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Computing Maximum Uniquely Restricted Matchings in Restricted Interval Graphs

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Abstract—A *uniquely restricted matching* is defined to be a matching *M* whose matched vertices induces a sub-graph which has only one perfect matching. In this paper, we make progress on the open question of the status of this problem on *interval graphs* (graphs obtained as the intersection graph of intervals on a line). We give an algorithm to compute maximum cardinality uniquely restricted matchings on certain sub-classes of interval graphs. We consider two sub-classes of interval graphs, the former contained in the latter, and give $\mathcal{O}(|E|^2)$ time algorithms for both of them. It is to be noted that both sub-classes are *incomparable* to *proper interval graphs* (graphs obtained as the intersection graph of intervals in which no interval completely contains another interval), on which the problem can be solved in polynomial time.

Keywords—Uniquely restricted matching, interval graph, design and analysis of algorithms, matching, induced matching, witness counting.

I. Introduction

THIS work focuses on two objects: *Interval Graphs* and *Uniquely Restricted Matchings*. A graph is an interval graph, as defined in [3] and characterized in [2], if it can be obtained as the intersection graph of intervals on a line. Interval graphs are used to represent resource allocation problems in operations research and scheduling theory. In these applications, each interval represents a request for a resource (such as a processing unit of a distributed computing system or a room for a class) for a specific period of time.

A matching in a graph is a set of edges in which no two edges shares an end point. Matchings have been researched extensively for many years. Lovasz and Plummer [10] have written a book dedicated to matchings. A *uniquely restricted matching* is defined to be a matching M whose matched vertices induces a subgraph which has only one perfect matching.

Given a matrix A, let A' be a matrix obtained by rearranging rows and columns of A. The size of largest upper triangular sub-matrix with non-zeros on the diagonal among all matrices A' is shown to be a lower bound on the rank of A [7]. This problem of finding such a non-zero upper triangular matrix is formulated equivalently in [7]. Since this formulation, the problem of determining a uniquely restricted matching attracted the attention of researchers and was studied in depth in [4]. This work showed that the problem of computing a maximum uniquely restricted matching is NP-Complete for arbitrary, bipartite and split graphs while for proper interval, trees, cacti and threshold graphs it can be computed in linear time. Levit and Mandrescu [8] showed that unicycle graphs having only uniquely restricted maximum matchings can be recognized in polynomial time and presented some poly time algorithms for the same in [9]. The technique of mapping a matching in a bipartite graph to one in a digraph has been successfully used in the context of the forcing set problem. In [12], it has been extended to show that determining a uniquely restricted matching in a bipartite graph is equivalent to recognizing an acyclic digraph. Specifically, the work by Penso, Dieter and Souza [11] showed a characterisation of graphs in which maximum matching are uniquely restricted and proved that corresponding graphs can be recognized in polynomial time. Uniquely restricted matching include the so-called *strong matchings* or *induced matchings* of [1], [5], [6].

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A. Our Results

In [4], the authors posed the following problem – "Is it possible to determine a maximum uniquely restricted matching in a given interval graph?". In this work, we address special cases of this open problem. First, we give an $\mathcal{O}(|E|^2)$ time algorithm for the sub-class of interval graphs where all the structures given in Fig. 1 are not present as maximal sub-graphs. We then show an $\mathcal{O}(|E|^2)$ time algorithm which works even if only the third structure given in Fig. 1 is disallowed. It is to be noted that both sub-classes are *incomparable* to *proper interval graphs* (graphs obtained as the intersection graph of intervals in which no interval completely contains another), on which the problem can be solved in polynomial time [4]. The sub-classes considered are recognizable in polynomial time ($\mathcal{O}(|V|^5)$), just by a brute-force search.

B. Organization

In Section II, we present the preliminaries and notation required to describe the algorithms. We also formally define the sub-classes of interval graphs we will give algorithms for. In Section III, we present sub-routines used by the algorithm for one of the sub-classes and the formal proof of correctness of the algorithm, which itself is described in Section IV. In Sections V and VI, we provide further sub-routines and show how to modify the algorithm to work for the second sub-class and provide the proof of the same. In Section VII, we present our conclusions.

II. Preliminaries

A. Definitions

1) Interval Graph:

A graph is an interval graph, as defined in [3], if it can be obtained as the intersection graph of intervals on a line. It has one vertex for each interval and an edge between every pair of vertices corresponding to intersecting intervals. A graph is an interval graph if and only if it is chordal and its complement is a comparability graph [3]. Formally, given a set of intervals $\{I_i\}_{1\leq i\leq k}$, the corresponding interval graph is $G = (V, E)$, where $V = \{v_i \mid 1 \leq i \leq k\}$ and $E = \{(v_i, v_j) \mid I_i \cap I_j \neq \phi\}$.

If the underlying interval set $\{I_i\}_{1\leq i\leq k}$ satisfies the property that $I_i \not\subseteq I_j$ for all $i \neq j$, then the interval graph is said to be a *proper interval graph*.

2) Restricted Interval Graph:

We call an interval graph a *Type-I Restricted Interval graph* if it has no maximal sub-graphs*(induced sub-grahs)* isomorphic to the graphs *(a)* and *(b)* in Fig. 1, and a *Type-II Restricted Interval graph* if it has no maximal sub-graphs*(induced sub-grahs)* isomorphic to the third of the graphs in Fig. 1. It is to be noted that these sub-classes of interval graphs is incomparable with the class of graphs which may have such maximal sub-graphs*(induced sub-grahs)*, for there exist Type-I and Type-II restricted interval graphs which are not proper interval graphs.

3) Uniquely Restricted Matching:

Let $G = (V, E)$ be a graph. A set of edges $M \subseteq E$ is said be a *matching* if no two edges of M share a vertex. An *alternating cycle with respect to a matching M* is a set of edges which forms a cycle and alternate edges of the cycle belong to the matching. A matching M in a graph G is said to be *uniquely restricted* if there is no other matching of the same size on the vertices spanned by M. A uniquely restricted matching M in a graph G M is said to be a *maximum uniquely restricted matching* if there is no other uniquely restricted matching in G of larger size.

The following result is known.

Theorem 1: [4] Let G be an interval graph and M a matching on G. M is uniquely restricted if and only if there is no alternating cycle of length four with respect to M.

B. Notation

For any $n \in \mathbb{N}$, $[n] = \{1, \ldots, n\}$. For any (ordered) set S, the i'th element is accessed as $S[i]$. We denote by $I_i = (\ell_i, r_i)$ a real interval indexed by i with left and right end points ℓ_i and r_i respectively, $\ell_i \leq r_i$, $\ell_i, r_i \in \mathbb{R}$. For any two intervals I_i and I_j, with [1] $i < j$, we define the ***segment*** $I_{i,j} = I_i \cap I_j$. Testing whether two intervals have a non-empty intersection can be done as:

$$I_{i,j} \neq \phi \iff (\ell_i - r_j \leq 0) \wedge (\ell_j - r_i \leq 0) \tag{1}$$

Let $r_{i,j}$ (redp. $\ell_{i,j}$) denote the right (resp. left) end point of the segment $I_{i,j}$, if [2] $I_{i,j} \neq \phi$, that is, $r_{i,j} = \max_{x\in I_{i,j}} x$ $\left(\text{resp. } \ell_{i,j} = \min_{x\in I_{i,j}} x\right)$. When $I_{i,j} \neq \phi$,

$$r_{i,j} = \min\{r_i, r_j\} \tag{2}$$

$$\ell_{i,j} = \max\{\ell_i, \ell_j\} \tag{3}$$

Consider an interval graph $G = (V, E)$. Without loss of generality, let $V = \{v_1, \ldots, v_n\}$ and $E = \{e_{i,j}\}_{i<j,i,j\in[n]}$, where we [3] associate (implicitly) vertex $v_i \in V$ with an interval I_i and edge $e_{i,j} = (v_i, v_j)$ with the segment $I_{i,j}$. The edges in E are indexed by k, that is, $E = \left\{e^k\right\}_{1\leq k\leq|E|}$. We have that for each $k \in [|E|]$, $e^k = e_{i(k),j(k)}$ for some $i(k)$ and $j(k)$.

C. Intuition

The main idea is to try a block-wise sweep of the graph and decide to take edges or drop them as we go. The crucial element is the ordering we impose on the edges. Section III presents an ordering which suffices to deal with Type-I restricted interval graphs. In Section IV, we refine the ordering to work for Type-II restricted interval graphs. It is also a question as to whether the block-wise sweep would work for the entire class of interval graphs.

III. Sub-Routines Used by the Algorithm for Type-I Restricted Interval Graphs

A. Compare

Algorithm 1 returns **true** if the first tuple is lexicographically lower than the second. Clearly, Algorithm 1 runs in time $\mathcal{O}(1)$.

Algorithm 1 Algorithm to compare tuples

```
procedure COMPARE((i, j), (i', j'))
    if i < i' then
        return true
    if (i = i') ∧ (j < j') then
        return true
    return false
```

B. Pre-Processing the Edge Set

Algorithm 2 processes the graph $G = (V, E)$ to obtain a processed edge set E^*. The processing sets up, for each edge, a tuple containing the right end points of the intervals whose segment is the edge and the edge and interval indices. Formally, for $k \in [|E|]$, corresponding to the edge $e^k \in E$, which, by the definition $e^k = e_{i(k),j(k)}$, is an edge between the vertices $v_{i(k)}$ and $v_{j(k)}$ (which correspond to intervals $I_{i(k)}$ and $I_{j(k)}$ respectively), we create the tuple $\hat{e}^k = \left(r_{i(k)}, r_{j(k)}, k, i(k), j(k)\right)$. The reason for the choice of those components in the processed edge set will be clear from the ordering defined in Section V-C – some of these components are not necessary for the algorithm for Type-I Restricted Interval Graphs, however, we are re-using this

[1] This can be assumed without loss of generality to ensure a unique nomenclature for an interval. In general, $I_{\min\{i,j\},\max\{i,j\}} = I_i \cap I_j = I_j \cap I_i$.

[2] We can assume that $r_{i,j}$ and $\ell_{i,j}$ is undefined when $I_{i,j} = \phi$.

Algorithm 2 Algorithm for Pre-processing the Edge set

procedure PRE-PROCESS($G = (V, E)$)
  $E^* = \phi$ ▷ E^* denotes the processed set of edges in G
  for k **in** $[|E|]$ **do**
    $\hat{e}^k = \left(r_{i(k)}, r_{j(k)}, k, i(k), j(k)\right)$
    $E^* = E^* \cup \left\{\hat{e}^k\right\}$
  return E^*

sub-routine and hence include all necessary components here itself.

Clearly, Algorithm 2 runs in time $\mathcal{O}(|V| + |E|)$, since this can be done alongside the recognition algorithm for the interval graph G.

C. Type-I Ordering the Edge Set

We induce an ordering on E in the following way. We define for $e_{i,j}, e_{i',j'} \in E$, $e_{i,j} \neq e_{i',j'}$,

$$e_{i,j} \prec e_{i',j'} \iff r_{i,j} < r_{i',j'} \vee (r_{i,j} = r_{i',j'} \wedge (i,j) < (i',j')) \quad (4)$$

Note that this ordering is well defined and defined for every pair of edges in E. Also, it is easy to see that $e_{i,j} \prec e_{i',j'} \iff e_{i',j'} \not\prec e_{i,j}$.

We describe an algorithm to set up such a total order on E. We order the edges of the graph by working on the processed set E^*. Algorithm 3 mimics the simple sequential sort algorithm while using the order relation defined by (4). It generates a sorted processed edge list L. For a tuple element $\hat{e}^k = \left(r_{i(k)}, r_{j(k)}, k, i(k), j(k)\right)$ of E^*, we use the following notation to access individual components: $\hat{e}^k[1] = r_{i(k)}$, $\hat{e}^k[2] = r_{j(k)}$, $\hat{e}^k[3] = k$, $\hat{e}^k[4] = i(k)$ and $\hat{e}^k[5] = j(k)$. The list L is indexed in a similar manner. Clearly, Algorithm 3 sets up a total order on E as described before. The correctness of the algorithm follows in a straight-forward manner from (4). Also, Algorithm 3 runs in time $\mathcal{O}\left(|E|^2\right)$.

D. Growing a Matching

Algorithm 4 checks whether an edge can be added to a matching so that the resulting set of edges is also a matching. The algorithm simply checks whether any of the edges already present share a vertex with the edge to be newly added. This algorithm assumes all edges to be provided in processed form. Clearly, Algorithm 4 runs in time $\mathcal{O}(|M^*|)$.

E. Check for Length Four Alternating Cycles

Algorithm 5 checks whether an edge can be added to a matching so that the resulting set of edges does not contain any length four alternating cycle with respect to the augmented set of edges. The algorithm simply checks whether any of the edges already present share can form such a cycle with the edge to be newly added. This algorithm assumes all edges to be provided in processed form. Clearly, Algorithm 5 runs in time $\mathcal{O}(|M^*|)$, if testing the membership of an edge in E can be done in $\mathcal{O}(1)$ time. This is possible since one can just set up a $|V| \times |V|$ binary indicator matrix for the edge set of the graph in $\mathcal{O}\left(|E|^2\right)$ time while pre-processing and look it up while testing for membership. We omit such details for the ease of exposition.

[3]Every interval graph has an equivalent interval representation which can be obtained in linear time as described in [4]. The intervals which we associate can be assumed to be the canonical intervals output by the algorithm.

Algorithm 3 Algorithm for Type-I Ordering the Pre-processed Edge set

procedure ORDER$^I(E^*)$
  $L = \phi$ ▷ L denotes the ordered set of edges in G
  for k **in** $[|E^*|]$ **do**
    $L[k] = \hat{e}^k$
  for k_1 **in** $[|E^*| - 1]$ **do**
    $\hat{e}^{k_1} = L[k_1]$
    for k_2 **in** $\{i+1, \dots, |E^*|\}$ **do**
      $\hat{e}^{k_2} = L[k_2]$
      if $\min\left\{\hat{e}^{k_1}[1], \hat{e}^{k_1}[2]\right\} > \min\left\{\hat{e}^{k_2}[1], \hat{e}^{k_2}[2]\right\}$ **then**
        swap $L[k_1]$ **and** $L[k_2]$ ▷ from (4)
      if $\min\left\{\hat{e}^{k_1}[1], \hat{e}^{k_1}[2]\right\} = \min\left\{\hat{e}^{k_2}[1], \hat{e}^{k_2}[2]\right\}$ **then**
        if COMPARE$((\hat{e}^{k_1}[4], \hat{e}^{k_1}[5]), (\hat{e}^{k_2}[4], \hat{e}^{k_2}[5]))$ **= false then**
          swap $L[k_1]$ **and** $L[k_2]$ ▷ from (4)
  return L ▷ L is the ordered set of edges of the graph

Algorithm 4 Algorithm to check for a matching

procedure IS-MATCHING($M^*, \hat{e}^k$)
  for k_1 **in** $[|M^*|]$ **do**
    $\hat{e}^{k_1} = M^*[k_1]$
    if $\hat{e}^{k_1}[4] = \hat{e}^k[4] \vee \hat{e}^{k_1}[4] = \hat{e}^k[5] \vee \hat{e}^{k_1}[5] = \hat{e}^k[4] \vee \hat{e}^{k_1}[5] = \hat{e}^k[5]$ **then**
      return false
  return true

IV. THE ALGORITHM FOR TYPE-I RESTRICTED INTERVAL GRAPHS

Given a Type-I restricted interval graph $G = (V, E)$, we induce an ordering on E as in (4). We now discuss the algorithm, given as Algorithm 6, to compute a maximum uniquely restricted matching in a restricted interval graph. Clearly M returned by Algorithm 6 is a matching. And by Theorem 1, M returned by Algorithm 6 is a uniquely restricted matching. Clearly, M is also a maximal uniquely restricted matching. We only have to show that M is maximum. We prove this as follows. Suppose that there is a uniquely restricted matching M' with $|M'| > |M|$. Then, we know that

$$M' = M \setminus (M - M') \cup (M' - M)$$

that is, we can obtain M' from M by removing certain edges (those in $M - M'$) and adding others (those in $M' - M$). Clearly, in this transformation, no removed edge is added back. Also, since $|M'| > |M|$, we must add more vertices than we

Algorithm 5 Algorithm to check for length four alternating cycles

procedure No-4-Cycle($E, M^*, \hat{e}^k$)
  for k_1 **in** $[|M^*|]$ **do**
    $\hat{e}^{k_1} = M^*[k_1]$
    $i = \hat{e}^{k_1}[4], j = \hat{e}^{k_1}[5]$
    $i' = \hat{e}^k[4], j' = \hat{e}^k[5]$
    if $(e_{i,i'} \in E \wedge e_{j,j'} \in E) \vee (e_{i,j'} \in E \wedge e_{j,i'} \in E)$ **then**
      return false
  return true

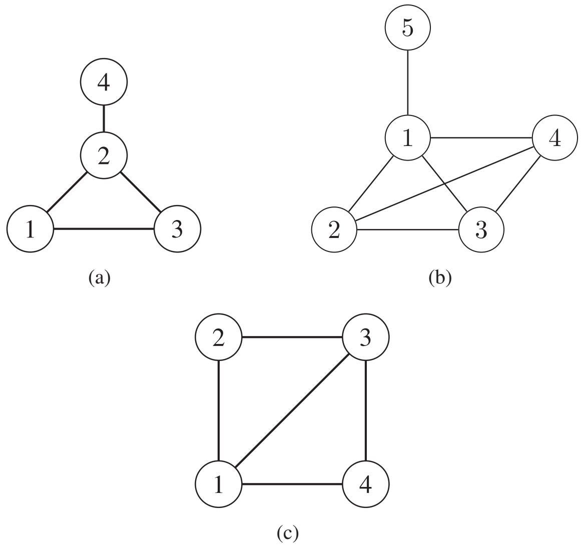


Fig. 1 Maximal subgraphs not present in the restricted interval graphs under consideration

have removed, that is, $|M - M'| < |M' - M|$. We claim that this is not possible. In fact, we claim that for any ℓ, it is not possible to remove ℓ edges and add $(\ell + 1)$ (different) edges to obtain a new uniquely restricted matching in order to prove the correctness of the algorithm.

Lemma 1: Suppose M is a uniquely restricted matching and $M' \subseteq M$. Then M' is also a uniquely restricted matching.

Proof: Suppose to the contrary that M' is not a uniquely restricted matching. By Theorem 1, G has an alternating cycle of length four with respect to M'. The same cycle would be present with respect to M as well since $M' \subseteq M$, which means that M is not a uniquely restricted matching, by Theorem 1, a contradiction. ■

Lemma 2: Suppose M is a matching, $e_{i,j}, e_{i',j'} \in M$, each of i, j, i', j' are distinct. Then, $I_{i,j} \cap I_{i',j'} \neq \phi$ if and only if the sub-graph induced by i, j, i', j' is isomorphic to K_4, the complete graph on four vertices if and only if there is an alternating cycle of length four with respect to M in the sub-graph induced by i, j, i', j'.

Proof: Consider the sub-graph induced by the vertices i, j, i', j'. Now, $I_{i,j} \cap I_{i',j'} = I_i \cap I_j \cap I_{i'} \cap I_{j'} \neq \phi$. This is true if and only if every pair of intervals overlap and the sub-graph is isomorphic to K_4. As M is a matching, every

Algorithm 6 Algorithm for a Maximum Uniquely Restricted Matching in Type-I Restricted Interval Graphs

procedure MaxURM($G = (V, E)$)
  $E^* = $ Pre-process(G)
  $L = $ Order^I (E^*)
  $\mathsf{ct} = 1$ ▷ ct denotes the number of edges in our matching
  $M^*[\mathsf{ct}] = L[1]$ ▷ M^* denotes the list of processed edges in our matching
  for k **in** $\{2, \ldots, |E^*|\}$ **do** ▷ k denotes the index of the currently inspected edge
    if Is-Matching $(M^*, L[k]) =$ **true then**
      if No-4-Cycle $(E, M^*, L[k]) =$ **true then**
        $\mathsf{ct} = \mathsf{ct} + 1$
        $M^*[\mathsf{ct}] = L[k]$
  $M = \phi$▷ M denotes the set of edges in our matching
  for k **in** $[|M^*|]$ **do**
    $\hat{e}^k = M^*[k]$
    $\hat{k} = \hat{e}^k[3]$
    $M = M \cup \{e^{\hat{k}}\}$
  return M ▷ M is a maximum uniquely restricted matching

edge in this sub-graph other than $e_{i,j}, e_{i',j'}$ do not belong to M and equivalently there is an alternating cycle of length four with respect to M in the sub-graph induced by i, j, i', j'. ■

Lemma 3: Let e, e', e'' be edges in the interval graph G satisfying $e \prec e' \prec e''$ and e'' forms an alternating cycle of length four with e. Then, e'' also forms an alternating cycle of length four with e'.

Proof: Suppose $e = e_{i,j}$, $e' = e_{i',j'}$ and $e'' = e_{i'',j''}$. By Lemma 2, $I_{i,j} \cap I_{i'',j''} \neq \phi$. Since $e \prec e' \prec e''$, $I_{i',j'} \cap I_{i'',j''} \neq \phi$ and by Lemma 2, e'' forms an alternating cycle of length four with e'. ■

Lemma 4: Suppose M was returned by Algorithm 6, and for $\ell \leq |M|$, $e^{1,0} \prec \ldots \prec e^{\ell,0} \in M$ and there exist $e^{1,1} \prec \ldots \prec e^{\ell+1,1} \notin M$. Then, $M \setminus \{e^{i,0}\}_{1 \leq i \leq \ell} \cup \{e^{i,1}\}_{1 \leq i \leq \ell+1}$ is not a uniquely restricted matching.

Proof: Suppose to the contrary that $M \setminus \{e^{i,0}\}_{1 \leq i \leq \ell} \cup \{e^{i,1}\}_{1 \leq i \leq \ell+1}$ is a uniquely restricted matching. Consider the edges $e^{1,1} \prec \ldots \prec e^{\ell+1,1} \notin M$. Each of them was dropped either because adding them meant that we no longer had a matching or because an alternating cycle of length four was formed. Note that we check for the matching invariant first, followed by checking for length four alternating cycles. Let $S_1 \subseteq \{e^{i,1}\}_{1 \leq i \leq \ell+1}$ denote the set of edges which were dropped because adding them meant we no longer had a matching and let $S_2 = \{e^{i,1}\}_{1 \leq i \leq \ell+1} \setminus S_1$ denote the set of edges which were dropped because adding them created a length four alternating cycle (they do not violate the matching invariant, however). Let $e^{i,1} = e_{j_i,k_i}$ for each i.

We define the notion of a *witness* for an edge in $\{e^{i,1}\}_{1 \leq i \leq \ell+1}$ as for all $1 \leq i \leq \ell + 1$, $e \in M$ is the *witness* for $e^{i,1}$ being dropped if $e^{i,1}$ was dropped as it shared a vertex with edge $e \in M$ and hence adding it meant that we no longer had a matching, or $e^{i,1}$ was dropped as it formed a length four alternating cycle with $e \in M$.

Let $w^i \in M$ denote the witness for $e^{i,1}$. We prove ahead that each of the witnesses must be distinct. Since we are removing only ℓ edges from M, *viz.*, $\{e^{i,0}\}_{1\leq i\leq \ell}$, at least one witness remains and the presence of a witness will either violate the matching invariant or induce a length four alternating cycle which means that $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a uniquely restricted matching. Hence, it only remains to show that for all $i \neq i'$, $w^i \neq w^{i'}$. Let $w^i = e_{j_{w^i},k_{w^i}}$ for each i.

a) Case 1: Consider $e^{i,1}, e^{i',1} \in S_1$. Without loss of generality, assume $e^{i,1} \prec e^{i',1}$. Suppose $w^i = w^{i'} = w = e_{j_w,k_w}$. If $e^{i,1}, e^{i',1}$ share a vertex, clearly $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a matching and hence not a uniquely restricted matching, which would be a contradiction. Hence, $e^{i,1}, e^{i',1}$ do not share any vertex. Hence, $j_i, k_i, j_{i'}, k_{i'}$ are all distinct. If $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} \neq \phi$, by Lemma 2, there is an alternating cycle of length four with respect to $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ and hence $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a uniquely restricted matching, which would be a contradiction. Hence, $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} = \phi$. By Lemma 2, the sub-graph induced by $j_i, k_i, j_{i'}, k_{i'}$ is not isomorphic to K_4. The sub-graph contains the edges $(j_i, k_i), (j_{i'}, k_{i'})$. We also know that the sub-graph also contains the vertices j_w, k_w. Let $p \in \{j_w, k_w\}$ be the vertex shared by $w, e^{i',1}$. Then, it is easy to see that $I_p \cap I_{j_i} \neq \phi$ and $I_p \cap I_{k_i} \neq \phi$. This would mean that the sub-graph is isomorphic to one of the graphs in Fig. 1, which cannot be the case. Hence, $w^i \neq w^{i'}$.

b) Case 2: Consider $e^{i,1}, e^{i',1} \in S_2$. Suppose $w^i = w^{i'} = w = e_{j_w,k_w}$. Note that $e^{i,1}, e^{i',1}$ did not violate the matching invariant and hence they share no vertex with any of the edges in M. Hence, $e^{i,1}, w$ share no vertices and $e^{i',1}, w$ share no vertices. Let $M_i = M \cup \{e^{i,1}\}$. We know that M_i is a matching and that there is an alternating cycle of length four in the sub-graph spanned by the vertices j_i, k_i, j_w, k_w, which are all distinct. By Lemma 2, $I_{j_i,k_i} \cap I_{j_w,k_w} \neq \phi$. On a similar account, $I_{j_{i'},k_{i'}} \cap I_{j_w,k_w} \neq \phi$. Combining, we have $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} \neq \phi$. By Lemma 2 applied on $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$, we obtain that there is a length four alternating cycle on the sub-graph induced by the vertices j_i, k_i, j_w, k_w, which by Theorem 1 implies that $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a uniquely restricted matching, a contradiction. Hence, $w^i \neq w^{i'}$.

c) Case 3: Consider $e^{i,1} \in S_1, e^{i',1} \in S_2$. Suppose $w^i = w^{i'} = w = e_{j_w,k_w}$. If $e^{i,1}, e^{i',1}$ share a vertex, clearly $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a matching and hence, not a uniquely restricted matching, which would be a contradiction. Hence, $e^{i,1}, e^{i',1}$ do not share any vertex. Hence, $j_i, k_i, j_{i'}, k_{i'}$ are all distinct. If $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} \neq \phi$, by Lemma 2, there is an alternating cycle of length four with respect to $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ and hence $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a uniquely restricted matching, which would be a contradiction. Hence, $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} = \phi$. By Lemma 2, the sub-graph induced by $j_i, k_i, j_{i'}, k_{i'}$ is not isomorphic to K_4. The sub-graph contains the edges $(j_i, k_i), (j_{i'}, k_{i'})$. We also know that the sub-graph also contains the one of the vertices j_w, k_w. Let $p \in \{j_w, k_w\}$ be the vertex shared by $w, e^{i',1}$. Then, it is easy to see that $I_p \cap I_{j_i} \neq \phi$ and $I_p \cap I_{k_i} \neq \phi$. This would mean that the sub-graph is isomorphic to one of the graphs in Fig. 1, which cannot be the case. Hence, $w^i \neq w^{i'}$.

d) Case 4: Consider $e^{i,1} \in S_2, e^{i',1} \in S_1$. Suppose $w^i = w^{i'} = w = e_{j_w,k_w}$. If $e^{i,1}, e^{i',1}$ share a vertex, clearly $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a matching and hence not a uniquely restricted matching, which would be a contradiction. Hence, $e^{i,1}, e^{i',1}$ do not share any vertex. Hence, $j_i, k_i, j_{i'}, k_{i'}$ are all distinct. If $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} \neq \phi$, by Lemma 2, there is an alternating cycle of length four with respect to $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ and hence $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a uniquely restricted matching, which would be a contradiction. Hence, $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} = \phi$. By Lemma 2, the sub-graph induced by $j_i, k_i, j_{i'}, k_{i'}$ is not isomorphic to K_4. The sub-graph contains the edges $(j_i, k_i), (j_{i'}, k_{i'})$. We also know that the sub-graph also contains the one of the vertices j_w, k_w. Let $p \in \{j_w, k_w\}$ be the vertex shared by $w, e^{i',1}$ and let $u \in \{j_w, k_w\}$ with $p \neq u$. Then, it is easy to see that $I_u \cap I_{j_i} \neq \phi$ and $I_u \cap I_{k_i} \neq \phi$. This would mean that the sub-graph is isomorphic to the second graph in Fig. 1, which cannot be the case. Hence, $w^i \neq w^{i'}$.

This exhausts all the cases and shows that for all $i \neq i'$, $w^i \neq w^{i'}$, completing the proof. ■

Theorem 2: Let G be a restricted interval graph. M output by Algorithm 6 is a maximum uniquely restricted matching of G.

Proof: Suppose to the contrary that there is a uniquely restricted matching M' with $|M'| > |M|$. Then, we know that $M' = M \setminus (M - M') \cup (M' - M)$. Also, since $|M'| > |M|$, $|M - M'| < |M' - M|$. Let $|M - M'| = \ell \leq |M|$ and let X be a subset of $(M' - M)$ such that $|X| = \ell + 1$. Since M' is a uniquely restricted matching, by Lemma 1, so is $M'' = M \setminus (M - M') \cup X$. However, by Lemma 4, M'' is not a uniquely restricted matching, which is a contradiction. ■

Theorem 2 proves the correctness of Algorithm 6. The time complexity is analyzed as follows. The initial ordering of the edges can be obtained in $\mathcal{O}(|E|^2)$ time. Each iteration of the **while** loop can be performed in $\mathcal{O}(|E|)$ time (the detection of the matching invariant as well as the alternating cycle of length four must only be done for the newly added edge with each of the existing edges). Hence, the algorithm runs in time $\mathcal{O}\left(|E|^2\right)$, which is polynomial time.

V. More Sub-Routines Used by the Algorithm for Type-II Restricted Interval Graphs

A. Growing Intervals

Algorithm 7 takes a set of intervals and produces an equivalent set of intervals (in terms of their intersection graphs) such that every interval is as long as it can be, that is, if it were extended in either direction, the intersection graph defined by the intervals would change. Clearly, Algorithm 7 runs in time $\mathcal{O}(n^2)$.

B. Single-Argmax

Algorithm 8 returns the index of *an* element of a collection which is the maximum. We assume the set S can be indexed.

Clearly, Algorithm 8 runs in time $\mathcal{O}(|S|)$.

Algorithm 7 Algorithm to Elongate Intervals

procedure ELONGATE(V) ▷ $V = \{v_i\}_{i=1}^n = \{I_i\}_{i=1}^n = \{(\ell_i, r_i)\}_{i=1}^n$
  for k **in** $[n-1]$ **do**
    for k' **in** $\{k+1, \dots, n\}$ **do**
      if $r_k > r_{k'}$ **then**
        swap I_k **and** $I_{k'}$ ▷ Sort the intervals in order of their right-end points
  for k **from** $n-1$ **to** 1 **do**
    $\mathsf{min} = r_n$ ▷ min denotes minimum left-end point a later interval would have
    for k' **in** $\{k+1, \dots, n\}$ **do**
      if $(\ell_{k'} < \mathsf{min}) \wedge (\ell_{k'} > r_k)$ **then**
        $\mathsf{min} = \ell_{k'}$
    if $r_k < \mathsf{min}$ **then**
      $r_k = \mathsf{min} - \epsilon$ ▷ ϵ is an extremely small positive parameter, that is $\epsilon \to 0$
  ▷ Repeat the entire procedure viewing left as right and vice-versa
  for k **in** $[n-1]$ **do**
    for k' **in** $\{k+1, \dots, n\}$ **do**
      if $\ell_k < \ell_{k'}$ **then**
        swap I_k **and** $I_{k'}$
  for k **from** $n-1$ **to** 1 **do**
    $\mathsf{max} = \ell_n$ ▷ max denotes maximum right-end point a previous interval would have
    for k' **in** $\{k+1, \dots, n\}$ **do**
      if $(r_{k'} > \mathsf{max}) \wedge (r_{k'} < \ell_k)$ **then**
        $\mathsf{max} = r_{k'}$
    if $\ell_k > \mathsf{max}$ **then**
      $\ell_k = \mathsf{max} + \epsilon$
  return $\{I_i\}_{i=1}^n$

Algorithm 8 Algorithm to compute a Single-argmax

procedure SINGLE-ARGMAX(S)
  $\mathsf{max} = 1$ ▷ max denotes *an* index of S which contains the maximum element
  for k **in** $\{2, \dots, |S|\}$ **do**
    if $S[k] > S[\mathsf{max}]$ **then**
      $\mathsf{max} = k$
  return max

C. Type-II Ordering the Edge Set

We induce a partial ordering on E in the following way. We define for $e_{i,j}, e_{i',j'} \in E$, $e_{i,j} \neq e_{i',j'}$,

$$e_{i,j} \prec e_{i',j'} \Leftarrow r_{i,j} < r_{i',j'} \tag{5}$$

To set up a total order on E, we break ties in the following way. Consider two edges corresponding to segments having the same right end point. If one of the intervals involved has a unique farthest right end point, then the segment constructed from it is considered to occur later in the ordering. Formally, $e_{i,j} \prec e_{i',j'} \Leftarrow r_{i,j} =$

$$r_{i',j'} \wedge \left(\{i,j\} \cap \underset{i,j,i',j'}{\arg\max} \{r_i, r_j, r_{i'}, r_{j'}\} = \phi \right) \tag{6}$$

Any further ties are broken arbitrarily. In this way, we extend the partial order to a total order, $\prec$, on E, ensuring that $e_{i,j} \prec e_{i',j'} \iff e_{i',j'} \not\prec e_{i,j}$.

We describe an algorithm to set up such a total order on E. We order the edges of the graph by working on the processed set E^*. Algorithm 9 mimics the simple sequential sort algorithm while using the order relation defined by (5) and (6). It generates a sorted processed edge list L. For a tuple element $\hat{e}^k = \left(r_{i(k)}, r_{j(k)}, k, i(k), j(k)\right)$ of E^*, we use the following notation to access individual components: $\hat{e}^k[1] = r_{i(k)}$, $\hat{e}^k[2] = r_{j(k)}$, $\hat{e}^k[3] = k$, $\hat{e}^k[4] = i(k)$ and $\hat{e}^k[5] = j(k)$. The list L is indexed in a similar manner.

Algorithm 9 Algorithm for Type-II Ordering the Pre-processed Edge set

procedure ORDER^{II}(E^*)
  $L = \phi$ ▷ L denotes the ordered set of edges in G
  for k **in** $[|E^*|]$ **do**
    $L[k] = \hat{e}^k$
  for k_1 **in** $[|E^*| - 1]$ **do**
    $\hat{e}^{k_1} = L[k_1]$
    for k_2 **in** $\{i+1, \dots, |E^*|\}$ **do**
      $\hat{e}^{k_2} = L[k_2]$
      if $\min\left\{\hat{e}^{k_1}[1], \hat{e}^{k_1}[2]\right\} > \min\left\{\hat{e}^{k_2}[1], \hat{e}^{k_2}[2]\right\}$ **then**
        swap $L[k_1]$ **and** $L[k_2]$ ▷ from (5)
      if $\min\left\{\hat{e}^{k_1}[1], \hat{e}^{k_1}[2]\right\} = \min\left\{\hat{e}^{k_2}[1], \hat{e}^{k_2}[2]\right\}$ **then**
        $\mathsf{maxright} =$ SINGLE-ARGMAX $\left(\left\{\hat{e}^{k_1}[1], \hat{e}^{k_1}[2], \hat{e}^{k_2}[1], \hat{e}^{k_2}[2]\right\}\right)$
        if $\mathsf{maxright} \leq 2$ **then**
          swap $L[k_1]$ **and** $L[k_2]$ ▷ from (6)
  return L ▷ L is the ordered set of edges of the graph

Clearly, Algorithm 9 sets up a total order on E as described before. The correctness of the algorithm follows in a straight-forward manner from (5) and (6). Also, Algorithm 9 runs in time $\mathcal{O}\left(|E|^2\right)$.

VI. The Algorithm for Type-II Restricted Interval Graphs

We now discuss the algorithm, given as Algorithm 10 , to compute a maximum uniquely restricted matching in a restricted interval graph. As before, M is also a maximal uniquely restricted matching. We only have to show that M is maximum. We claim that for any ℓ, it is not possible to remove ℓ edges and add $(\ell+1)$ (different) edges to obtain a new uniquely restricted matching in order to prove the correctness of the algorithm.

Lemma 5: Suppose M was returned by Algorithm 10, and for $\ell \leq |M|$, $e^{1,0} \prec \ldots \prec e^{\ell,0} \in M$ and there exist $e^{1,1} \prec \ldots \prec e^{\ell+1,1} \notin M$. Then, $M \setminus \{e^{i,0}\}_{1 \leq i \leq \ell} \cup \{e^{i,1}\}_{1 \leq i \leq \ell+1}$ is not a uniquely restricted matching.

Algorithm 10 Algorithm for a Maximum Uniquely Restricted Matching in Type-II Restricted Interval Graphs

```
procedure MaxURM(G = (V, E))
    V* = Elongate(V)
    E* = Pre-process(G* = (V*, E))
    L = Order^II (E*)
    ct = 1          ▷ ct denotes the number of edges in our matching
    M*[ct] = L[1]     ▷ M* denotes the list of processed edges in our matching
    for k in {2, ..., |E*|} do    ▷ k denotes the index of the currently inspected edge
        if Is-Matching (M*, L[k]) = true then
            if No-4-Cycle (E, M*, L[k]) = true then
                ct = ct + 1
                M*[ct] = L[k]
    M = φ▷ M denotes the set of edges in our matching
    for k in [|M*|] do
        ê^k = M* [k]
        k̂ = ê^k[3]
        M = M ∪ {e^k̂}
    return M       ▷ M is a maximum uniquely restricted matching
```

Proof: Suppose to the contrary that $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is a uniquely restricted matching. Consider the edges $e^{1,1} \prec \ldots \prec e^{\ell+1,1} \notin M$. Each of them was dropped either because adding them meant that we no longer had a matching or because an alternating cycle of length four was formed. Note that we check for the matching invariant first, followed by checking for length four alternating cycles. Let $S_1 \subseteq \{e^{i,1}\}_{1\leq i\leq \ell+1}$ denote the set of edges which were dropped because adding them meant we no longer had a matching and let $S_2 = \{e^{i,1}\}_{1\leq i\leq \ell+1} \setminus S_1$ denote the set of edges which were dropped because adding them created a length four alternating cycle (they do not violate the matching invariant, however). Let $e^{i,1} = e_{j_i,k_i}$ for each i.

We define the notion of a *witness* for an edge in $\{e^{i,1}\}_{1\leq i\leq \ell+1}$ as for all $1 \leq i \leq \ell+1$, $e \in M$ is the *witness* for $e^{i,1}$ being dropped if $e^{i,1}$ was dropped as it shared a vertex with a edge $e \in M$ and hence adding it meant that we no longer had a matching, or $e^{i,1}$ was dropped as it formed a length four alternating cycle with $e \in M$.

Let $w^i \in M$ denotes the witness for $e^{i,1}$. We prove ahead that each of the witnesses must be distinct. Since we are removing only ℓ edges from M, *viz.*, $\{e^{i,0}\}_{1\leq i\leq \ell}$, at least one witness remains and the presence of a witness will either violate the matching invariant or induce a length four alternating cycle which means that $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a uniquely restricted matching. Hence, it only remains to show that for all $i \neq i'$, $w^i \neq w^{i'}$. Let $w^i = e_{j_{w^i},k_{w^i}}$ for each i. ***We stress here that it is the <u>witness</u> seen by the calls to*** Is-Matching ***and*** No-4-Cycle***, and in that order, and the witness being found by inspecting the ordered set of matched edges.***

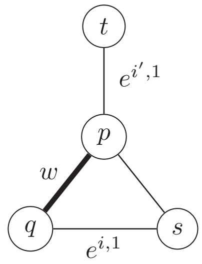


Fig. 2 Sub-graph induced by $j_i, k_i, j_{i'}, k_{i'}$ in Case 1

a) Case 1: Consider $e^{i,1}, e^{i',1} \in S_1$. Without loss of generality, assume $e^{i,1} \prec e^{i',1}$. Suppose $w^i = w^{i'} = w = e_{j_w,k_w}$. Then, we know that $w \prec e^{i,1} \prec e^{i',1}$. If $e^{i,1}, e^{i',1}$ share a vertex, clearly $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a matching and hence not a uniquely restricted matching, which would be a contradiction. Hence, $e^{i,1}, e^{i',1}$ do not share any vertex. Hence, $j_i, k_i, j_{i'}, k_{i'}$ are all distinct. If $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} \neq \phi$, by Lemma 2, there is an alternating cycle of length four with respect to $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ and hence $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a uniquely restricted matching, which would be a contradiction. Hence, $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} = \phi$. By Lemma 2, the sub-graph induced by $j_i, k_i, j_{i'}, k_{i'}$ is not isomorphic to K_4. The sub-graph contains the edges $(j_i, k_i), (j_{i'}, k_{i'})$. We also know that the sub-graph also contains the vertices j_w, k_w. Let $p \in \{j_w, k_w\}$ be the vertex shared by $w, e^{i',1}$. Then, it is easy to see that $I_p \cap I_{j_i} \neq \phi$ and $I_p \cap I_{k_i} \neq \phi$. This would mean that the sub-graph is isomorphic to the graph in Fig. 2.

Consider the case in Fig. 2. Let $q \in \{j_w, k_w\}$ with $q \neq p$, $t \in \{j_{i'}, k_{i'}\}$ with $t \neq p$, and $s \in \{j_i, k_i\}$ with $s \neq q$. Then, we have

$$r_{p,q} \leq r_{q,s} \tag{7}$$

Since we have assumed that $e^{i,1} \prec e^{i',1}$ and since $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} = \phi$, we have $\ell_{j_{i'},k_{i'}} > r_{j_i,k_i}$, that is,

$$\ell_{p,t} > r_{q,s} \tag{8}$$

Since there is an edge between the vertices p and t, the corresponding intervals I_p and I_t intersect. From (1), we have $(\ell_p - r_t \leq 0) \wedge (\ell_t - r_p \leq 0)$, and hence

$$r_p \geq \ell_t \tag{9}$$

Since there is no edge between the vertices t and s, the corresponding intervals I_t and I_s do not intersect. From (1), we have $(\ell_t - r_s > 0) \vee (\ell_s - r_t > 0)$. If $\ell_s - r_t > 0$, then

$$r_{p,t} \leq r_t < \ell_s \leq \ell_{q,s} \leq r_{q,s}$$

which implies from (5) that $e^{i',1} \prec e^{i,1}$, which is a contradiction. Hence

$$\ell_t > r_s \tag{10}$$

Since there is no edge between the vertices t and q, the corresponding intervals I_t and I_q do not intersect. From (1), we have $(\ell_t - r_q > 0) \vee (\ell_q - r_t > 0)$. If $\ell_q - r_t > 0$, then

$$r_{j_{i'},k_{i'}} \leq r_t < \ell_q \leq \ell_{j_i,k_i} \leq r_{j_i,k_i}$$

which implies from (5) that $e^{i',1} \prec e^{i,1}$, which is a contradiction. Hence

$$\ell_t > r_q \tag{11}$$

From (9) and (11), $r_{p,q} = r_q$. From (7), $r_q \leq r_s$ and hence $r_{q,s} = r_q$. Now, from (6), $e^{i,1} \prec w$, which is a contradiction. Hence, $w^i \neq w^{i'}$.

b) Case 2: Consider $e^{i,1}, e^{i',1} \in S_2$. Suppose $w^i = w^{i'} = w = e_{j_w,k_w}$. Note that $e^{i,1}, e^{i',1}$ did not violate the matching invariant and hence they share no vertex with any of the edges in M. Hence, $e^{i,1}, w$ share no vertices and $e^{i',1}, w$ share no vertices. Let $M_i = M \cup \{e^{i,1}\}$. We know that M_i is a matching and that there is an alternating cycle of length four in the sub-graph spanned by the vertices j_i, k_i, j_w, k_w, which are all distinct. By Lemma 2, $I_{j_i,k_i} \cap I_{j_w,k_w} \neq \phi$. On a similar account, $I_{j_{i'},k_{i'}} \cap I_{j_w,k_w} \neq \phi$. Combining, we have $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} \neq \phi$. By Lemma 2 applied on $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$, we obtain that there is a length four alternating cycle on the sub-graph induced by the vertices j_i, k_i, j_w, k_w, which by Theorem 1 implies that $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a uniquely restricted matching, a contradiction. Hence, $w^i \neq w^{i'}$.

c) Case 3: Consider $e^{i,1} \in S_1, e^{i',1} \in S_2$. Suppose $w^i = w^{i'} = w = e_{j_w,k_w}$. If $e^{i,1}, e^{i',1}$ share a vertex, clearly $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a matching and hence not a uniquely restricted matching, which would be a contradiction. Hence, $e^{i,1}, e^{i',1}$ do not share any vertex. Hence, $j_i, k_i, j_{i'}, k_{i'}$ are all distinct. If $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} \neq \phi$, by Lemma 2, there is an alternating cycle of length four with respect to $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ and hence $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a uniquely restricted matching, which would be a contradiction. Hence, $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} = \phi$. By Lemma 2, the sub-graph induced by $j_i, k_i, j_{i'}, k_{i'}$ is not isomorphic to K_4. The sub-graph contains the edges $(j_i, k_i), (j_{i'}, k_{i'})$. We also know that the sub-graph also contains one of the vertices j_w, k_w. Let $s \in \{j_w, k_w\}$ be the vertex shared by $w, e^{i,1}$ and let $u \in \{j_w, k_w\}$ with $s \neq u$. Then, it is easy to see that the sub-graph is isomorphic to one of the graphs in Fig. 3.

d) Case 3 (a): Consider the case in Fig. 3(a)[4]. Let $q \in \{j_i, k_i\}$ with $q \neq s$, and $t \in \{j_{i'}, k_{i'}\}$ with $t \neq p$. As in Case 1 (a), we have

$$r_{s,u} \leq r_{q,s} \tag{12}$$

$$\ell_{p,t} > r_{q,s} \tag{13}$$

$$r_p \geq \ell_t \tag{14}$$

$$\ell_t > r_s \tag{15}$$

$$\ell_t > r_q \tag{16}$$

Since there is an edge between the vertices t and u, the corresponding intervals I_t and I_u intersect. From (1), we have $(\ell_t - r_u \leq 0) \wedge (\ell_u - r_t \leq 0)$ and hence

$$r_u \geq \ell_t \tag{17}$$

[4]This case is disallowed, however, we show that it does not arise anyway.

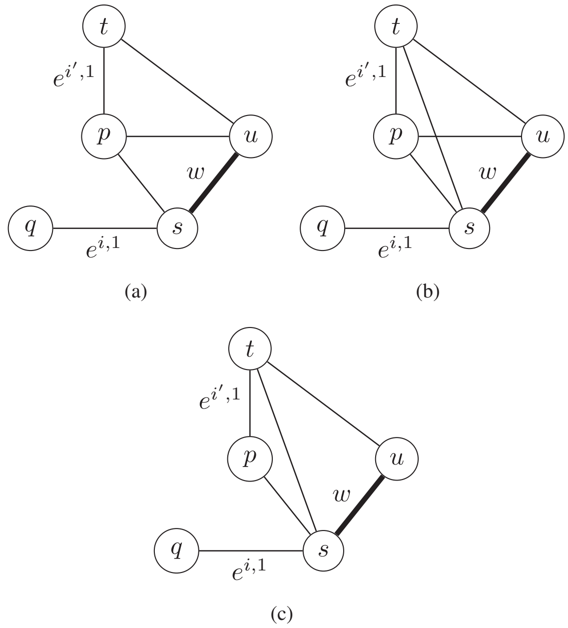


Fig. 3 Partial Sub-graph ($e_{p,q}$ and $e_{q,u}$ may be present) induced by $j_i, k_i, j_{i'}, k_{i'}, j_w, k_w$ (not all unique) in Case 3

From (15) and (17), $r_{s,u} = r_s$. From (12), $r_s \leq r_q$ and hence $r_{q,s} = r_s$. From (16) and (17), $r_u > r_q$. Now, from (6), $e^{i,1} \prec w$, which is a contradiction. Hence, $w^i \neq w^{i'}$.

e) Case 3 (b): Consider the case in Fig. 3(b). As in Case 3 (a), we have

$$\ell_t > r_q \tag{18}$$

$$r_u \geq \ell_t \tag{19}$$

Since there is an edge between the vertices t and s, the corresponding intervals I_s and I_t intersect. From (1), we have $(\ell_s - r_t \leq 0) \wedge (\ell_t - r_s \leq 0)$ and hence

$$r_s \geq \ell_t \tag{20}$$

From (18) and (19), $r_u > r_q$. From (18) and (20), $r_s > r_q$. Hence, $r_q < r_{s,u}$ and $r_{q,s} = r_q$. Hence, $r_{q,s} < r_{s,u}$. Now, from (6), $e^{i,1} \prec w$, which is a contradiction. Hence, $w^i \neq w^{i'}$.

f) Case 3 (c): Consider the case in Fig. 3(c)[5]. Since there is an edge between the vertices t and p, the corresponding intervals I_p and I_t intersect. From (1), we have $(\ell_p - r_t \leq 0) \wedge (\ell_t - r_p \leq 0)$ and hence

$$r_t \geq \ell_p \tag{21}$$

Since there is no edge between the vertices p and u, the corresponding intervals I_p and I_u do not intersect. From (1), we have $(\ell_p - r_u > 0) \vee (\ell_u - r_p > 0)$. Assume $\ell_p > r_u$. As in Case 3 (b),

$$r_u \geq \ell_t \tag{22}$$

$$\ell_t > r_q \tag{23}$$

[5]This case is disallowed, however, we show that it does not arise anyway.

$$r_s \geq \ell_t \tag{24}$$

From (21), (22), (23) and our assumption, $r_t > r_q \geq \ell_q \implies r_t \geq \ell_q$. Again, from (23) and (24), $r_s > r_q$. Hence, $r_{q,s} = r_q$. As in Case 3 (a),

$$r_{s,u} \leq r_{q,s} \tag{25}$$

From (25), $r_q \geq r_u$. From (21), $r_q \geq \ell_t$. From (1), we have that the intervals q and t intersect, which is a contradiction. Assume $\ell_u > r_p$. Then, $r_p < \ell_u \leq r_u$. Since there is an edge between the vertices s and u, the corresponding intervals I_s and I_u intersect. From (1), we have $(\ell_s - r_u \leq 0) \wedge (\ell_u - r_s \leq 0)$ and hence

$$r_s \geq \ell_u \tag{26}$$

From (26) and our assumption, $r_p < r_u$. Hence, $r_p < r_{s,u}$. Since there is an edge between the vertices t and u, the corresponding intervals I_t and I_u intersect. From (1), we have $(\ell_t - r_u \leq 0) \wedge (\ell_u - r_t \leq 0)$ and hence

$$r_t \geq \ell_u \tag{27}$$

From (27) and our assumption, $r_p < r_t$. Hence, $r_p = r_{p,t}$. Hence, $r_{p,t} < r_{s,u}$. Now, from (6), $e^{i',1} \prec w$, which is a contradiction. Hence, $w^i \neq w^{i'}$.

g) Case 4: Consider $e^{i,1} \in S_2, e^{i',1} \in S_1$. Suppose $w^i = w^{i'} = w = e_{j_w,k_w}$. If $e^{i,1}, e^{i',1}$ share a vertex, clearly $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a matching and hence not a uniquely restricted matching, which would be a contradiction. Hence, $e^{i,1}, e^{i',1}$ do not share any vertex. Hence, $j_i, k_i, j_{i'}, k_{i'}$ are all distinct. If $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} \neq \phi$, by Lemma 2, there is an alternating cycle of length four with respect to $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ and hence $M \setminus \{e^{i,0}\}_{1\leq i\leq \ell} \cup \{e^{i,1}\}_{1\leq i\leq \ell+1}$ is not a uniquely restricted matching, which would be a contradiction. Hence, $I_{j_i,k_i} \cap I_{j_{i'},k_{i'}} = \phi$. By Lemma 2, the sub-graph induced by $j_i, k_i, j_{i'}, k_{i'}$ is not isomorphic to K_4. The sub-graph contains the edges $(j_i, k_i), (j_{i'}, k_{i'})$. We also know that the sub-graph also contains the one of the vertices j_w, k_w. Let $p \in \{j_w, k_w\}$ be the vertex shared by $w, e^{i',1}$ and let $u \in \{j_w, k_w\}$ with $p \neq u$. Then, it is easy to see that $I_u \cap I_{j_i} \neq \phi$ and $I_u \cap I_{k_i} \neq \phi$. This would mean that the sub-graph is isomorphic to the graph in Fig. 4.

Consider the case in Fig. 4(a)[6]. Let $q \in \{j_i, k_i\}$ with $q \neq s$, and $t \in \{j_{i'}, k_{i'}\}$ with $t \neq p$. Since there is no edge between the vertices p and s, the corresponding intervals I_p and I_s do not intersect. If $\ell_s - r_p > 0$, then

$$r_{p,t} \leq r_p < \ell_s \leq \ell_{q,s} \leq r_{q,s}$$

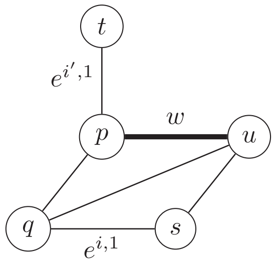
which implies from (5) that $e^{i',1} \prec e^{i,1}$, which is a contradiction. Hence

$$\ell_p > r_s \tag{28}$$

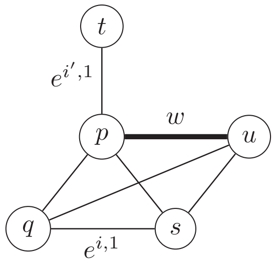
Since there is an edge between the vertices p and q, the corresponding intervals I_p and I_q intersect. From (1), we have $(\ell_p - r_q \leq 0) \wedge (\ell_q - r_p \leq 0)$ and hence

$$r_q \geq \ell_p \tag{29}$$

[6]This case is disallowed, however, we show that it does not arise anyway.



(a) $e_{u,t}$ may be present



(b)

Fig. 4 Partial Sub-graph induced by $j_i, k_i, j_{i'}, k_{i'}, j_w, k_w$ (not all unique) in Case 4

From (28) and (29), $r_s < r_q$. Hence, $r_{q,s} = r_s$. As in Case 1 (a),

$$\ell_t > r_s \tag{30}$$

$$r_p \geq \ell_t \tag{31}$$

From (30) and (31), $r_s < r_p$. Since there is an edge between the vertices p and u, the corresponding intervals I_p and I_u intersect. From (1), we have $(\ell_p - r_u \leq 0) \wedge (\ell_u - r_p \leq 0)$ and hence

$$r_u \geq \ell_p \tag{32}$$

From (28) and (32), $r_s < r_u$. Hence, $r_s < r_{p,u}$, that is, $r_{q,s} < r_{p,u}$. Now, from (6), $e^{i,1} \prec w$, which is a contradiction. Hence, $w^i \neq w^{i'}$.

Consider the case in Fig. 4(b). As in Case 1 (a),

$$r_p \geq \ell_t \tag{33}$$

$$\ell_t > r_s \tag{34}$$

$$\ell_t > r_q \tag{35}$$

Since there is no edge between the vertices t and u, the corresponding intervals I_t and I_u do not intersect. From (1), we have $(\ell_t - r_u > 0) \vee (\ell_u - r_t > 0)$. If $\ell_u - r_t > 0$, then

$$r_{j_{i'},k_{i'}} \leq r_t < \ell_u \leq \ell_{j_w,k_w} \leq r_{j_w,k_w}$$

which implies from (5) that $e^{i',1} \prec w$, which is a contradiction. Hence

$$\ell_t > r_u \tag{36}$$

From (33), (34), (35) and (36), $r_p = \max\{r_p, r_q, r_s, r_u\}$. Since $w \prec e^{i,1}$,

$$r_{p,u} \leq r_{q,s} \tag{37}$$

From (33) and (36), $r_u < r_p$. Hence $r_{p,u} = r_u$. Assume $r_{p,u} < r_{q,s}$. This is true inspite of the call to the ELONGATE algorithm. Hence, there must exist an interval α with $r_u < \ell_\alpha \leq r_{q,s}$. Since there is an edge between the vertices q and u, the corresponding intervals I_q and I_u intersect. From (1), we have $(\ell_q - r_u \leq 0) \wedge (\ell_u - r_q \leq 0)$ and hence

$$r_u \geq \ell_q \tag{38}$$

Since there is an edge between the vertices s and u, the corresponding intervals I_s and I_u intersect. From (1), we have $(\ell_s - r_u \leq 0) \wedge (\ell_u - r_s \leq 0)$ and hence

$$r_u \geq \ell_s \tag{39}$$

From (38), $\ell_q < \ell_\alpha \leq r_\alpha$. Also, $\ell_\alpha \leq r_{q,s} \implies \ell_\alpha \leq r_q$. From (1), we have that the intervals q and α intersect. From (39), $\ell_s < \ell_\alpha \leq r_\alpha$. Also, $\ell_\alpha \leq r_{q,s} \implies \ell_\alpha \leq r_s$. From (1), we have that the intervals s and α intersect. Since $\ell_\alpha > r_u$, from (1), we have that the intervals u and α do not intersect. Thus, the sub-graph induced by the vertices q, s, u and α is isomorphic to the third graph in Fig. 1, which is a contradiction. Hence, $w^i \neq w^{i'}$. This exhausts all the cases and shows that for all $i \neq i'$, $w^i \neq w^{i'}$, completing the proof. ■

Theorem 3: Let G be a restricted interval graph. M output by Algorithm 10 is a maximum uniquely restricted matching of G.

Proof: Suppose to the contrary that there is a uniquely restricted matching M' with $|M'| > |M|$. Then, we know that $M' = M \setminus (M - M') \cup (M' - M)$. Also, since $|M'| > |M|$, $|M - M'| < |M' - M|$. Let $|M - M'| = \ell \leq |M|$ and let X be a subset of $(M' - M)$ such that $|X| = \ell + 1$. Since M' is a uniquely restricted matching, by Lemma 1, so is $M'' = M \setminus (M - M') \cup X$. However, by Lemma 5, M'' is not a uniquely restricted matching, which is a contradiction. ■

Theorem 3 proves the correctness of Algorithm 10. The time complexity is analyzed as follows. The elongation of intervals can be done in $\mathcal{O}(|V|^2)$ time. The initial ordering of the edges can be obtained in $\mathcal{O}(|E|^2)$ time. Each iteration of the **while** loop can be performed in $\mathcal{O}(|E|)$ time (the detection of the matching invariant as well as the alternating cycle of length four must only be done for the newly added edge with each of the existing edges). Hence, the algorithm runs in time $\mathcal{O}\left(|E|^2\right)$, which is polynomial time.

VII. Conclusions and Future Work

In this work, we have considered two sub-classes of interval graphs and described poly time algorithms for each, which solve the problem of computing a maximum uniquely restricted matching for both sub-classes. This makes progress on the question as to whether maximum uniquely restricted matchings of interval graphs can be determined in polynomial time. However, the exact complexity of this problem on interval graphs remains an open question. The structures described in Fig. 1 have the property that at most one edge can be chosen from them in any uniquely restricted matching. The difficulty lies in the choice of which of the edges are taken in order to obtain a maximum uniquely restricted matching. If indeed the problem is NP-Complete, these structures should qualify as hard instances since locally, a choice of an edge from such a structure would determine the global outcome of the algorithm as maximum or not. It is also interesting to note that only the last case of the proofs really need restrictions on the graph while the other cases go through just based on the ordering.

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