

# $L^1$ -Convergence of Modified Trigonometric Sums

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**Abstract**—The existence of sine and cosine series as a Fourier series, their  $L^1$ -convergence seems to be one of the difficult question in theory of convergence of trigonometric series in  $L^1$ -metric norm. In the literature so far available, various authors have studied the  $L^1$ -convergence of cosine and sine trigonometric series with special coefficients. In this paper, we present a modified cosine and sine sums and criterion for  $L^1$ -convergence of these modified sums is obtained. Also, a necessary and sufficient condition for the  $L^1$ -convergence of the cosine and sine series is deduced as corollaries.

**Keywords**—Conjugate Dirichlet kernel, Dirichlet kernel,  $L^1$ -convergence, modified sums.

## I. INTRODUCTION

LET

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \quad (1)$$

and

$$g(x) = \sum_{k=1}^{\infty} b_k \sin kx \quad (2)$$

be the trigonometric cosine and sine series.

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^n a_k \cos kx$$

and

$$\tilde{S}_n(x) = \sum_{k=1}^n b_k \sin kx$$

be the partial sums of the series (1) and (2) respectively.

**Convex sequence.** ([1], Vol. I, p. 4) A sequence  $\{a_k\}$  is said to be convex if  $\Delta^2 a_k \geq 0$ .

where  $\Delta^2 a_k = \Delta(\Delta a_k)$  and  $\Delta a_k = a_k - a_{k+1}$ .

**Quasi-convex sequence.** ([1], Vol. II, p. 202) A sequence  $\{a_k\}$  is said to be quasi-convex if

$$\sum_{k=1}^{\infty} n |\Delta^2 a_k| < \infty$$

The work on  $L^1$ -convergence of trigonometric series with special coefficients was introduced by Young [2] and Kolmogorov [3] by taking classes of convex sequences and quasi-convex sequences.

**Theorem 1.** [2], [3] *If  $\{a_k\} \downarrow 0$  and  $\{a_k\}$  is convex or even quasi-convex, then for the convergence of the series (1) in the metric space  $L^1$ , it is necessary and sufficient that  $a_k \log k = o(1)$ ,  $k \rightarrow \infty$ .*

**Class S.** [4], [5] A sequence  $\{a_k\}$  belongs to class S, if  $a_k \rightarrow \infty$  as  $k \rightarrow \infty$  and there exists a sequence of numbers

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$\{A_k\}$  such that

(i)  $A_k \downarrow 0$  as  $k \rightarrow \infty$ .

(ii)  $\sum_{k=0}^{\infty} A_k < \infty$ .

(iii)  $|\Delta a_k| \leq A_k$ , for all  $k$ .

The class S is usually called as Sidon-Teljakovskii class.

A quasi-convex null sequence satisfies conditions of the class S by choosing:

$$A_k = \sum_{m=k}^{\infty} |\Delta^2 a_m|.$$

Teljakovskii [5] generalized Theorem 1 for the cosine series (1) with coefficients satisfying the conditions of the class S in the following form:

**Theorem 2.** [5] *Let  $\{a_k\}$  be the sequence of the cosine series (1) belongs to the class S, then a necessary and sufficient condition for  $L^1$ -convergence of (1) is  $a_k \log k = o(1)$ ,  $k \rightarrow \infty$ .*

It is well known that if a trigonometric series converges in  $L^1$ -metric to a function  $f \in L^1(T)$ , then it is the Fourier series of the function  $f$ . Riesz ([1], Vol. II, Ch. VIII, 22) gave a counter example to show that in  $L^1$ -metric, the converse of the above said result does not hold good.  $L^1$ -convergence of trigonometric series with special coefficients have been studied by various authors. During the literature survey, It can be observed that many authors have introduced modified trigonometric sums "as these sums approximate their limits better than the classical trigonometric series in the sense that these sums converge in  $L^1$ -metric to the sum of trigonometric series whereas the classical series itself may not". Rees and Stanojevic [6], Kumari and Ram [7], [8], Hooda, Ram and Bhatia [9], Kaur, Bhatia and Ram [10], Kaur and Bhatia [11], [12], Braha and Xhevat [13], Krasniqi [14], [15] have introduced new modified trigonometric sums and studied their  $L^1$ -convergence under various classes of coefficient sequences.

Rees and Stanojevic [6] have introduced modified cosine sum as

$$g_n(x) = \frac{1}{2} \sum_{k=0}^n \Delta a_k + \sum_{k=1}^n \sum_{j=k}^n (\Delta a_j) \cos kx \quad (3)$$

Garrett and Stanojevic [16], Ram [17], Singh and Sharma [18] studied the  $L^1$ -convergence of cosine sum (3) under different set of conditions on the coefficients.

Kumari and Ram ([7], [8]) introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^n \sum_{j=k}^n \Delta \left( \frac{a_j}{j} \right) k \cos kx$$

and

$$g_n(x) = \sum_{k=1}^n \sum_{j=k}^n \Delta \left( \frac{b_j}{j} \right) k \sin kx$$

and have studied their  $L^1$ -convergence under the condition that the coefficients  $\{a_n\}$  and  $\{b_n\}$  belong to different classes of sequences.

Hooda, Ram and Bhatia [9] introduced a new modified cosine sums as

$$f_n(x) = \frac{1}{2} \left( a_1 + \sum_{k=0}^n \Delta^2 a_k \right) + \sum_{k=1}^n \left( a_{k+1} + \sum_{j=k}^n \Delta^2 a_j \right) \cos kx$$

and have studied its  $L^1$ -convergence to a cosine trigonometric series and also deduced a result of Teljakovskii [19] as a corollary.

Kaur, Bhatia and Ram [10] introduced new modified sine sums as

$$K_n(x) = \frac{1}{2 \sin x} \sum_{k=1}^n \sum_{j=k}^n (\Delta b_{j-1} - \Delta b_{j+1}) \sin kx$$

and have studied the  $L^1$ -convergence of modified sine sums under a different class.

Kaur and Bhatia [11] have introduced new modified cosine and sine sums as

$$\frac{a_0}{2} + \sum_{k=1}^n \sum_{j=k}^n \Delta(a_j \cos jx)$$

and

$$\sum_{k=1}^n \sum_{j=k}^n \Delta(b_j \sin jx)$$

and studied their integrability and  $L^1$ -convergence.

Krasniqi [14] have introduced new modified cosine and sine sums as

$$H_n(x) = -\frac{1}{2 \sin x} \sum_{k=0}^n \sum_{j=k}^n \Delta[(a_{j-1} - a_{j+1}) \cos jx]$$

and

$$G_n(x) = \frac{1}{2 \sin x} \sum_{k=1}^n \sum_{j=k}^n \Delta[(b_{j-1} - b_{j+1}) \sin jx]$$

and studied the  $L^1$ -convergence of these modified cosine and sine sums with semi convex coefficients.

In 2013, Krasniqi [15] have introduced new modified cosine sum as

$$G_n(x) = \frac{a_0}{2} + \sum_{k_1=1}^n \sum_{k_2=k_1}^n \sum_{k_3=k_2}^n \Delta^2(a_{k_3} \cos k_3x),$$

where  $\Delta^2 a_k = \Delta(\Delta a_k) = a_k - 2a_{k+1} + a_{k+2}$

and studied its  $L^1$ -convergence.

We introduce here a new modified cosine and sine sums as

$$f_n(x) = \sum_{k=1}^n \left[ \sum_{j=k}^n \left( \Delta a_{j+1} + \sum_{i=j}^n \Delta^3 a_i \right) \right] \cos kx$$

where  $\Delta^3 a_k = \Delta^2 a_k - \Delta^2 a_{k+1}$

and

$$g_n(x) = \sum_{k=1}^n \left[ \sum_{j=k}^n \left( \Delta b_{j+1} + \sum_{i=j}^n \Delta^3 b_i \right) \right] \sin kx$$

where  $\Delta^3 b_k = \Delta^2 b_k - \Delta^2 b_{k+1}$

and study the  $L^1$ -convergence of these modified cosine and sine sums under the class S.

As usual  $D_n(x)$  and  $\tilde{D}_n(x)$  will denote the Dirichlet and its conjugate kernels respectively, defined by:

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^n \cos kx$$

and

$$\tilde{D}_n(x) = \sum_{k=1}^n \sin kx$$

## II. LEMMAS

The following lemmas are used in the proof of main results:

**Lemma 1.** [20] Let  $n \geq 1$  and let  $r$  be a nonnegative integer,  $x \in [\epsilon, \pi]$ . Then  $|\tilde{D}_n^r(x)| \leq C_\epsilon \frac{n^r}{x}$  where  $C_\epsilon$  is a positive constant depending on  $\epsilon$ ,  $0 < \epsilon < \pi$  and  $\tilde{D}_n(x)$  is the conjugate Dirichlet kernel.

**Lemma 2.** [20]  $\|D_n^r(x)\|_{L^1} = O(n^r \log n)$ ,  $r = 0, 1, 2, 3, \dots$  where  $D_n^r(x)$  represents the  $r^{\text{th}}$  derivative of Dirichlet kernel.

**Lemma 3.** [20]  $\|\tilde{D}_n^r(x)\|_{L^1} = O(n^r \log n)$ ,  $r = 0, 1, 2, 3, \dots$  where  $\tilde{D}_n^r(x)$  represents the  $r^{\text{th}}$  derivative of conjugate Dirichlet kernel.

## III. MAIN RESULTS

We establish the following results.

**Theorem 3.** Let  $\{a_k\}$  belong to the class S and  $n^2 a_n = o(1)$ , then  $\|f - f_n\|_{L^1} = o(1)$  as  $n \rightarrow \infty$ .

*Proof.* we have

$$\begin{aligned} f_n(x) &= \sum_{k=1}^n \left[ \sum_{j=k}^n \left( \Delta a_{j+1} + \sum_{i=j}^n \Delta^3 a_i \right) \right] \cos kx \\ &= \sum_{k=1}^n \left[ \sum_{j=k}^n (\Delta a_{j+1} + \Delta^2 a_j - \Delta^2 a_{n+1}) \right] \\ &= \sum_{k=1}^n [a_k - a_{n+1} - \Delta^2 a_{n+1}(n - k + 1)] \cos kx \\ &= \sum_{k=1}^n a_k \cos kx - a_{n+1} D_n(x) - (n+1) \Delta^2 a_{n+1} D_n(x) \\ &\quad + \Delta^2 a_{n+1} \tilde{D}'_n(x) \end{aligned}$$

Apply Abel's transformation, we get

$$\begin{aligned} &= \sum_{k=1}^n \Delta a_k D_k(x) - (n+1)(a_{n+1} - 2a_{n+2} + a_{n+3}) D_n(x) \\ &\quad + (a_{n+1} - 2a_{n+2} + a_{n+3}) \tilde{D}'_n(x) \end{aligned}$$

Since  $D_n(x)$  is bounded and  $|\Delta a_k| \leq A_k \quad \forall k = 1, 2, \dots$

Hence by given hypothesis and Lemma 1:

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \text{ exists in } (0, \pi)$$

Next, we consider

$$f(x) - f_n(x) = \sum_{k=n+1}^{\infty} a_k \cos kx + (n+1) \Delta^2 a_{n+1} D_n(x)$$

$$+ a_{n+1}D_n(x) - \Delta^2 a_{n+1}\tilde{D}'_n(x)$$

Apply Abel's transformation, we have

$$\begin{aligned} &= \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) + (n+1)\Delta^2 a_{n+1}D_n(x) - \Delta^2 a_{n+1}\tilde{D}'_n(x) \\ &\int_0^{\pi} |f(x) - f_n(x)|dx = \\ &\int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) + (n+1)a_{n+1}D_n(x) \right. \\ &\quad \left. - 2(n+1)a_{n+2}D_n(x) + (n+1)a_{n+3}D_n(x) - a_{n+1}\tilde{D}'_n(x) \right. \\ &\quad \left. + 2a_{n+2}\tilde{D}'_n(x) - a_{n+3}\tilde{D}'_n(x) \right| dx \\ &\leq \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx + (n+1)|a_{n+1}| \int_0^{\pi} |D_n(x)|dx \\ &+ 2(n+1)|a_{n+2}| \int_0^{\pi} |D_n(x)|dx + (n+1)|a_{n+3}| \int_0^{\pi} |D_n(x)|dx \\ &\quad + |a_{n+1}| \int_0^{\pi} |\tilde{D}'_n(x)|dx + 2|a_{n+2}| \int_0^{\pi} |\tilde{D}'_n(x)|dx \\ &\quad + |a_{n+3}| \int_0^{\pi} |\tilde{D}'_n(x)|dx \end{aligned}$$

It is well known that  $\int_0^{\pi} |D_n(x)|dx \sim \log n$  (see [1]) and using Lemma 3, we get

$$\begin{aligned} &\leq \sum_{k=n+1}^{\infty} \Delta A_k \int_0^{\pi} \left| \sum_{i=0}^k \frac{\Delta a_i}{A_i} D_i(x) \right| dx + (n+1)|a_{n+1}| \log n \\ &+ 2(n+1)|a_{n+2}| \log n + (n+1)|a_{n+3}| \log n + |a_{n+1}|n \log n \\ &\quad + 2|a_{n+2}|n \log n + |a_{n+3}|n \log n \\ &\leq C \sum_{k=n+1}^{\infty} (k+1)\Delta A_k + n(n+1)|a_{n+1}| + 2n(n+1)|a_{n+2}| \\ &\quad + n(n+1)|a_{n+3}| + n^2|a_{n+1}| + 2n^2|a_{n+2}| + n^2|a_{n+3}| \end{aligned}$$

Since,  $\{a_k\} \in S$  then  $\sum_{k=n+1}^{\infty} (k+1)\Delta A_k = o(1)$  as  $n \rightarrow \infty$  and by given hypothesis

The conclusion of main result holds.

**Corollary 1.** Let the coefficient  $\{a_k\}$  belong to the class  $S$  and  $n^2 a_n = o(1)$  as  $n \rightarrow \infty$ , then the necessary and sufficient condition for the  $L^1$ -convergence of cosine series (1) is  $\lim_{n \rightarrow \infty} a_n \log n = 0$ .

*Proof.* we notice that

$$\begin{aligned} \|f - S_n\|_{L^1} &= \|f - f_n + f_n - S_n\|_{L^1} \\ &\leq \|f - f_n\|_{L^1} + \|f_n - S_n\|_{L^1} \\ &\leq \int_0^{\pi} |f(x) - f_n(x)|dx + \int_0^{\pi} |a_{n+1}D_n(x) - \Delta^2 a_{n+1}\tilde{D}'_n(x) \\ &\quad + (n+1)\Delta^2 a_{n+1}D_n(x)|dx \\ &\leq \int_0^{\pi} |f(x) - f_n(x)|dx + \int_0^{\pi} |a_{n+1}D_n(x) + (n+1)(a_{n+1} \\ &\quad - 2a_{n+2} + a_{n+3})D_n(x) - (a_{n+1} - 2a_{n+2} + a_{n+3})\tilde{D}'_n(x)|dx \\ &\leq \int_0^{\pi} |f(x) - f_n(x)|dx + |a_{n+1}| \int_0^{\pi} |D_n(x)|dx \\ &+ (n+1)|a_{n+1}| \int_0^{\pi} |D_n(x)|dx + 2(n+1)|a_{n+2}| \int_0^{\pi} |D_n(x)|dx \end{aligned}$$

$$\begin{aligned} &+ (n+1)|a_{n+3}| \int_0^{\pi} |D_n(x)|dx + |a_{n+1}| \int_0^{\pi} |\tilde{D}'_n(x)|dx \\ &+ 2|a_{n+2}| \int_0^{\pi} |\tilde{D}'_n(x)|dx + |a_{n+3}| \int_0^{\pi} |\tilde{D}'_n(x)|dx \end{aligned}$$

We know  $\int_0^{\pi} |D_n(x)|dx \sim \log n$  (see [1]) and using Lemma 3, we have:

$$\begin{aligned} &\leq \int_0^{\pi} |f(x) - f_n(x)|dx + |a_{n+1}| \log n + (n+1)|a_{n+1}| \log n \\ &+ 2(n+1)|a_{n+2}| \log n + (n+1)|a_{n+3}| \log n + |a_{n+1}|n \log n \\ &\quad + 2|a_{n+2}|n \log n + |a_{n+3}|n \log n \\ &\leq \int_0^{\pi} |f(x) - f_n(x)|dx + |a_{n+1}| \log n + n(n+1)|a_{n+1}| \\ &+ 2n(n+1)|a_{n+2}| + n(n+1)|a_{n+3}| + n^2|a_{n+1}| \log n \\ &\quad + 2n^2|a_{n+2}| + n^2|a_{n+3}| \end{aligned}$$

Since,  $\int_0^{\pi} |f(x) - f_n(x)|dx = 0$  as  $n \rightarrow \infty$  and by given conditions.

The conclusion of the corollary follows.

**Theorem 4.** Let the coefficient  $\{b_k\}$  belong to the class  $S$  and  $n^2 b_n = o(1)$ , then  $\|g - g_n\|_{L^1} = o(1)$  as  $n \rightarrow \infty$ .

**Corollary 2.** Let  $\{b_k\}$  belong to the class  $S$  and  $n^2 b_n = o(1)$  as  $n \rightarrow \infty$ , then the necessary and sufficient condition for the  $L^1$ -convergence of sine series (2) is  $\lim_{n \rightarrow \infty} b_n \log n = 0$ .

The proofs of Theorem 4 and Corollary 2 are similar to the proofs of Theorem 3 and Corollary 1, therefore we omit.

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