Optimal Linear Quadratic Digital Tracker for the Discrete-Time Proper System with an Unknown Disturbance

Jason Sheng-Hong Tsai, Faezeh Ebrahimzadeh, Min-Ching Chung, Shu-Mei Guo, Leang-San Shieh, Tzong-Jiy Tsai, Li Wang

Abstract—In this paper, we first construct a new state and disturbance estimator using discrete-time proportional plus integral observer to estimate the system state and the unknown external disturbance for the discrete-time system with an input-to-output direct-feedthrough term. Then, the generalized optimal linear quadratic digital tracker design is applied to construct a proportional plus integral observer-based tracker for the system with an unknown external disturbance to have a desired tracking performance. Finally, a numerical simulation is given to demonstrate the effectiveness of the new application of our proposed approach.

Keywords—Optimal linear quadratic tracker, proportional plus integral observer, state estimator, disturbance estimator.

I. INTRODUCTION

IN the real world, there is usually an unknown external disturbance which occurs at the plant input and results in a performance [1]. Due to this issue, Chang [2] used the discrete-time proportional plus integral observer (PIO) to develop a state and disturbance estimator for the strictly proper discrete-time system with an unknown external disturbance. In addition, Gao et al. [3] applied the PIO to develop an observer-based regulator for the discrete-time strictly proper system with an unknown external disturbance.

In this study, we construct a state and disturbance estimator using discrete-time PIO to estimate the proper system state and the external disturbance, when the system state is not measurable and the disturbance is unknown. Then, by applying the generalized optimal linear quadratic digital tracker (LQDT) design, presented in our previous works [4], [5], we construct a

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PIO-based LQDT with a high-gain property to have a desired tracking performance.

The paper is organized as follows. A generalized optimal LQDT for discrete-time proper system with known system disturbances or a compensatory signal is briefly described in Section II. A new PIO-based LQDT for the discrete-time proper system with an unknown external disturbance is presented in Section III. A numerical simulation is given in Section IV to demonstrate the effectiveness of our proposed approach. Finally, conclusion is given in Section V.

II. A GENERALIZED OPTIMAL TRACKER FOR THE PROPER SYSTEM WITH SYSTEM DISTURBANCES

A generalized optimal LQDT with pre-specified output and control input trajectories for the linear, controllable and observable discrete-time proper system with known system disturbances has been presented in [4]. In this section, we briefly describe the generalized LQDT as follows.

Consider the linear discrete-time minimum phase system described by

$$x_d(k+1) = Gx_d(k) + Hu_d(k) + d(k), \qquad (1a)$$

$$y_d(k) = Cx_d(k) + Du_d(k) + s(k), \qquad (1b)$$

where $x_d(k) \in \Re^n$ is the state vector, $u_d(k) \in \Re^m$ is the control input vector, $y_d(k) \in \Re^p$ is the measured output vector, $m \geq p$, $d(k) \in \Re^n$ is the process disturbance, and $s(k) \in \Re^p$ is the measurement disturbance. The objective is to determine the optimal control sequence $u_d(0)$, $u_d(1)$, $u_d(2)$, $\cdots$ $u_d(N-1)$ that minimizes the following linear quadratic performance index for a finite time process ($0 \leq k \leq N$)

$$
\begin{aligned}
J(y_d, u_d) = & \frac{1}{2}\left[y_d(N) - r_d(N)\right]^T S \left[y_d(N) - r_d(N)\right] \\
& + \frac{1}{2}\sum_{k=0}^{N-1}\left\{\left[y_d(k) - r_d(k)\right]^T Q_d \left[y_d(k) - r_d(k)\right]\right. \\
& \left. + \left[u_d(k) - u_d^*(k)\right]^T R_d \left[u_d(k) - u_d^*(k)\right]\right\},
\end{aligned}
\qquad (2)
$$

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where Q_d is a $p \times p$ positive definite or positive semi-definite real symmetric matrix, R_d is an $m \times m$ positive definite real symmetric matrix, S is a $p \times p$ positive definite or positive semi-definite real symmetric matrix, $r_d(k)$ is a pre-specified output trajectory, and $u_d^*(k)$ is an approximate pre-specified control input trajectory if it is available. An interesting application of the generalized optimal LQDT design for the input-constraint problem has been studied in our previous work [4], where $u_d^*(k)$ is set as the pre-specified control input saturation bound, whenever the pre-determined control input exceeds the practical saturation bound at any time instant.

The minimization problem subjected to equality constraint in (1a) can be solved by adjoining the equality constraint to the quadratic performance index to be minimized by use of Lagrange multiplier [6]. Hence, if the final time $k = N$ goes to infinity, the optimal control vector $u_d(k)$ can be determined as

$$u_d(k) = -K_d x_d(k) + E_d r_d(k) + C_d(k) + C_{u_d} u_d^*(k), \quad (3)$$

where

$$K_d = \tilde{R}_d^{-1} \overline{P}_d,$$

$$E_d = \tilde{R}_d^{-1} \left\{ D^T + H^T \left[I_n - (G - HK_d)^T \right]^{-1} (C - DK_d)^T \right\} Q_d,$$

$$C_d(k) = Z_d d(k) + Z_s s(k),$$

$$C_{u_d} = \tilde{R}_d^{-1} \left\{ H^T \left[(G - HK_d)^T - I_n \right]^{-1} K_d^T + I_m \right\} R_d,$$

in which

$$Z_d = \tilde{R}_d^{-1} H^T \left\{ \left[(G - HK_d)^T - I_n \right]^{-1} (G - HK_d)^T - I_n \right\} P_d,$$

$$Z_s = \tilde{R}_d^{-1} \left\{ H^T \left[(G - HK_d)^T - I_n \right]^{-1} (C - DK_d)^T - D^T \right\} Q_d,$$

$$\overline{R}_d = R_d + D^T Q_d D, \quad N_d = C^T Q_d D, \quad \tilde{R}_d = \overline{R}_d + H^T P_d H,$$

$$\overline{P}_d = N_d^T + H^T P_d G,$$

and P_d is the positive definite solution of the following algebraic Riccati equation (ARE)

$$P_d = G^T P_d G + C^T Q_d C - \left(H^T P_d G + N_d^T \right)^T$$
$$\times \left[\overline{R}_d + H^T P_d H \right]^{-1} \left(H^T P_d G + N_d^T \right).$$

III. Observer-Based Optimal Tracker for the Proper System with an Unknown Disturbance

Consider the linear MIMO discrete-time system with a direct-feedthrough term and an unknown disturbance which occurs at the plant input described by

$$x_d(k+1) = Gx_d(k) + Hu(k) = Gx_d(k) + H[u_d(k) + d(k)], \quad (4a)$$

$$y_d(k) = Cx_d(k) + Du(k) = Cx_d(k) + D[u_d(k) + d(k)], \quad (4b)$$

where $x_d(k) \in \Re^n$ is the state vector, $u_d(k) \in \Re^m$ is the control input vector, $y_d(k) \in \Re^p$ is the measured output vector, and $d(k) \in \Re^m$ is the unknown external disturbance vector.

The objective is to design a state and disturbance estimator using discrete-time PIO, and then apply (3) to have a PIO-based optimal LQDT such that the controlled system in (4) has a desired tracking performance for a given arbitrary reference trajectory $r_d(k)$ with some drastic variations.

In order to design the state and disturbance estimator using PIO, the following assumptions are presumed.

Assumption 1 [2]: The sampling time T_s is sufficiently small such that the disturbance does not vary too much between two consecutive sampling instants.

If the previous assumption holds, one can obtain $\Delta d(k) = d(k+1) - d(k) \in O(T_s^2) \; \forall \, k$.

Assumption 2 [2]: The pair (G, C) is observable, and

$$\text{rank}\left(\begin{bmatrix} G_{n \times n} - I_n & H_{n \times m} \\ C_{p \times n} & D_{p \times m} \end{bmatrix} \right) = n + m.$$

Since the PIO has two feedback loops, a proportional loop and an integral loop of the output estimation error, namely, it allows us to estimate not only the state $x_d(k)$ but also the unknown external disturbance $d(k)$. A PIO is given as

$$\hat{x}_d(k+1) = G\hat{x}_d(k) + H[u_d(k) + \hat{d}(k)] + K_P[y_d(k) - \hat{y}_d(k)], \quad (5a)$$

$$\hat{d}(k+1) = \hat{d}(k) + K_I[y_d(k) - \hat{y}_d(k)], \quad (5b)$$

$$\hat{y}_d(k) = C\hat{x}_d(k) + D[u_d(k) + \hat{d}(k)], \quad (5c)$$

where $\hat{x}_d(k) \in \Re^n$ is the estimation of the state vector, $\hat{d}(k) \in \Re^m$ is the estimation of the disturbance, $K_P \in \Re^{n \times p}$ and $K_I \in \Re^{m \times p}$ are the proportional and integral gain matrices, respectively, to be designed.

In order to derive the estimation error dynamic equations, the estimation errors are defined by $\tilde{y}_d(k) = y_d(k) - \hat{y}_d(k)$, $\tilde{x}_d(k) = x_d(k) - \hat{x}_d(k)$, and $\tilde{d}(k) = d(k) - \hat{d}(k)$. Then, the estimation error dynamic equations can be derived as:

$$\begin{bmatrix} \tilde{x}_d(k+1) \\ \tilde{d}(k+1) \end{bmatrix} = (\hat{G} - L\hat{C}) \begin{bmatrix} \tilde{x}_d(k) \\ \tilde{d}(k) \end{bmatrix} + \begin{bmatrix} 0_{n \times 1} \\ \Delta d(k) \end{bmatrix}, \quad (6a)$$

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$$\tilde{y}_d(k) = \hat{C}\begin{bmatrix} \tilde{x}_d(k) \\ \tilde{d}(k) \end{bmatrix}, \tag{6b}$$

where

$$\hat{G} = \begin{bmatrix} G & H \\ 0_{m \times n} & I_m \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} C & D \end{bmatrix}, \quad L = \begin{bmatrix} K_P \\ K_I \end{bmatrix}.$$

From (6), one can see that the matrix $(\hat{G} - L\hat{C})$ can be stabilized if the pair $(\hat{G}, \hat{C})$ is observable. An observer gain L can be found such that all eigenvalues of the matrix $(\hat{G} - L\hat{C})$ are inside the unit circle. As for the observability condition for the pair $(\hat{G}, \hat{C})$, we may state in Lemma 1.

Lemma 1: Under Assumptions 1 and 2, the pair $(\hat{G}, \hat{C})$ is observable and the estimation errors are constrained in the small region of $O(T_s^2)$.

$\square$

Proof of Lemma 1: By Popov-Belevitch-Hautus rank test [2], the pair $(\hat{G}, \hat{C})$ is observable if the matrix $\begin{bmatrix} \lambda I_{n+m} - \hat{G} \\ \hat{C} \end{bmatrix}$ has full column rank for all λ in $\mathbb{C}$. Observe that

$$\begin{aligned}
\text{rank}\left(\begin{bmatrix} \lambda I_{n+m} - \hat{G} \\ \hat{C} \end{bmatrix}\right) &= \text{rank}\left(\begin{bmatrix} \lambda I_n - G & -H \\ 0_{m \times n} & \lambda I_m - I_m \\ \hline C & D \end{bmatrix}\right) \\
&= \begin{cases} \text{rank}\left(\begin{bmatrix} \lambda I_n - G \\ C \end{bmatrix}\right) + m, & \lambda \neq 1 \\ \text{rank}\left(\begin{bmatrix} G - I_n & H \\ C & D \end{bmatrix}\right), & \lambda = 1 \end{cases}
\end{aligned} \tag{7}$$

From (7), one can conclude that the pair $(\hat{G}, \hat{C})$ is observable if Assumption 2 holds. In addition, the estimation errors will be constrained in the small region of $O(T_s^2)$ since the sampling time T_s is assumed to be sufficiently small.

$\square$

To have a desired observer gain L for the PIO in (5) such that the closed-loop observer error system poles are optimally assigned inside a circle with a pre-specified radius α ($0 < \alpha \leq 1$), let us perform the transformations $\tilde{G} = \hat{G}/\alpha$, $\tilde{C} = \hat{C}/\alpha$, and $0 < \alpha \leq 1$, leading to a transformed system given by

$$\begin{aligned}
\begin{bmatrix} \tilde{x}_d^t(k+1) \\ \tilde{d}^t(k+1) \end{bmatrix} &= \frac{1}{\alpha}(\hat{G} - L\hat{C})\begin{bmatrix} \tilde{x}_d^t(k) \\ \tilde{d}^t(k) \end{bmatrix} + \begin{bmatrix} 0_{n \times 1} \\ \Delta d^t(k) \end{bmatrix} \\
&= (\tilde{G} - L\tilde{C})\begin{bmatrix} \tilde{x}_d^t(k) \\ \tilde{d}^t(k) \end{bmatrix} + \begin{bmatrix} 0_{n \times 1} \\ \Delta d^t(k) \end{bmatrix}.
\end{aligned} \tag{8}$$

Then, by solving the following steady-state algebraic Riccati equation

$$P = \tilde{G}P\tilde{G}^T - (\tilde{G}P\tilde{C}^T)(R_o + \tilde{C}P\tilde{C}^T)^{-1}(\tilde{G}P\tilde{C}^T)^T + Q_o, \tag{9a}$$

the desired observer gain is given by

$$L = (\tilde{G}P\tilde{C}^T)(R_o + \tilde{C}P\tilde{C}^T)^{-1}, \tag{9b}$$

where the weighting matrices $Q_o \in \mathfrak{R}^{(n+m) \times (n+m)}$ and $R_o \in \mathfrak{R}^{p \times p}$ are selected with a high-gain property. Hence, the eigenvalues of $(\tilde{G} - L\tilde{C})$ are guaranteed to be inside the unit circle. The obtained observer gain L is then applied to the original observer system in (6), leading to the closed-loop characteristic equation

$$\det\left[zI_{(n+m)} - (\hat{G} - L\hat{C})\right] = \det\left[zI_{(n+m)} - (\tilde{G} - L\tilde{C}) \times \alpha\right] = 0, \tag{10}$$

which means that the eigenvalues of $(\hat{G} - L\hat{C})$ are equal to the eigenvalues of $(\tilde{G} - L\tilde{C})$ multiplied by the factor α. Clearly, it's desired to choose a small value of α to speed up the convergence of the estimation errors.

Then, by applying (3), the PIO-based LQDT is given by

$$u_d(k) = -K_d \hat{x}_d(k) + E_d r_d(k) + \hat{C}_d(k), \tag{11}$$

where

$$K_d = \tilde{R}_d^{-1}\overline{P}_d,$$

$$E_d = \tilde{R}_d^{-1}\left\{D^T + H^T\left[I_n - (G - HK_d)^T\right]^{-1}(C - DK_d)^T\right\}Q_d,$$

$$\hat{C}_d(k) = Z_{\hat{d}}\hat{d}(k),$$

in which

$$Z_{\hat{d}} = \tilde{R}_d^{-1}H^T\left\{\left[(G - HK_d)^T - I_n\right]^{-1}(G - HK_d)^T - I_n\right\}P_d H$$

$$+ \tilde{R}_d^{-1}\left\{H^T\left[(G - HK_d)^T - I_n\right]^{-1}(C - DK_d)^T - D^T\right\}Q_d D,$$

$$\overline{R}_d = R_d + D^T Q_d D, \quad N_d = C^T Q_d D, \quad \tilde{R}_d = \overline{R}_d + H^T P_d H,$$

$$\overline{P}_d = N_d^T + H^T P_d G,$$

and P_d is the positive definite solution of the following algebraic Riccati equation

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$$P_d = G^T P_d G + C^T Q_d C - \left(H^T P_d G + N_d^T \right)^T$$
$$\times \left[\overline{R}_d + H^T P_d H \right]^{-1} \left(H^T P_d G + N_d^T \right).$$

The structure of the PIO-based LQDT for the discrete-time system with an unknown external disturbance is shown in Fig. 1.
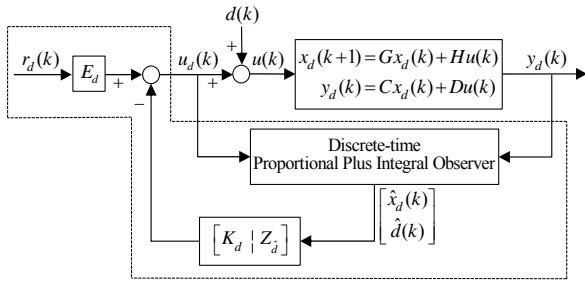
Proportional Plus Integral Observer-Based Linear Quadratic Digital Tracker

Fig. 1 PIO-based optimal LQDT for the discrete-time system with an unknown external disturbance

Lemma 2: If the ratio of Q_d to R_d tends to infinity, and the direct-feed through term D has full row rank, then the system output in (4b) is reduced to $y_d(k) \approx r_d(k)$ when $\tilde{d}(k) \to 0$ and $\tilde{x}_d(k) \to 0$.

$\square$

Proof of Lemma 2: Let $Q_d = \mu I_p$, $R_d = I_m$, and $\mu \to \infty$. Then, one has

$$K_d \approx \left(D^T Q_d D \right)^{\dagger} \left(D^T Q_d C \right), \qquad (12)$$

where $\dagger$ denotes the pseudo inverse. Since $\text{rank}(D) = p$, it can be obtained that

$$(C - DK_d) = C - D \left(D^T Q_d D \right)^{\dagger} \left(D^T Q_d C \right)$$
$$= C - DD^{\dagger} \left[\left(D^T Q_d \right)^{\dagger} \left(D^T Q_d \right) \right] C \quad (13)$$
$$\cong 0.$$

Also, one has

$$DZ_{\hat{d}} \approx D\tilde{R}_d^{-1} \left\{ H^T \left[(G - HK_d)^T - I_n \right]^{-1} (C - DK_d)^T - D^T \right\} Q_d D$$
$$\approx D \left(D^T Q_d D \right)^{\dagger} \left(-D^T \right) Q_d D$$
$$= -I_p D, \qquad (14)$$

which implies $Z_{\hat{d}} = -I_m$. So, the control input of the discrete-time system in (4) becomes

$$u(k) = u_d(k) + d(k)$$
$$\approx \left[-K_d \hat{x}_d(k) + E_d r_d(k) \right] + \left[d(k) - \hat{d}(k) \right] \qquad (15)$$
$$\approx \left[-K_d x_d(k) + E_d r_d(k) \right],$$

as $\tilde{d}(k) \to 0$, and $\tilde{x}_d(k) \to 0$. In addition, the forward gain matrix is reduced to

$$E_d \approx \left(D^T Q_d D \right)^{\dagger} D^T Q_d, \qquad (16)$$

which implies that $DE_d \approx I_p$ since $\text{rank}(D) = p$. Hence, by substituting (15) into (4b) and using the above results $(C - DK_d) \approx 0_{p \times n}$ and $DE_d \approx I_p$, we have

$$y_d(k) = Cx_d(k) + Du(k)$$
$$\approx Cx_d(k) + D\left[-K_d x_d(k) + E_d r_d(k) \right] \qquad (17)$$
$$\approx (C - DK_d) x_d(k) + DE_d r_d(k)$$
$$\approx r_d(k),$$

as $\tilde{d}(k) \to 0$, and $\tilde{x}_d(k) \to 0$.

$\square$

The design procedure for the PIO-based optimal LQDT consists of four steps and it may be stated as follows.
Step 1: Select a small value of α and an appropriate weighting matrix pair $\{Q_o, R_o\}$ which has a high-gain property.
Step 2: Solve the steady-state algebraic Riccati equation in (9a) to obtain the desired observer gain L.
Step 3: Construct the discrete-time PIO in (5) to estimate the state and disturbance.
Step 4: By applying (11), design the PIO-based optimal LQDT with a high-gain property to have a desired tracking performance.

IV. AN ILLUSTRATIVE EXAMPLE

Consider the discrete-time proper system with an unknown external disturbance in (4), where

$$G = \begin{bmatrix} -0.2726 & -0.3065 & 0.5746 & 0.5600 \\ 0.5557 & -0.1779 & -0.2320 & -0.2503 \\ -0.2221 & 0.0775 & 0.7158 & -0.0729 \\ 0.7324 & 0.4277 & -0.2562 & -1.1044 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.0825 & 0.1487 \\ -0.1640 & -0.2068 \\ 0.2977 & 0.7479 \\ 0.6286 & -0.8047 \end{bmatrix},$$

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$$C = \begin{bmatrix} -0.5632 & -0.7568 & -0.8830 & -0.6352 \\ 1.1420 & 0.8442 & -0.2290 & -0.7266 \end{bmatrix},$$

$$D = \begin{bmatrix} -0.4686 & -1.7327 \\ 0.7448 & 0.2237 \end{bmatrix},$$

with open-loop system poles $\{-1.2102, -0.2915, 0.1304,$ $0.5322\}$ and finite control zeros $\{-0.1364 \pm j0.3373,$ $0.2446 \pm j0.8115\}$ [4]. Assume the system state is not measurable. The unknown external disturbance $d(k) = \begin{bmatrix} d_1(k) & d_2(k) \end{bmatrix}^T$ is generated by

$$d_1(k) = \begin{cases} 5 & , 0.0 \le k < 0.6 \text{ sec} \\ -10k + 11 & , 0.6 \le k < 1.2 \text{ sec} \\ -1 & , 1.2 \le k < 1.8 \text{ sec} \\ 10k - 19 & , 1.8 \le k < 2.4 \text{ sec} \\ 5 & , 2.4 \le k \le 3.0 \text{ sec} \end{cases},$$

$$d_2(k) = -1.2 \cos(3\pi k) - 1.0 \sin(\pi k) - 0.8 \cos(0.5\pi k)$$
$$- 0.6 \sin(0.33\pi k) - 0.4 \cos(0.25\pi k).$$

The initial condition and sampling time are $x_d(0) = \begin{bmatrix} 1.2 & 2.3 & -0.6 & 1.5 \end{bmatrix}^T$ and $T_s = 0.01$ sec, respectively. The desired output trajectory is given as

$$r_d(k) = \begin{cases} \begin{bmatrix} \cos(2\pi k) - 5 \\ 5.5 \cdot k^2 \cdot (1-k) + 3 \end{bmatrix}, & 0 \le k < 1 \text{ sec} \\ \begin{bmatrix} 1.8 \cdot k^2 \cdot (1-k) \\ \cos(3\pi k) \end{bmatrix}, & 1 \le k < 2 \text{ sec}. \\ \begin{bmatrix} 0.5 \cdot \cos(4\pi k) + 1 \\ 0.5 \cdot \sin(2\pi k) - 3 \end{bmatrix}, & 2 \le k \le 3 \text{ sec} \end{cases}$$

The objective is to design a state and disturbance estimator using proportional plus integral observer, and then apply (11) to have a proportional plus integral observer-based optimal LQDT such that the controlled system in (4) has a desired tracking performance.

First, for comparison, we assume the system state is measurable. Then, apply the traditional optimal LQDT for the weighting matrix pair $\{Q_o, R_o\} = \{10^6 I_6, I_2\}$ [7], [8]. Simulation results given in Fig. 2 show that the traditional optimal LQDT demonstrates a poor tracking performance due to the missing compensatory signal $C_d(k)$.

Next, we show the proposed approach for the originally specified system as follows.

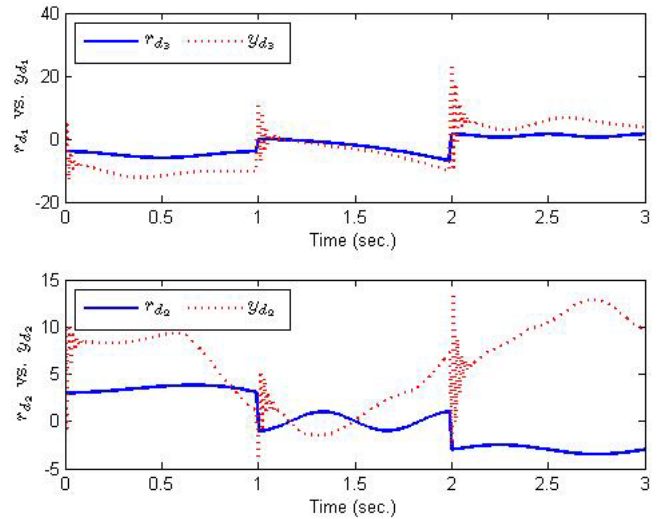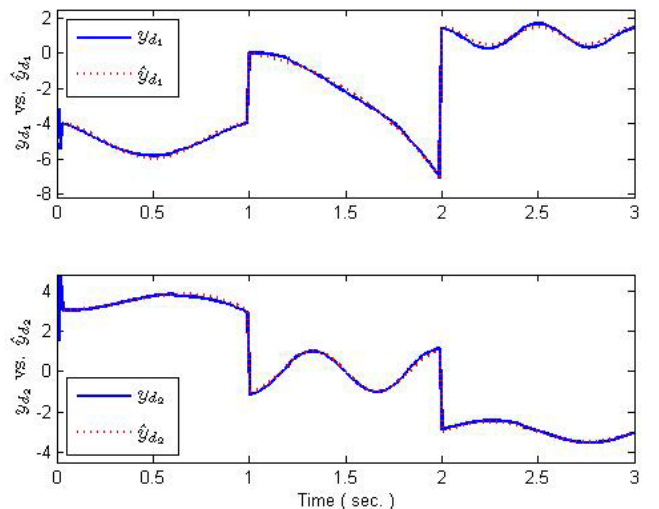Step 1: Construct the discrete-time PIO to estimate the state and disturbance.

Fig. 2 Tracking responses of the traditional optimal LQDT-based controlled system: $r_{di}(k)$ vs. $y_{di}(k)$, $i = 1, 2$
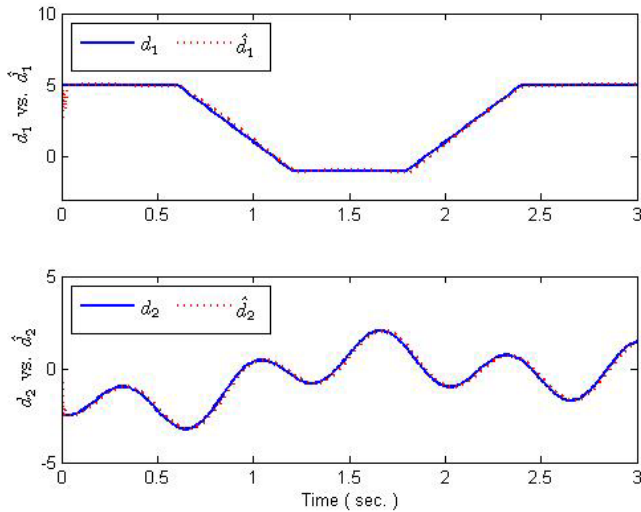
By selecting the radius $\alpha = 0.1$ and the weighting matrices $\{Q_o, R_o\} = \{10^6 I_6, I_2\}$, we obtain the PIO gain

$$L = \begin{bmatrix} K_P^T & \vdots & K_I^T \end{bmatrix}^T$$
$$= \begin{bmatrix} -0.3108 & 0.2352 & -0.6650 & 0.3879 & \vdots & -0.2107 & -0.4567 \\ -0.0350 & 0.2308 & -0.3346 & 1.0797 & \vdots & 0.8032 & -0.5511 \end{bmatrix}^T,$$
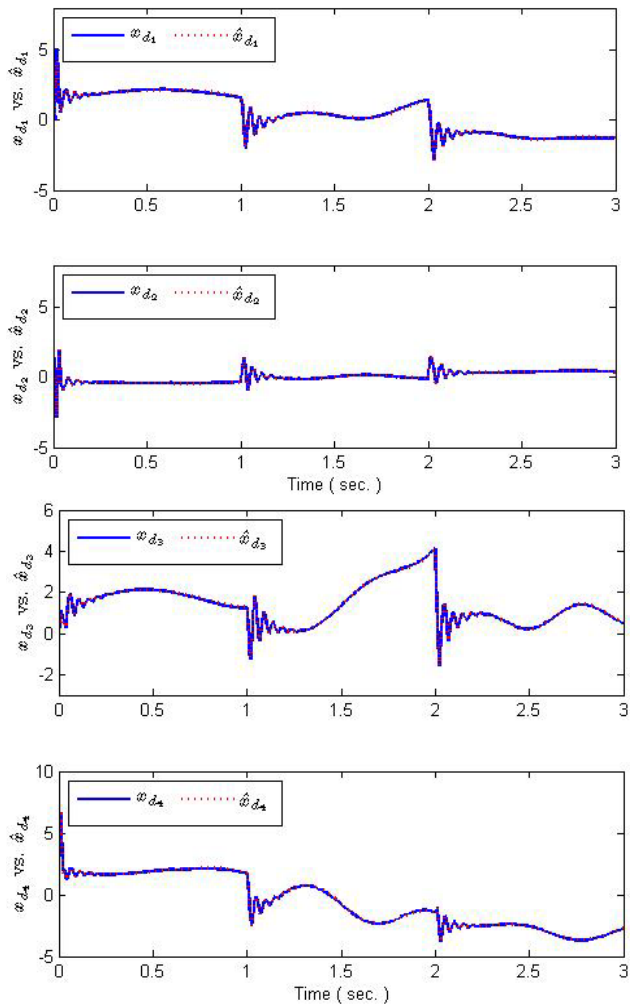
which results in closed-loop observer poles at $\{ 0.0000,$ $0.0000, -0.0082, 0.0080, 0.0056 \pm j0.0075 \}$. Under the initial conditions $\hat{x}_d(0) = 0_{4\times1}$ and $\hat{d}(0) = 0_{2\times1}$, Fig. 3 shows responses of the various estimations via the proposed disc-rete-time PIO. Obviously, the system state $x_d(k)$ and unknown external disturbance $d(k)$ are well estimated after $t \ge 0.04$ sec.

(a)

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(b)

(c)

Fig. 3 The closed-loop responses of the proportional plus integral observer: (a) $y_d(k)$ vs. $\hat{y}_d(k)$, (b) $d(k)$ vs. $\hat{d}(k)$, (c) $x_d(k)$ vs. $\hat{x}_d(k)$

Step 2: Design a PIO-based optimal LQDT with a high-gain property to have a desired tracking performance.

Then, choose the weighting matrix pair $\{ Q_d , R_d \} = \{ 10^6 I_2 , I_2 \}$, which has high-gain property, to have the PIO-based optimal LQDT in (11), where the state-feedback and forward gain matrices are

$$K_d = \begin{bmatrix} 1.5626 & 1.0909 & -0.5013 & -1.1816 \\ -0.0976 & 0.1417 & 0.6452 & 0.6862 \end{bmatrix},$$

$$E_d = \begin{bmatrix} 0.1887 & 1.4613 \\ -0.6282 & -0.3952 \end{bmatrix}, \quad Z_{\hat{d}} \approx -I_2.$$

The closed-loop responses shown in Fig. 4 demonstrate that the system output $y_d(k)$ well tracks the reference trajectory $r_d(k)$ after $t \geq 0.04$ sec.

(a)
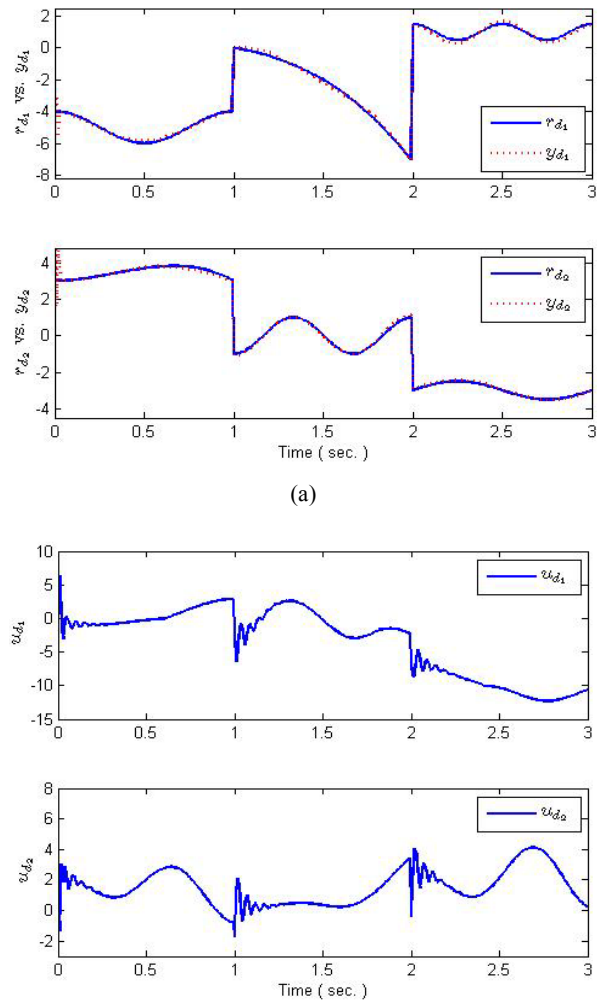
(b)

Fig. 4 The closed-loop responses of the system with an unknown disturbance: (a) tracking response $r_d(k)$ vs. $y_d(k)$, (b) control input $u_d(k)$

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V. CONCLUSION

In this paper, the generalized optimal linear quadratic digital tracker design for the proper system with a known disturbance has been extended to the system with an unknown disturbance. In the real world, there is usually an unknown external disturbance which occurs at the plant input, leading to a poor tracking performance. To overcome this issue, we construct a state and disturbance estimator using discrete-time proportional plus integral observer to estimate the system state and the unknown external disturbance. Then, by applying the generalized optimal LQDT design, we propose a PIO-based optimal LQDT with a high-gain property to have a desired tracking performance for a given arbitrary reference trajectory with some drastic variations. Finally, a numerical simulation is given to show the effectiveness of our proposed approach.

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