Recovering the Boundary Data in the Two Dimensional Inverse Heat Conduction Problem
Using the Ritz-Galerkin Method

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Abstract—This article presents a numerical method to find the heat flux in an inhomogeneous inverse heat conduction problem with linear boundary conditions and an extra specification at the terminal. The method is based upon applying the satisfier function along with the Ritz-Galerkin technique to reduce the approximate solution of the inverse problem to the solution of a system of algebraic equations. The instability of the problem is resolved by taking advantage of the Landweber’s iterations as an admissible regularization strategy. In computations, we find the stable and low-cost results which demonstrate the efficiency of the technique.

Keywords—Inverse problem, parabolic equations, heat equation, Ritz-Galerkin method, Landweber iterations.

I. INTRODUCTION

Heat conduction problems are of vital importance in many areas of applied sciences [1], [8] e.g. heat exchangers, mathematical finance, various chemical and biological systems [1], [5], [6]. The variety of applications has drawn attention of many researchers which in turn led to significant progress in this area [3], [4]. However, multi-dimensional problems with irregular domains have not examined sufficiently [7], [13]. The organization of heat conduction problems is twofold:

- If the heat flux or temperature histories are known as functions of time, the problem of finding temperature distribution is interpreted as a direct problem.
- If the description of a surface condition is sufficient, the problem of approximating temperature histories from transient temperature measurements at some interior locations is called the inverse problem.

Of particular interest to us is the inverse problem for the heat equation which describes the heat conduction procedure in a given inhomogeneous medium.

In the rectangle $\Omega_2 = \{(x, y, t) \in \mathbb{R}^3 | 0 \leq x \leq 1, 0 \leq y \leq 1, 0 < t < T\}$, we seek the approximation of the solution of the Cauchy problem described by the following heat conduction equation in two dimensions [15]:

$$\frac{\partial A(x, y, t)}{\partial t} = \frac{\partial^2 A(x, y, t)}{\partial x^2} + \frac{\partial^2 A(x, y, t)}{\partial y^2} + F(x, y, t), \quad \text{in} \ \Omega_2,$$

with the initial condition:

$$A(x, y, 0) = f(x, y), \quad (x, y) \in [0, 1] \times [0, 1],$$

and the known boundary data:

$$A(0, y, t) = h_1(y, t), \quad A(1, y, t) = h_2(y, t), \quad (y, t) \in [0, 1] \times [0, 1] \times (0, T].$$

II. SOLUTION OF INVERSE PROBLEM VIA RITZ-GALERKIN METHOD

Assume that the given functions satisfy some consistent conditions such as:

$$h_1(y, 0) = f(0, y), \quad h_2(y, 0) = f(1, y), \quad \varphi(x, 0) = f_y(x, 1),$$

$$E(0) + \alpha(0) = f(x^*, y^*).$$

Since (1)-(5) are linear, we employ the classical Ritz-Galerkin method [11]-[13]. Hence, we define:

$$B_1(x, y, t) = (y - 1)\varphi(x, t),$$

$$B_2(x, y, t) = h_1(y, t) + x(h_2(y, t) - h_1(y, t)),$$

$$B_3(x, y, t) = B_1(x, y, t) + B_2(x, y, t) - (y - 1)\frac{\partial B_2(x, y, t)}{\partial y}|_{y=1},$$

the satisfier function which fulfills (2)-(4) is:

$$SF2(x, y, t) = B_3(x, y, t) + f(x, y) - B_3(x, y, 0).$$
and the Ritz-type approximation for $A(x,y,t)$ is sought in the form of the truncated series

$$A(x,y,t) = \sum_{i_1,i_2,i_3=0}^{N_1,N_2,N_3} c_{i_1,i_2,i_3} x(i-1)(y-1)^2 \psi_{i_1,i_2,i_3}(x,y,t) + SF2(x,y,t).$$

(11)

The expansion coefficients $c_{i_1,i_2,i_3}$ are determined by the simultaneous Galerkin equations [12], [16]:

$$GEs \{ \begin{array}{l}
< (\text{Residual}(x,y,t)), \Psi_{i_1,i_2,i_3}(x,y,t) > = 0,
 (\text{Residual}(x^*,y^*,\tau_{i_3})) = E(\tau_{i_3}) + \alpha(\tau_{i_3}),
\end{array} \}
$$

where $<,>$ denotes the inner product defined by

$$< f, g > = \int_{[0,T]} \int_{[0,1]} \int_{[0,1]} f(x,y,t)g(x,y,t)dxdydt,$$

$$i_3 \in [1,2,3], i_3 = 1, N_3 = \frac{s+3}{s+2}, i_3 = 1, N_1 N_2 \text{~and~} \delta(t - \tau_{i_3}) \text{~is~the~Dirac~delta~corresponding~to~the~grids~come~from~the~domain~of~the~problem,~}\psi_{i_1,i_2,i_3}(x,y,t) = \psi_{i_1}(x)\psi_{i_2}(y)\psi_{i_3}(t),$n $$i_3 \in [1,2,3], i_3 = 1, N_3$$ are the basis functions and

$$\text{Residual}(x,y,t) := \frac{\partial A(x,y,t)}{\partial t} - \frac{\partial^2 A(x,y,t)}{\partial x^2} - \frac{\partial^2 A(x,y,t)}{\partial y^2} + F(x,y,t),$$

(13)

or equivalently

$$\text{Residual}(x,y,t) := A(x,y,t) - f(x,y)$$

(14)

By applying the Galerkin equations (12), we obtain a linear system of equations $Ac = b$ which will be solved by the Landweber’s iterations to find the stable solution [8].

Following, we introduce the basic Gronwall’s lemma [14] formulated in its simplest form which will be used in the sequel.

**Lemma 1:** For a function $g(t)$ satisfying the inequality:

$$\frac{dg}{dt} \leq ag(t) + b(t), \quad t > 0, a = \text{const}, \quad b(t) \geq 0,$$

(15)

the following estimate holds:

$$g(t) \leq \exp(at)(g(0) + \int_{[0,t]} \exp(-a\tau)b(\tau)d\tau).$$

**Theorem 1:** Consider $A(x,y,t)$ as the regularized solution for $A(x,y,t)$ obtained by (7)-(13) corresponding to inverse problem. Then the existence of a priori information

$$\sup_{x \in (0,1)} A(x^*,y^*,t) - \frac{\partial A(x,0,t)}{\partial y}|_{y=0} \leq \delta^*(t), \quad |\delta^*(t)| < \infty,$$

about the solution guarantees solution stability with respect to the right-hand side and boundary input data. Here $A(x^*,y^*,t) = E(t) + \alpha(t)$ is exactly given as the overdetermination (5).

**III. Numerical Examples**

This example [15] illustrates the applicability of the proposed method for solving the inverse problem with input data given by:

$$h_i(y,t) = iy^2, \quad i \in \{1,2\}, \quad \varphi(x,t) = (x+1)^2,$$

(16)

$$E(t) = \frac{3t^2}{4}, \quad (x^*,y^*) = (\frac{1}{2}, \frac{1}{2}), \quad T = 1.$$

The unknown data which is sought, is given by:

$$g(x,t) = (x+1)^2, \quad (x,t) \in [0,1] \times [0,1].$$

(17)

We use the scheme presented by (7)-(14) to solve this problem numerically and for the exact data i.e. $\alpha(t) = 0$ we derive the exact solution. It can be expected because the analytical solution of this problem is separable with respect to the variables $x,y,t$ and our procedure employs the satisfier function which produces accurately the exact solution.

As the second example, we consider the inverse heat conduction problem with the exact solution:

$$A(x,y,t) = \sin(x+y+t) - (y-1)^2 \cos(t), \quad 0 \leq t \leq 1,$$

(18)

and we are going to find the approximation to the boundary data:

$$g(x,t) = \cos(x+t) + 2 \cos(t), \quad (x,t) \in [0,1] \times [0,1],$$

(19)

using the extra condition given at the $(x^*,y^*) = (\frac{1}{2}, \frac{1}{2})$.

First we contaminate the boundary data using the function:

$$\alpha(t) = \lambda \% \sin\left(\frac{t}{\lambda \%}\right), \quad \lambda \% = \lambda \times 10^{-2} \in \{3, 5, 9\} \times 10^{-2},$$

(20)

and then use the Bernstein multi-scaling functions and the Landweber’s iterations with the parameters $N_1 = N_2 = 1, N_3 = 20, a = 1, m = 10^4$, we find the the approximations depicted by Figs. (1a)-(1c). It is seen as $\lambda$ decreases, the accuracy of the approximations gradually becomes better.

**IV. Conclusion**

The paper addresses a technique for boundary identification of the two dimensional inverse heat conduction problems. Focusing on the numerical aspects, the discontinuity between the input and output data has been discussed. Appropriate application of the Ritz-Galerkin method along with an iterative procedure, namely, the Landweber’s iterations, yield satisfactory productions that contain the qualifications of accurate, stable and low cost numerical solution.

**Appendix A**

**Proof of Theorem 1**

Generally speaking, the approximate solution (11) satisfies the following partial differential equation

$$\frac{\partial A(x,y,t)}{\partial t} = \left(\frac{\partial^2 A(x,y,t)}{\partial x^2} + \frac{\partial^2 A(x,y,t)}{\partial y^2}\right) = F(x,y,t),$$

(20)
we define a Hilbert space $\mathcal{H} = L^2(\Omega^*_2)$, equipped with the norm
\[
\|v\|_\mathcal{H} = \langle v, v \rangle^{\frac{1}{2}} = \left(\int_{\Omega^*_2} v^2(x) dx\right)^{\frac{1}{2}}, \quad x \in \Omega^*_2. \tag{23}
\]
Thus taking $u \in \mathcal{H}$ and introducing the operator
\[
\mathcal{F} u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 < x < 1, \quad 0 < y < 1, \tag{24}
\]
taking into account (21)-(22) we have
\[
\frac{de}{dt} + \mathcal{F} e = \delta F(t), \quad e(0) = 0, \quad 0 < t < T.
\]
By multiplying (21) by $e(t)$, we have
\[
\langle \frac{de}{dt}, e \rangle + \langle \mathcal{F} e, e \rangle = < \delta F, e >. \tag{25}
\]
Firstly, applying the Cauchy-Bunyakovsky inequality [14], we have:
\[
(I) = \int_{[0,1]} \int_{[0,1]} \delta F(x, y, t) e(x, y, t) dx dy \leq \|\delta F\|_\mathcal{H} \|e\|_\mathcal{H}. \tag{26}
\]
Secondly, applying integration by parts and taking the homogeneous conditions (22) we have
\[
(II) = -\int_{[0,1]} \int_{[0,1]} \frac{\partial e}{\partial x} \frac{\partial e}{\partial x} dx dy - \int_{[0,1]} \int_{[0,1]} \frac{\partial e}{\partial y} \frac{\partial e}{\partial y} dx dy - \int_{[0,1]} \int_{[0,1]} e(x, 0, t) \frac{\partial e}{\partial y}|_{y=0} dx dy.
\]
Finally, considering
\[
(III) = \int_{[0,1]} \int_{[0,1]} \frac{de}{dt} dx dy = \frac{1}{2} \frac{d}{dt} \left(\int_{[0,1]} \int_{[0,1]} e^2 dx dy\right)
\]
we get
\[
\|e\|_\mathcal{H} \frac{d}{dt} \|e\|_\mathcal{H} + \int_{[0,1]} \int_{[0,1]} \left(\frac{\partial e}{\partial x}\right)^2 + \left(\frac{\partial e}{\partial y}\right)^2 dx dy \geq 0
\]
\[
\leq \|\delta F\|_\mathcal{H} \|e\|_\mathcal{H} + \int_{[0,1]} e(x, 0, t) \frac{\partial e}{\partial y}|_{y=0} dx dy.
\]

Applying the mean-value theorem for $e(x, 0, t) \frac{\partial e(x, y, t)}{\partial y}|_{y=0}$ and taking $x^t \in (0, 1)$ we have
\[
\int_{[0,1]} e(x, 0, t) \frac{\partial e}{\partial y}|_{y=0} dx = \left| e(x^t, 0, t) \frac{\partial e(x^t, y, t)}{\partial y}|_{y=0}\right|
\]
\[
\leq \|e\|_\mathcal{H} \left| \frac{\partial e(x^t, y, t)}{\partial y}|_{y=0}\right|.
\]
Considering the extra specification (5) and assuming the priori condition
\[
\sup_{x \in (0, 1)} \left| A(x^*, y^*, t) \frac{\partial A(x, 0, t)}{\partial y}|_{y=0}\right| \leq \delta^*(t),
\]
we have
\[
\frac{de}{dt} + \mathcal{F} e = \delta F(t), \quad e(0) = 0, \quad 0 < t < T.
\]
we get
\[
\frac{\partial e(x^1,y,t)}{\partial y}\bigg|_{y=0} \leq (|\delta^*(t) - \bar{\delta}^*(t)| + |E(t) - \bar{E}(t)|),
\]

where
\[
\sup_{x \in (0,1)} A(x^*, y^*, t) - \frac{\partial A(x,0,t)}{\partial y} \bigg|_{y=0} \leq \bar{\delta}^*(t),
\]

Consequently
\[
\frac{de}{dt} \leq |\delta F(t)|_H + \|\delta^*(t) - \bar{\delta}^*(t)|_H + \|E(t) - \bar{E}(t)|_H,
\]

by using the Gronwall’s lemma with
\[
a = 0, \ b(t) = |\delta F(t)|_H + \|\delta^*(t) - \bar{\delta}^*(t)|_H + \|E(t) - \bar{E}(t)|_H,
\]

one obtains
\[
\|e(t)|_H \leq \int_{[0,t]} \left( |\delta F(s)|_H + \|\delta^*(s) - \bar{\delta}^*(s)|_H \right) ds,
\]

For a given \( \epsilon > 0 \), such that \( \sup_{0 \leq t \leq T} \left( |\delta F(t)|_H + \|\delta^*(t) - \bar{\delta}^*(t)|_H + \|E(t) - \bar{E}(t)|_H \right) \leq \epsilon \) we deduce
\[
\|e(t)|_H \leq T \epsilon.
\]

The latter inequality guarantees continuous dependence of the solution of problem (1)-(5) on the right-hand side and boundary conditions.

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for carefully reading the paper.

REFERENCES