$, we have $\mu(x,y)\geq\alpha_2$. Because $\alpha_1\leq\alpha_2$, $\mu(x,y)\geq\alpha_1$. That is $y\in NL_A^{\alpha_1}(x)$. So $NL_A^{\alpha_2}(x)\subseteq NL_A^{\alpha_1}(x)$.

For $\forall y\in\overline{NL_A^{\alpha_2}}(X)$, according to the Definition 12, we have $NL_A^{\alpha_2}(y)\cap X\neq\varnothing$. Because from i we have $NL_A^{\alpha_2}(y)\subseteq NL_A^{\alpha_1}(y)$; therefore, $NL_A^{\alpha_1}(y)\cap X\neq\varnothing$. It follows that $y\in\overline{NL_A^{\alpha_1}}(X)$. Thus, $\overline{NL_A^{\alpha_2}}(X)\subseteq\overline{NL_A^{\alpha_1}}(X)$ for $y\in\overline{NL_A^{\alpha_2}}(X)$ is arbitrarily chosen.

For $\forall y\in\underline{NL_A^{\alpha_1}}(X)$, according to the Definition 12, we have $NL_A^{\alpha_1}(y)\subseteq X$. Because from i we have $NL_A^{\alpha_2}(y)\subseteq NL_A^{\alpha_1}(y)$; therefore, $NL_A^{\alpha_2}(y)\subseteq X$. It follows that $y\in\underline{NL_A^{\alpha_2}}(X)$. Thus, $\underline{NL_A^{\alpha_1}}(X)\subseteq\underline{NL_A^{\alpha_2}}(X)$ for $\forall y\in\underline{NL_A^{\alpha_1}}(X)$ is arbitrarily chosen.

Theorem 5. Let S be an incomplete information system. $\forall A\subseteq AT$. For $\forall x\in U$, $\forall X\subseteq U$, then

i. $\underline{NL_A^\alpha}(X)\subseteq X\subseteq\overline{NL_A^\alpha}(X)$ (31)

ii. $V_A^{-1,\alpha}(x)\subseteq NL_A^\alpha(x)\subseteq T_A(x)$ (32)

iii. $\overline{V_A^\alpha}(X)\subseteq\overline{NL_A^\alpha}(X)\subseteq\overline{T_A}(X)$ (33)

iv. $\underline{T_A}(X)\subseteq\underline{NL_A^\alpha}(X)\subseteq\underline{V_A^\alpha}(X)$ (34)

Proof.

i. It can be proved to be true by the definition.

ii. For any $y\in V_A^{-1,\alpha}(x)$, by Definition 11 and 12, we have $(x,y)\in V_A^{-1,\alpha}$, that is, we have:

① $\forall a\in P_A(x)\cap P_A(y)(f_a(x)=f_a(y))$.

② $|P_A(x)\cap P_A(y)|/|P_A(x)|\geq\alpha$.

Notice that ② can be transformed to

$$\frac{|P_A(x)\cap P_A(y)|}{\min\{|P_A(x)|,|P_A(y)|\}}\cdot\frac{\min\{|P_A(x)|,|P_A(y)|\}}{|P_A(x)|}\geq\alpha .$$

That is

$$\frac{|P_A(x)\cap P_A(y)|}{\min\{|P_A(x)|,|P_A(y)|\}}\geq\alpha\cdot\frac{|P_A(x)|}{\min\{|P_A(x)|,|P_A(y)|\}}.$$

Due to A is a subset of attributes, we have $\frac{|P_A(x)|}{\min\{|P_A(x)|,|P_A(y)|\}}\geq 1$. That is $\frac{|P_A(x)\cap P_A(y)|}{\min\{|P_A(x)|,|P_A(y)|\}}\geq\alpha$.

In summary, we can get $(x,y)\in NL_A^\alpha$, so we have $y\in NL_A^\alpha(x)$. Thus $V_A^{-1,\alpha}(x)\subseteq NL_A^\alpha(x)$.

For any $y\in NL_A^\alpha(x)$, by Definition 11 and 12, we have $(x,y)\in NV_A^\alpha$, that is, we have:

$$\forall a\in P_A(x)\cap P_A(y)(f_a(x)=f_a(y))\wedge\frac{|P_A(x)\cap P_A(y)|}{\min\{|P_A(x)|,|P_A(y)|\}}\geq\alpha$$

Therefore, we have

$$\forall a\in P_A(x)\cap P_A(y)(f_a(x)=f_a(y)),\text{ i.e. }(x,y)\in T_A.$$

So $NL_A^\alpha(x)\subseteq T_A(x)$ for $y\in NL_A^\alpha(x)$ is arbitrarily selected from $NL_A^\alpha(x)$.

iii. For $\forall y\in\overline{V_A^\alpha}(X)=\cup_{x\in X}V_A^\alpha(x)$, by Definition 6,7, we have $\exists x\in X(y\in V_A^\alpha(x))$. i.e. $(y,x)\in V_A^\alpha$, so

$$y=x\vee\forall a\in P_A(y)\cap P_A(x)(f_a(y)=f_a(x))\wedge|P_A(y)\cap P_A(x)|/|P_A(y)|\geq\alpha.$$

In the same proof of (1) ②, we have $y\in NL_A^\alpha(x)$, $x\in NL_A^\alpha(y)$. Thus $NL_A^\alpha(y)\cap X\supseteq\{x\}\neq\varnothing$. So $y\in\overline{NL_A^\alpha}(X)$. Therefore, $\overline{V_A^\alpha}(X)\subseteq\overline{NL_A^\alpha}(X)$ is right.

For $\forall y\in\overline{NL_A^\alpha}(X)$, by Definition 8, we have $NL_A^\alpha(y)\cap X\neq\varnothing$. By (ii), we have $NL_A^\alpha(y)\subseteq T_A(y)$. So $T_A(y)\cap X\neq\varnothing$. By Definition 6, we have $y\in\overline{T_A}(X)$. So $\overline{NL_A^\alpha}(X)\subseteq\overline{T_A}(X)$ is right since y is arbitrarily selected from $\overline{NL_A^\alpha}(X)$.

iv. For $\forall y\in\underline{T_A}(X)$, by Definition 8, we have $T_A(y)\subseteq X$. By ii, we have $NL_A^\alpha(y)\subseteq T_A(y)$, so $NL_A^\alpha(y)\subseteq X$. By Definition 6, we have $y\in\underline{NL_A^\alpha}(X)$. So $\underline{T_A}(X)\subseteq\underline{NL_A^\alpha}(X)$ is right for y is arbitrarily selected from $\underline{T_A}(X)$. For $\forall y\in\underline{NL_A^\alpha}(X)$, by Definition 8, we have $NL_A^\alpha(y)\subseteq X$. By ii, we have $V_A^{-1,\alpha}(y)\subseteq NL_A^\alpha(y)$, so $V_A^{-1,\alpha}(y)\subseteq X$. By Definition 6, we have $y\in\underline{V_A^\alpha}(X)$. So $\underline{NL_A^\alpha}(X)\subseteq\underline{V_A^\alpha}(X)$ is right for y is arbitrarily selected from $\underline{NL_A^\alpha}(X)$.

Theorem 6. Let S be an incomplete information system. For $\forall A\subseteq AT$. $\forall X,Y\subseteq U$, then

i. $\underline{NL_A^\alpha}(X)\cap\underline{NL_A^\alpha}(Y)=\underline{NL_A^\alpha}(X\cap Y)$ (35)

ii. $\underline{NL_A^\alpha}(X)\cup\underline{NL_A^\alpha}(Y)\subseteq\underline{NL_A^\alpha}(X\cup Y)$ (36)

Proof.

i. $y\in\underline{NL_A^\alpha}(X)\cap\underline{NL_A^\alpha}(Y)$

$$\Leftrightarrow y\in\underline{NL_A^\alpha}(X)\wedge y\in\underline{NL_A^\alpha}(Y)\Leftrightarrow NL_A^\alpha(y)\subseteq X\wedge NL_A^\alpha(y)\subseteq Y$$
$$\Leftrightarrow NL_A^\alpha(y)\subseteq X\cap Y\Leftrightarrow y\in\underline{NL_A^\alpha}(X\cap Y)$$

ii. $y\in\underline{NL_A^\alpha}(X)\cup\underline{NL_A^\alpha}(Y)\Leftrightarrow y\in\underline{NL_A^\alpha}(X)\vee y\in\underline{NL_A^\alpha}(Y)$

$$\Leftrightarrow NL_A^\alpha(y)\subseteq X\vee NL_A^\alpha(y)\subseteq Y\Rightarrow NL_A^\alpha(y)\subseteq X\cup Y$$
$$\Leftrightarrow y\in\underline{NL_A^\alpha}(X\cup Y)$$

Theorem 7. Let S be an incomplete information system. For $\forall A\subseteq AT$. For $\forall X,Y\subseteq U$, then

i. $\overline{NL_A^\alpha}(X\cap Y)\subseteq\overline{NL_A^\alpha}(X)\cap\overline{NL_A^\alpha}(Y)$ (37)

ii. $\overline{NL_A^\alpha}(X\cup Y)=\overline{NL_A^\alpha}(X)\cup\overline{NL_A^\alpha}(Y)$ (38)

Proof.

i. $y\in\overline{NL_A^\alpha}(X\cap Y)$

$$\Leftrightarrow NL_A^\alpha(y)\cap(X\cap Y)\neq\varnothing$$
$$\Rightarrow NL_A^\alpha(y)\cap X\neq\varnothing\wedge NL_A^\alpha(y)\cap Y\neq\varnothing$$
$$\Leftrightarrow y\in\overline{NL_A^\alpha}(X)\wedge y\in\overline{NL_A^\alpha}(Y)$$
$$\Leftrightarrow y\in\overline{NL_A^\alpha}(X)\cap\overline{NL_A^\alpha}(Y)$$

ii. $y\in\overline{NL_A^\alpha}(X\cup Y)\Leftrightarrow NL_A^\alpha(y)\cap(X\cup Y)\neq\varnothing$

$$\Leftrightarrow NL_A^\alpha(y)\cap X\neq\varnothing\vee NL_A^\alpha(y)\cap Y\neq\varnothing$$
$$\Leftrightarrow y\in\overline{NL_A^\alpha}(X)\vee y\in\overline{NL_A^\alpha}(Y)\Leftrightarrow y\in\overline{NL_A^\alpha}(X)\cup\overline{NL_A^\alpha}(Y)$$

IV. Rule Generalization and Case Study

The key problem in rough set is knowledge reduction and rule generalization. Through simplified information system, we can obtain intuitive decision algorithm and make decision or classification. Under the leading guidance of variable precision rough set, we can often design some heuristic reduction algorithms to get reducts and then to generate rules from them. However, the reducts are ordinarily non-exact and the number of reducts is many, so rules may also diverse. In order to deduce the whole determinative and probable rules, an effective approach is to use discriminatory matrices on the given information system by applying upper and lower approximations of decision class.

Definition 13. An incomplete decision system $S=(U,AT=C\cup D,V,f)$ is given, where C is the conditional attribute set, D is the decision attribute set, $AT=C\cup D$ is the whole set of attributes. $C\cap D=\varnothing$, $V=\cup_{a\in AT}V_a$ is the value set and V_a is the subset of values at attribute a. $*\notin V_d(d\in D)$. Suppose $A\subseteq C$, $U/IND(D)=\{D_1,D_2,...,D_m\}$ is a partition on U. Referring to [13], [14], a matrix with respect to the decision class $D_k(k=1,2,...,m)$ with $|\underline{L_A^\alpha}(D_k)|$ rows and $|U-D_k|$ columns is formed by defining its element $M^k_{x,y}$ as:

$$M^k_{x,y} = \begin{cases} \{(a, f_a(x)) : f_a(x) \neq * \wedge f_a(y) \neq * \\ \qquad \wedge f_a(x) \neq f_a(y)\} \\ \varnothing, \qquad otherwise \end{cases} \tag{39}$$

where $x \in \underline{L}^{\alpha}_A(D_k), y \in U - D_k (k = 1, 2, \ldots, m)$, $a \in P_A(x) \cap P_A(y)$. Let $B_k = \underset{y}{\wedge} \vee M^k_{x,y} (M^k_{x,y} \neq \varnothing)$. B_k is called a decision function referring to D_k. B_k is simplified to a disjunction normal formula using absorbing law in logic. Each conjunctive factor makes a rule which is determine, but may not absolutely determine, due to the model is variable precision with α.

In a very similar way, if we alternatively use $x \in \overline{L}^{\alpha}_A(D_k)$, $y \in U - \overline{L}^{\alpha}_A(D_k)(k = 1, 2, \ldots, m)$, $a \in P_A(x) \cap P_A(y)$ and $D_k(x) \neq D_k(y)$, as the condition to construct elements referring to D_k and then form another similar discernibility matrix, we can generate probable rules.

In order to comparatively analyze, we adopt a real incomplete information system in [4] shown in Table I to perform some computations, where $AT = C \cup D$, $C = \{a, b, c, d\}$, $D = \{e\}$. At first we have

$$D_{\Phi} = \{O_1, O_2, O_4, O_7, O_{10}, O_{12}\},\ D_{\Psi} = \{O_3, O_5, O_6, O_8, O_9, O_{11}\}.$$

TABLE I
AN IIS

U	a	b	d	e
O_1	3	2	0	Φ
O_2	2	3	0	Φ
O_3	2	3	0	Ψ
O_4	*	2	1	Φ
O_5	*	2	1	Ψ
O_6	2	3	1	Ψ
O_7	3	*	3	Φ
O_8	*	0	*	Ψ
O_9	3	2	3	Ψ
O_{10}	1	*	*	Φ
O_{11}	*	2	*	Ψ
O_{12}	3	2	*	Φ

Let $A = C, \alpha = 0$. According to Definition 11, we obtain:

$$NL^0_A(O_1) = \{O_1, O_{11}, O_{12}\},\ NL^0_A(O_2) = \{O_2, O_3\},\ NL^0_A(O_3) = \{O_2, O_3\},$$
$$NL^0_A(O_4) = \{O_4, O_5, O_{10}, O_{11}, O_{12}\},\ NL^0_A(O_5) = NL^0_A(O_4),$$
$$NL^0_A(O_6) = \{O_6\},\ NL^0_A(O_7) = \{O_7, O_8, O_9, O_{11}, O_{12}\},$$
$$NL^0_A(O_8) = \{O_7, O_8, O_{10}\},\ NL^0_A(O_9) = \{O_7, O_9, O_{11}, O_{12}\},$$
$$NL^0_A(O_{10}) = \{O_4, O_5, O_8, O_{10}, O_{11}\},\ NL^0_A(O_{11})$$
$$= \{O_1, O_4, O_5, O_7, O_9, O_{10}, O_{11}, O_{12}\},$$
$$NL^0_A(O_{12}) = \{O_1, O_4, O_5, O_7, O_9, O_{11}, O_{12}\}.$$

This result is the same as $T_A(x)$ $(x = O_1, O_2, \ldots, O_{12})$ defined by the tolerance relation in Definition 1 and $V^{-1,0}_A(x)$ $(x = O_1, O_2, \ldots, O_{12})$ defined in Definition 7 at $\alpha = 0$.

According to Definition 11, we obtain:

$$NL^1_A(O_1) = \{O_1, O_{11}, O_{12}\},\ NL^1_A(O_2) = \{O_2, O_3\},$$
$$NL^1_A(O_3) = \{O_2, O_3\}, NL^1_A(O_4) = \{O_4, O_5, O_{11}, O_{12}\},$$
$$NL^1_A(O_5) = NL^1_A(O_4),\ NL^1_A(O_6) = \{O_6\},\ NL^1_A(O_7)$$
$$= \{O_7, O_9, O_{12}\},\ NL^1_A(O_8) = \{O_8\},\ NL^1_A(O_9)$$
$$= \{O_7, O_9, O_{11}, O_{12}\},\ NL^1_A(O_{10}) = \{O_{10}\},\ NL^1_A(O_{11})$$
$$= \{O_4, O_5, O_9, O_{11}, O_{12}\},\ NL^1_A(O_{12}) = \{O_1, O_4, O_5, O_7, O_9, O_{11}, O_{12}\}.$$

According to Definition 11, we obtain:

$$NL^{0.5}_A(O_1) = \{O_1, O_{11}, O_{12}\},\ NL^{0.5}_A(O_2) = \{O_2, O_3\},$$
$$NL^{0.5}_A(O_3) = \{O_2, O_3\},\ NL^{0.5}_A(O_4) = \{O_4, O_5, O_{11}, O_{12}\},$$
$$NL^{0.5}_A(O_5) = NL^{0.5}_A(O_4),\ NL^{0.5}_A(O_6) = \{O_6\},$$
$$NL^{0.5}_A(O_7) = \{O_7, O_9, O_{12}\},\ NL^{0.5}_A(O_8) = \{O_8\},$$
$$NL^{0.5}_A(O_9) = \{O_7, O_9, O_{11}, O_{12}\},\ NL^{0.5}_A(O_{10}) = \{O_{10}\},$$
$$NL^{0.5}_A(O_{11}) = \{O_1, O_4, O_5, O_9, O_{11}, O_{12}\},\ NL^{0.5}_A(O_{12}) = \{O_1, O_4,$$
$$O_5, O_9, O_{11}, O_{12}\}.\ \underline{NL}^{0.5}_A(D_{\Phi}) = \{O_{10}\},\ \overline{NL}^{0.5}_A(D_{\Phi})$$
$$= \{O_1, O_2, O_3, O_4, O_5, O_7, O_9, O_{10}, O_{11}, O_{12}\},\ \underline{NL}^{0.5}_A(D_{\Psi})$$
$$= \{O_6, O_8\},\ \overline{NL}^{0.5}_A(D_{\Psi}) = \{O_1, O_2, O_3, O_4, O_5, O_6, O_7,$$
$$O_8, O_9, O_{11}, O_{12}\}.$$

So, Φ's discernibility matrix for relatively determine rule generation by using $x \in \underline{NL}^{0.5}_A(D_{\Phi}), y \in U - D_{\Phi}$ is as in Table II. Thus, relatively determine rules generated from Table II. are: $(a,1) \to (e, \Phi)$. In the same way, we can also construct Ψ's discernibility matrix for relatively determine rule generation by using $x \in \underline{NL}^{0.5}_A(D_{\Psi}),\ y \in U - D_{\Psi}$ in Table III.

TABLE II
DISCERNIBILITY MATRIX FOR RELATIVELY DETERMINE RULE GENERATION TO Φ

	O_3	O_5	O_6	O_8	O_9	O_{11}
O_{10}	$(a,1)$		$(a,1)$		$(a,1)$	O_{10}

Blank means null (the same in other tables).

Relatively determine rules for decision class Ψ gotten from Table III for decision class Ψ are: $(a,2) \wedge (b,3) \wedge (d,1) \to (e, \Psi);\ (b,0) \to (e, \Psi)$.

TABLE III
DISCERNIBILITY MATRIX FOR RELATIVELY DETERMINE RULE GENERATION TO DECISION CLASS Ψ

	O_1	O_2	O_4	O_7	O_{10}	O_{12}
O_6	$(a,2)$ $(b,3)$ $(c,2)$ $(d,1)$	$(d,1)$	$(b,3)$	$(a,2)$ $(d,1)$	$(a,2)$	$(a,2)$ $(b,3)$ $(c,2)$
O_8	$(b,0)$ $(c,0)$	$(b,0)$ $(c,0)$	$(b,0)$			$(b,0)$ $(c,0)$

TABLE IV
DISCERNIBILITY MATRIX FOR RELATIVELY PROBABLE RULE GENERATION TO Φ

	O_6	O_8
O_1	$(a,3)(b,2)(c,1)(d,0)$	$(b,2)(c,1)$
O_2	$(d,0)$	$(b,3)(c,2)$
O_4	$(b,2)$	$(b,2)$
O_7	$(a,3)$ $(d,3)$	
O_{10}	(a,1)	
O_{12}	(a,3)(b,2)(c,1)	(b,2) (c,1)

Φ's discernibility matrix for relatively probable rule generation by using $x \in \overline{NL_A^{0.5}}(D_\Phi), y \in U - \overline{NL_A^{0.5}}(D_\Phi)$ and $f_e(x) \neq f_e(y)$ is as in Table IV. Thus, relatively probable rules generated from Table IV are: $(b,2)\vee(c,1)\rightarrow(e,\Phi)$; $(b,3)\wedge(d,0)\rightarrow(e,\Phi)$; $(c,2)\wedge(d,0)\rightarrow(e,\Phi)$; $(b,2)\rightarrow(e,\Phi)$; $(a,3)\wedge(d,3)\rightarrow(e,\Phi)$; $(a,1)\rightarrow(e,\Phi)$. In the same way, we can also construct Ψ's discernibility matrix for relatively probable rule generation by using $x \in \overline{NL_A^{0.5}}(D_\Psi)$, $y \in U - \overline{NL_A^{0.5}}(D_\Psi)$ and $f_e(x) \neq f_e(y)$ in Table V.

TABLE V
DISCERNIBILITY MATRIX FOR RELATIVELY PROBABLE RULE GENERATION TO Ψ

	O_{10}
O_3	$(a,2)$
O_5	
O_6	$(a,2)$
O_8	
O_9	$(a,3)$
O_{11}	

Relatively probable rules for decision class Ψ gotten from Table V for decision class Ψ are: $(a,2)\rightarrow(e,\Psi)$; $(a,3)\rightarrow(e,\Psi)$.

V. CONCLUSION

Due to the incompleteness of data in the real world, different extended rough set models are proposed. The tolerance relation and similarity relation are more commonly used. The variable precision rough set model in [6] controls classification of the incomplete system by setting a threshold value, so that the model is more general and more flexible to get the granularity of knowledge, but it is not symmetric. Although the limited and variable precision tolerance model in [11] is symmetric, but the two models consider that two small probability equivalent objects are indiscernible. The model proposed in this paper overcomes this shortcoming and gets a more accurate and reasonable result. Based on this work, the next step is to do further exploration on this new model and makes the knowledge representation simpler and efficient.

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