

# Hamiltonian Related Properties with and without Faults of the Dual-Cube Interconnection Network and Their Variations

Shih-Yan Chen, Shin-Shin Kao

**Abstract**—In this paper, a thorough review about dual-cubes,  $DC_n$ , the related studies and their variations are given.  $DC_n$  was introduced to be a network which retains the pleasing properties of hypercube  $Q_n$  but has a much smaller diameter. In fact, it is so constructed that the number of vertices of  $DC_n$  is equal to the number of vertices of  $Q_{2n+1}$ . However, each vertex in  $DC_n$  is adjacent to  $n+1$  neighbors and so  $DC_n$  has  $(n+1) \times 2^{2n}$  edges in total, which is roughly half the number of edges of  $Q_{2n+1}$ . In addition, the diameter of any  $DC_n$  is  $2n+2$ , which is of the same order of that of  $Q_{2n+1}$ . For self-completeness, basic definitions, construction rules and symbols are provided. We chronicle the results, where eleven significant theorems are presented, and include some open problems at the end.

**Keywords**—Hypercubes, dual-cubes, fault-tolerant hamiltonian property, dual-cube extensive networks, dual-cube-like networks.

## I. INTRODUCTION

THE hypercube family  $Q_n$  is one of the most well-known interconnection networks in parallel computers due to its many pleasing properties such as vertex/edge symmetry, recursive structure, easy routing, high degree of fault tolerance, and so on. See [1]–[3]. However,  $Q_n$  does not have the smallest diameter possible for its resources, which results in the less efficiency and cost-effectivity of interprocessor communication. Therefore, a variety of hypercube-like interconnection networks has been introduced to achieve a lower diameter by exchanging or “twisting” the endvertices of some edges. These hypercube variations, including Twisted cubes, Multiple-twisted cubes, Crossed cubes, Flip MCubes, Moebius cubes etc., have a diameter of roughly  $n/2$ , which is half the diameter of  $Q_n$ . A comparison of diameters among these networks can be found in [4]. On the other hand, the dual-cube family  $DC_n$  for  $n \geq 1$ , introduced by Li and Peng [5], is able to achieve the similar diameter of  $Q_n$  with much less edges. Li et al. make  $2n+1$  copies of  $Q_n$  and divide them into two classes, Class 0 and Class 1. Each class consists of  $2n$  copies of  $Q_n$  and each copy is called a cluster. By properly adding edges, they connect every pair of clusters from the opposite classes with an edge and prove that  $DC_n$  is a  $(n+1)$ -regular, vertex symmetric graph that contains some properties superior to hypercubes.

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Notice that the number of vertices of  $DC_n$  is equal to the number of vertices of  $Q_{2n+1}$ . Since each vertex in  $Q_{2n+1}$  is adjacent to  $2n+1$  neighbors, the total number of edges of  $Q_{2n+1}$  is  $(2n+1) \times 2^{2n}$ . However, each vertex in  $DC_n$  is adjacent to  $n+1$  neighbors and so  $DC_n$  has  $(n+1) \times 2^{2n}$  edges in total. Although any dual-cube  $DC_n$  has much less edges than the hypercube  $Q_{2n+1}$ , its diameter,  $2n+2$ , is of the same order of the diameter of  $Q_{2n+1}$ , which is  $2n+1$ . In addition, it is proved that  $DC_n$  is vertex symmetric, and any  $DC_n$  is a spanning subgraph of  $Q_{2n+1}$  for  $n \geq 1$ .

Ever since  $DC_n$  was introduced, it attracts many studies. See [6]–[9], for instance. In addition, researchers extend the main structure of  $DC_n$  to invent new variations of network topologies with nice properties. In this paper, we chronicle these results and include some open problems. This paper is organized as follows. In Section II, for self-completeness, notations, the precise definitions of  $Q_n$  and  $DC_n$  are given. In Section III, we recall the hamiltonian properties of  $DC_n$  and fault-tolerant hamiltonicities of  $DC_n$ . In Section IV, the dual-cube extensive networks, abbreviated as DCENs, dual-cube-like networks, and their associated nice properties will be reviewed. Finally, a brief conclusion is given in Section V.

## II. PRELIMINARY: HYPERCUBES AND DUAL-CUBES

Throughout this paper, we follow [10] for the graph definitions and notations. The sets of vertices and edges of a graph  $G$  are denoted by  $V(G)$  and  $E(G)$ , respectively. The cardinality of  $V(G)$  is denoted by  $|V(G)|$ . Two vertices  $u, v$  of  $G$  are *adjacent* in  $G$  if there is an edge  $e = (u, v) \in E(G)$  between  $u$  and  $v$ . The *degree* of a vertex  $u$  is the number of vertices adjacent to  $u$ . A graph  $G$  is *r-regular* if the degree of any vertex of  $G$  is  $r$ . A *path*  $P$  between two vertices  $v_0$  and  $v_k$  is represented by  $P = \langle v_0, v_1, \dots, v_k \rangle$ , where every two consecutive vertices are connected by an edge. We also write the path  $P = \langle v_0, v_1, \dots, v_k \rangle$  as  $\langle v_0, v_1, \dots, v_i, P', v_j, v_{j+1}, \dots, v_k \rangle$ , where  $P'$  denotes the path  $\langle v_i, v_{i+1}, \dots, v_j \rangle$ . A *hamiltonian path* between  $u$  and  $v$ , where  $u$  and  $v$  are two distinct vertices of  $G$ , is a path joining  $u$  to  $v$  that visits every vertex of  $G$  exactly once. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of  $G$  is a cycle that traverses every vertex of  $G$  exactly once. A *hamiltonian graph* is a graph with a hamiltonian cycle.

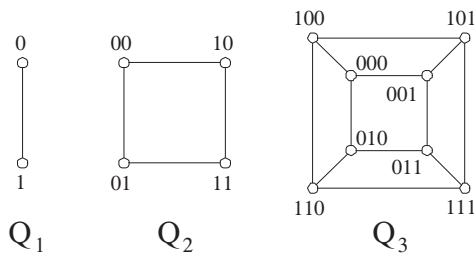


Fig. 1 Hypercubes  $Q_1$ ,  $Q_2$ , and  $Q_3$

A graph  $G$  is *connected* if there is a path between any two distinct vertices in  $G$  and is *hamiltonian connected* if there is a hamiltonian path between any two distinct vertices in  $G$ .

A graph  $H$  is *bipartite* if  $V(H) = B \cup W$  and  $E(H)$  is a subset of  $\{(u, v) | u \in B, v \in W\}$ . If  $|B| = |W|$ , then  $H$  is a *balanced bipartite graph*. Let  $H$  be an arbitrary balanced bipartite graph. Since any hamiltonian path in  $H$  consists of the same number of vertices of the two partite sets, there exists no hamiltonian path between two vertices belonging to the same partite set of  $H$ . Thus  $H$  is not hamiltonian connected. We say that a bipartite graph  $H$  is *hamiltonian laceable* if there is a hamiltonian path between any two distinct vertices from the opposite partite sets of  $H$ . In the sequel, all graphs, bipartite or nonbipartite, are simple and undirected. And for bipartite graphs, we only consider balanced bipartite ones.

An  $n$ -dimensional hypercube  $Q_n$  is a graph with the vertex set  $\{0, 1\}^n$  and there is an edge between any two vertices that differ in exactly one bit position. The label  $\{0, 1\}^n$  of each vertex of  $Q_n$  is called the *vertex id*. See Fig. 1 for an illustration.

The dual-cube family,  $DC_n$ ,  $n \geq 1$ , was first introduced by Li and Peng [5]. A dual-cube  $DC_n$  is obtained from a basic component  $Q_n$  as follows. Make  $2^{n+1}$  copies of  $Q_n$  and divide them into two *classes*, Class 0 and Class 1. Each class consists of  $2^n$  copies of  $Q_n$  and each copy is called a *cluster*. We shall label the  $2^n$  clusters in each class by  $\{0, 1\}^n$ , called the *cluster id*. Then any vertex  $u \in V(DC_n)$  is given a *vertex id*, which is a  $(2n+1)$ -bit sequence of the form  $u = (u_{2n}, u_{2n-1}, \dots, u_0)$ , according to the following rule:

(1)  $u \in$  Class 0 :

$$0 \underbrace{u_{2n-1}u_{2n-2} \dots u_n}_{n \text{ bits}} \underbrace{u_{n-1}u_{n-2} \dots u_0}_{n \text{ bits}} ;$$

(2)  $u \in$  Class 1 :

$$1 \underbrace{u_{2n-1}u_{2n-2} \dots u_n}_{n \text{ bits}} \underbrace{u_{n-1}u_{n-2} \dots u_0}_{n \text{ bits}} .$$

Given two vertices  $u = (u_{2n}, u_{2n-1}, \dots, u_0)$  and  $v = (v_{2n}, v_{2n-1}, \dots, v_0)$ . There is an edge between  $u$  and  $v$  in

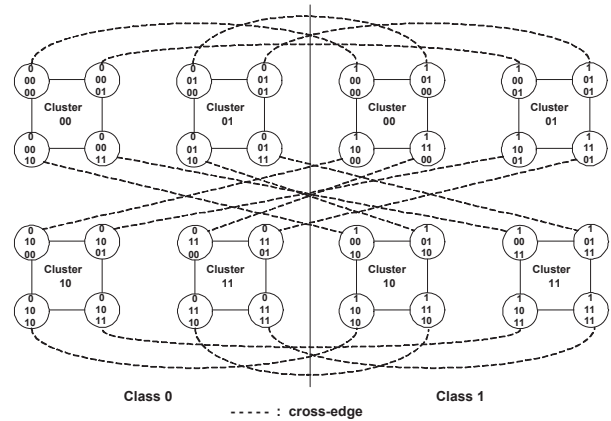


Fig. 2 Dual-cubes  $DC_2$

$DC_n$  if and only if the following conditions are satisfied:

- $u$  and  $v$  differ in exactly one bit position  $i$ , where  $0 \leq i \leq 2n$ ;
- if  $0 \leq i \leq n-1$ , then  $u_{2n} = v_{2n} = 0$ ;
- if  $n \leq i \leq 2n-1$ , then  $u_{2n} = v_{2n} = 1$ .

The example of  $DC_2$  is shown in Fig. 2. By the definition of  $DC_n$ , we can see that  $|V(DC_n)| = 2^{2n+1}$  and the degree of each vertex in  $DC_n$  is  $n+1$ , in which  $n$  edges are within the cluster and one edge reaches a vertex in some cluster of the other class. There is no edge between the clusters of the same class. The edges connecting two clusters of distinct classes are called *cross-edges*. If two vertices are in the same cluster, or belong to two clusters of distinct classes, the distance between the two vertices is equal to the Hamming distance (the number of bits where the two vertex id's have different values). Otherwise, it is equal to the Hamming distance plus two: one for entering a cluster of the other class and one for leaving. Moreover, it is easy to see that  $DC_n$  is a bipartite graph for any integer  $n$  with  $n \geq 1$ .

### III. PROPERTIES OF DUAL-CUBES

In this section, many results about fault-tolerant hamiltonian properties for  $DC_n$  will be presented. We shall start from definitions. A graph  $G$  is  $k$ -vertex-fault-tolerant hamiltonian (resp.  $k$ -edge-fault-tolerant hamiltonian) if  $G - F$  remains hamiltonian for any subset  $F$  of  $V(G)$  (resp.  $F$  as a subset of  $E(G)$ ) with  $|F| \leq k$ , where  $k \leq \delta(G) - 2$  and  $\delta(G)$  denotes the minimum degree of  $G$ . A graph  $G$  is  $k$ -vertex-fault-tolerant hamiltonian connected (resp.  $k$ -edge-fault-tolerant hamiltonian connected) if  $G - F$  is hamiltonian connected for any subset  $F$  of  $V(G)$  (resp.  $F$  as a subset of  $E(G)$ ) with  $|F| \leq k$ , where  $k \leq \delta(G) - 3$ . A graph  $G$  is  $k$ -fault-tolerant hamiltonian (resp.  $k$ -fault-tolerant hamiltonian connected) if  $G - F$  is hamiltonian (resp. hamiltonian connected) for any subset  $F$  belonging to  $V(G) \cup E(G)$ , where  $|F| \leq k$  and  $k \leq \delta(G) - 2$  (resp.  $k \leq \delta(G) - 3$ ). For bipartite graphs, a hamiltonian laceable graph  $H$  is  $k$ -edge-fault-tolerant hamiltonian laceable if  $H - F$  remains hamiltonian laceable for any subset  $F$  of  $E(H)$  with  $|F| \leq k$  and  $k \leq \delta(G) - 3$ .

In 2002, Li et al. proved that for  $n \geq 2$ ,  $DC_n$  contains a fault-free hamiltonian cycle even if it contains up to  $n - 1$  faulty edges. Later on, in 2005, they showed that there exists a fault-free cycle containing at least  $2^{2n+1} - 2f$  vertices in  $DC_n$ , where  $n \geq 3$  and  $f \leq n$  is the number of faulty vertices. See [11] and [9] for details. Inspired by the above-mentioned works, Lai et. al in 2008 defined high-level  $k$ -cycles and proved the following theorem [7].

**Definition 1.** A cycle of  $\langle u_0, v_0, u_1, v_1, \dots, u_{k-1}, v_{k-1}, u_k \rangle$  is called to be a *high-level  $k$ -cycle* if  $(u_i, v_i)$  is an edge inside a cluster and  $(v_i, u_{i+1})$  is an edge connecting two nodes in two clusters of distinct classes. Here  $C_{u_i} \neq C_{u_j}$  for  $i \neq j$ ,  $0 \leq i, j \leq k - 1$ , and  $u_k = u_0$ .

**Theorem 1.** For  $n \geq 3$ , every vertex of  $DC_n$  lies on a cycle of every even length from 4 to  $2^{2n+1}$ , inclusive. Note that  $DC_n$  is a bipartite graph, it contains no odd cycle.

The idea of high-level  $k$ -cycles turns out to be very easily applied in studying issues related to cycle embedding in dual-cubes and its variations. Theorem 1 implies that  $DC_n$  is *vertex-bipancyclic*, which is especially useful in the network broadcasting and message transmissions. More specifically, a network with  $DC_n$  as its underlying topology can use any vertex as the center, and the center can transmit messages to a circle with the required length such that news (or resources) can be more efficiently distributed (or utilized).

In 2010, Shih and his coauthors studied the existence of *mutually independent hamiltonian cycles* in  $DC_n$  [12]. The concept of mutually independent hamiltonian cycles arises from the following application [10]. If  $k$  pieces of data must be sent from a message center  $u$ , and the data must be processed at each intermediate receiver (and the process takes time) before they are sent back to the center, then the existence of mutually independent cycles from  $u$  guarantees that there will be no waiting time for the parallel processing. More precisely, given a graph  $G$ , let  $\langle v_0, v_1, v_2, \dots, v_{|V(G)|-1}, v_0 \rangle$  be a hamiltonian cycle of  $G$ . By saying  $G$  has  $l$  mutually independent hamiltonian cycles, we mean that for any vertex  $v_0$  of  $G$ , there exist hamiltonian cycles of the form  $\langle v_0, v_1^k, v_2^k, \dots, v_{|V(G)|-1}^k, v_0 \rangle$  such that  $v_i^k \neq v_i^{k'}$  wherever  $k \neq k'$  for  $1 \leq k \leq l$ . It is obvious that the number of mutually independent hamiltonian cycles of a graph is bounded by its minimum degree  $\delta(G)$ . In [12], the following theorem for  $DC_n$  is derived.

**Theorem 2.** For  $n \geq 2$ ,  $DC_n$  has  $n + 1$  mutually independent hamiltonian cycles. The result is optimal since each vertex of  $DC_n$  has only  $n + 1$  neighbors.

Processors of a multiprocessor system are connected according to a given interconnection network design. It is inevitable to have failures of certain network components. For this reason, various fault-tolerant measures have been studied in the literature, such as fault diameter, fault hamiltonicity, fault pancyclicity, fault hamiltonian laceability, etc.. See [14]

and its references. On the other hand, it is reasonable to assume that the possibility of all faulty elements being adjacent to the same node (vertex) is nearly zero. Thus the concept of *conditionally faulty tolerance* arises. Namely, any fault-tolerant property is discussed under the assumption that each node is incident with at least two fault-free links (edges). It was shown in [14], [15] that a  $k$ -ary,  $k \geq 3$ ,  $n$ -dimensional hypercube (resp. a hypercube  $Q_n$ ) contains a fault-free hamiltonian cycle even if there exists up to  $4n - 5$  (resp.  $2n - 5$ ) faulty links. With this result for  $Q_n$ , Chen and Tsai proved the following theorem for dual-cubes [13], which is optimal with respect to the number of tolerant edge faults.

**Theorem 3.** For  $n \geq 2$ ,  $DC_n$  contains a fault-free hamiltonian cycle provided  $f_e \leq 2n - 3$  and every vertex is incident with at least two fault-free edges, where  $f_e$  denotes the number of faulty edges in  $DC_n$ .

Recently, *1-perfect codes* constitute a significant field of study due to their wide applications in multiprocessor systems, and a number of other areas in the digital world. Readers can refer to [16] and its references. They have the capability to detect two or fewer errors, and even correct a single error. Among various types of 1-perfect codes, the Hamming codes, based on the topology of hypercubes, are the most well-known ones. Meanwhile the question of determining whether a given graph supports a 1-perfect code is a NP-complete problem even for planar 3-regular graphs. By saying that a graph  $G$  supports a 1-perfect code, it means there exists a subset  $C$  of  $V(G)$  such that the 1-balls centered at the vertices if  $C$  form a partition of  $V(G)$ . In 2015, P. Jha derived the theorem below [16].

**Theorem 4.** For  $n \geq 2$ ,  $DC_n$  admits a 1-perfect code if and only if  $n = 2^k - 2$  for  $k \geq 2$ .

The result in Theorem 4 parallels the existence of hamming codes on the hypercube  $Q_n$ . In [16], Jha further developed an algorithm for a vertex partition of  $Q_{(n+1)}$  into hamming codes using a Latin square, and showed that Theorem 4 leads to tight bounds on domination numbers of dual-cubes and exchanged hypercubes.

#### IV. DCENS AND DUAL-CUBE-LIKE NETWORKS

In 2010, the authors introduced a new kind of graphs, called *dual-cube extensive networks*, abbreviated as DCENs. See [17]. The idea of DCEN comes from dual-cubes. Instead of using the hypercube  $Q_n$  as its basic component as in  $DC_n$ , DCEN( $G$ ) takes any graph  $G$  as the basic component and is then obtained by the similar structure as in  $DC_n$ . Let the graph  $G$  with  $V(G) = n$  be the basic component of DCEN( $G$ ) and the vertices of  $G$  be labeled from 1 to  $n$ . Then DCEN( $G$ ) consists of two *classes*, Class 1 and Class 2. For  $i \in \{1, 2\}$ , Class  $i$  has  $n$  copies of  $G$ , namely  $G^{i,1}, \dots, G^{i,n}$ , and each  $G^{i,j}$  is called a *cluster*. We shall label any vertex in  $G^{i,j}$  of DCEN( $G$ ) by  $(i, j, k)$ , where  $k$  is the *vertex id* in  $G$ . The vertices  $(i, j, k)$  and  $(i', j', k')$  are adjacent in DCEN( $G$ )

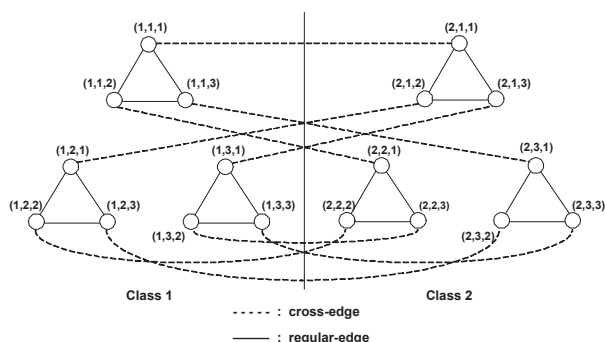


Fig. 3 An example for DCEN( $G$ ) where  $G = K_3$

if and only if one of the following conditions is satisfied:

- (1)  $i = i', j = j'$ , and the vertices  $k$  and  $k'$  are adjacent in  $G$ ;
- (2)  $|i - i'| = 1, j = k',$  and  $k = j'$ .

An example of DCEN is depicted in Fig. 3. The edges satisfying (2) are *cross-edges*, which connect different pairs of clusters belonging to the two classes. Vertices in a certain cluster use cross-edges to reach vertices in distinct clusters in the opposite class. Therefore, by regarding each cluster as a vertex, DCEN( $G$ ) becomes a complete bipartite graph  $K_{n,n}$ . Every cross-edge has the corresponding endvertices in the two clusters of the opposite classes. For example, the cross-edge connecting the clusters  $G^{1,i}$  and  $G^{2,j}$  has endvertices  $(1, i, j) \in G^{1,i}$  and  $(2, j, i) \in G^{2,j}$ . It was shown in [17] that if  $G$  is a nonbipartite graph, then so is DCEN( $G$ ). Besides, if  $H$  is a bipartite graph, then so is DCEN( $H$ ). The following four theorems are derived in [17].

**Theorem 5.** Let a graph  $G$  with  $|V(G)| \geq 3$  be the basic component of DCEN( $G$ ). If  $G$  is a nonbipartite and hamiltonian connected graph, then DCEN( $G$ ) is hamiltonian connected.

**Theorem 6.** Let a bipartite graph  $H$  with  $|V(H)| \geq 4$  be the basic component of DCEN( $H$ ). If  $H$  is a hamiltonian laceable graph, then DCEN( $H$ ) is hamiltonian laceable.

**Theorem 7.** Let a graph  $G$  with  $|V(G)| \geq 4$  be the basic component of DCEN( $G$ ). If  $G$  is a nonbipartite, hamiltonian connected, and globally 3\*-connected graph, then DCEN( $G$ ) is globally 3\*-connected.

**Theorem 8.** Let a balanced bipartite graph  $H$  with  $|V(H)| \geq 6$  be the basic component of DCEN( $H$ ). If  $H$  is a hamiltonian laceable, and globally bi-3\*-connected graph, then DCEN( $H$ ) is globally bi-3\*-connected.

In Theorems 7 and 8, a nonbipartite graph  $G$  is said to be *globally 3\*-connected* if for any given pair of distinct vertices  $\{u, v\}$  of  $G$ , there exist three paths  $\{P_i | i = 1, 2, 3\}$  such that  $P_i \cap P_j = \{u, v\}$  for  $i \neq j$  and  $\cup_{i=1}^3 P_i = V(G)$ . A balanced bipartite graph  $H$  is *globally bi-3\*-connected* if there exists a set of three paths  $\{P_i | i = 1, 2, 3\}$  between any two distinct vertices  $\{b, w\}$  from the opposite partite sets of

$H$ , and  $P_i \cap P_j = \{b, w\}$  for  $i \neq j$  and  $\cup_{i=1}^3 P_i = V(H)$ . The existence of globally  $r^*$ -connectivity in interconnection networks allows messages being processed simultaneously and independently. The above four theorems show that the nice properties of the basic component ( $G$  or  $H$ ) are well preserved by making copies of the basic component and properly adding links using the skeleton of  $DC_n$ . Theorems and applications about globally  $r^*$ -connectivity can be found in Chapter 14 of [10].

In 2013, Angjeli et al. introduced another variation of dual cubes [18], which they called dual-cube-like networks. Rather than giving freedom in choosing the basic component as of DCENs, they give freedom in choosing edges between distinct clusters. More specifically, a graph DCL( $n$ ) is a dual-cube-like network of order  $n$  if it has the following structure:

- DCL( $n$ ) consists of  $2^n$  disjoint copies of the  $(n - 1)$ -dimensional hypercube with additional edges only between different copies of hypercubes. These hypercubes are called clusters of DCL( $n$ ), and the edges between them are called cross edges.
- Every vertex of DCL( $n$ ) is incident to exactly one cross edge, so the cross edges form a perfect matching of DCL( $n$ ).
- Between any two different clusters of DCL( $n$ ) there is at most one cross edge, and if we define the underlying graph (basic component) as we did for  $DC_n$ , we get a  $2^{n-1}$ -regular, maximally connected graph (this could be  $K_{2^{n-1}, 2^{n-1}}$  or some other graph satisfying these conditions).

It is obvious that  $DC_n$  is in fact a special case of DCL( $n$ ). And even if one restricts the underlying graph to be a complete bipartite graphs, dual-cube-like networks still offer a much larger class of graphs. To present the main results of [18], some terminologies have to be defined. A non-complete graph  $G$  with at least  $r + 1$  vertices is *r-connected* if the removal of any set of at most  $r$  vertices results in a connected graph. A complete graph with  $r + 1$  vertices is *k-connected* for  $k \leq r$ . An  $r$ -regular graph is *maximally connected* if it is  $r$ -connected. If an  $r$ -regular graph  $G$  is maximally connected and has the property that for every  $T \subset V$  with  $|T| = r$  the graph  $G - T$  is either connected or has two components, one of which is a singleton, then  $G$  is called *tightly super-connected*. The following theorems about the fault-tolerant connectivity of DCL( $n$ ) were proved in [18].

**Theorem 9.** Let  $n$  be an integer with  $n \geq 3$ . Then any dual-cube-like network of order  $n$ , DCL( $n$ ), is maximally connected. Moreover, it is tightly super-connected.

**Theorem 10.** Let  $n$  and  $k$  be integers such that  $n \geq 3$  and  $1 \leq k \leq n - 1$ . Let DCL( $n$ ) be a dual-cube-like network of order  $n$  and  $T \subset V(DCL(n))$ . If  $|T| \leq kn - (k(k + 1))/2$ , then DCL( $n$ ) -  $T$  is either connected or it has a large connected component and small components with at most  $k - 1$  vertices in total. Moreover, there is a set of vertices  $T$  in DCL( $n$ ) such that  $|T| = kn - k(k + 1)/2 + 1$  and

$DCL(n) - T$  has a component containing exactly  $k$  vertices.

Theorem 10 already gave the largest  $m$  for which a dual-cube-like network is super  $m$ -connected of order  $k - 1$ , and the result is sharp for  $k \leq n$ . In [18], this theorem was applied to derive additional results such as the cyclic vertex-connectivity and the restricted vertex-connectivity of  $DCL(n)$ .

## V. CONCLUSION

In this paper, a thorough review about dual-cubes, the related studies and its variations is given. We start from the very basic definition about how it was constructed from the well-known hypercube topology, and then present many nice properties of hypercubes which are preserved in dual-cubes. We also review two different interconnection networks based on the idea of dual-cubes, namely DCENs and  $DCL(n)$ , and their properties about hamiltonian connectivity and fault-tolerant connectivities. Obviously, many issues remain open for dual-cubes and its variations. For example, the existence of ordered hamiltonian cycles/paths with or without faulty elements, the globally  $r^*$ -connectivity (or globally  $bi-r^*$ -connectivity) for  $r \geq 4$  under possible edge/vertex faults, the preservation of 1-perfect codes on DCENs,  $DCL(n)$  or other variations. These are all interesting topics and need further investigations.

## ACKNOWLEDGMENT

This work is partially supported by the Ministry of Science and Technology of the Republic of China under contract MOST-104-2115-M-033-002-.

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