Frequency Domain Analysis for Hopf Bifurcation in a Delayed Competitive Web-site Model

Changjin Xu, Yusen Wu

Abstract—In this paper, applying frequency domain approach, a delayed competitive web-site system is investigated. By choosing the parameter α as a bifurcation parameter, it is found that Hopf bifurcation occurs as the bifurcation parameter α passes a critical values. That is, a family of periodic solutions bifurcate from the equilibrium when the bifurcation parameter exceeds a critical value. Some numerical simulations are included to justify the theoretical analysis results. Finally, main conclusions are given.

Keywords—Web-site system, stability, Nyquist criterion, Hopf bifurcation, frequency domain.

I. INTRODUCTION

IN the past few years, many models of the competition dynamics of Internet, and other phenomena related to the World Wide Web have emerged [1]-[3]. The emergence of an information era mediated by the Internet brings about a great many novel and interesting economic problems. Thus, the dynamics properties of the competitive web-site models which have significant practical background have received much attention from many applied mathematicians, economists and engineers. Many interesting results have been reported, for example, based on the Lotka-Volterra model, Maurer and Huberman [2] explored the effects of competition among web and determined how they affect the nature of markets of the following dynamical model of web site growth

$$\frac{dx_i}{dt} = x_i \left(\alpha_i \beta_i - \alpha_i x_i - \sum_{i \neq j} \gamma_{ij} x_j \right), \qquad (1)$$

where x_i is the fraction of the market which is a customer of site i, α_i stands for the growth rate which measures the capacity of site i to grow, β_i is the maximum capacity which is related to the saturation value of x_i (the maximum value x_i can reach) and γ_{ij} denotes the competition rate between sites i and j. The parameter values should be satisfied $\alpha_i \geq 0, 0 \leq \beta_i \leq 1$ and $\gamma_{ij} \geq 0$.

For simplifying the analysis for model (1), assuming that $\beta_i = 1$ and there are only three competitors in system (1), furthermore, the first site has delayed self-feedback, Xiao and Cao [3] obtained the following delayed competitive web sites

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model

$$\begin{cases} \dot{x}_1(t) = x_1[\alpha - \alpha x_1(t - \tau) - \gamma x_2(t) \\ \qquad - \gamma x_3(t)], \\ \dot{x}_2(t) = x_2[\alpha - \alpha x_2(t) - \gamma x_1(t) \\ \qquad - \gamma x_3(t)], \\ \dot{x}_3(t) = x_3[\alpha - \alpha x_3(t) - \gamma x_1(t) \\ \qquad - \gamma x_2(t)] \end{cases} \qquad (2)$$

and investigate the linear stability and the behaviours (e.g., the existence, the direction and the stability of the bifurcated periodic solutions etc.) of Hopf bifurcation. Based on the work of Xiao and Cao [3], Zhang et al. [4] argued that, in fact, each site has self delayed self-feedback and different sites have time delay (for simplification, assume that different sites have the same time delay), then a competitive web-site system with reflexive and competitive delays is obtained as follows:

$$\begin{cases} \dot{x}_1(t) = x_1[\alpha - \alpha x_1(t - \tau) - \gamma x_2(t - \tau) \\ \qquad - \gamma x_3(t - \tau)], \\ \dot{x}_2(t) = x_2[\alpha - \alpha x_2(t - \tau) - \gamma x_1(t - \tau) \\ \qquad - \gamma x_3(t - \tau)], \\ \dot{x}_3(t) = x_3[\alpha - \alpha x_3(t - \tau) - \gamma x_1(t - \tau) \\ \qquad - \gamma x_2(t - \tau)]. \end{cases} \qquad (3)$$

Zhang et al. [4] had investigated the stability and the existence of Hopf bifurcation of system (3).

Motivated by the papers [3], [4], we assume that each site has the same delayed self-feedback, then we have the competitive web-site system which takes the form:

$$\begin{cases} \dot{x}_1(t) = x_1[\alpha - \alpha x_1(t - \tau) - \gamma x_2(t) \\ \qquad - \gamma x_3(t)], \\ \dot{x}_2(t) = x_2[\alpha - \alpha x_2(t - \tau) - \gamma x_1(t) \\ \qquad - \gamma x_3(t)], \\ \dot{x}_3(t) = x_3[\alpha - \alpha x_3(t - \tau) - \gamma x_1(t) \\ \qquad - \gamma x_2(t)], \end{cases} \qquad (4)$$

where $\alpha > 0, \beta > 0$. It is well known that the research on the existence of Hopf bifurcation is very critical. To obtain a deep and clear understanding of dynamics of competitive web-site system with time delay, in this paper, we shall investigate the existence of Hopf bifurcation for system (4). We must point out that many early work on Hopf bifurcation of the delayed differential equations is used the state-space formulation for delayed differential equations, known as the "time domain" approach [3], [4]. But in this paper, we will use another approach that comes from the theory of feedback systems known as frequency domain method which was initiated and developed by Allwright [5], Mees and Chua [6] and Moiola and Chen [6], [7] and is familiar to control engineers. This

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:9, No:2, 2015

alternative representation applies the engineering feedback systems theory and methodology: an approach described in the "frequency domain"–the complex domain after the standard Laplace transforms having been taken on the state-space system in the time domain. This new methodology has some advantages over the classical time-domain methods [8]-[11].

In this paper, we will devote our attention for seeking the Hopf bifurcation point for models (4) by means of the frequency-domain approach. It is found that if the coefficient α is used as a bifurcation parameter, then Hopf bifurcation occurs for the model (4). That is, a family of periodic solutions bifurcate from the equilibrium when the bifurcation parameter exceeds a critical value. Some numerical simulations are perform to illustrate the theoretical analysis. To the best of our knowledge, there are very few papers that are concerned with the Hopf bifurcation by the frequency-domain approach.

The remainder of the paper is organized as follows: in Section II, applying the frequency-domain approach formulated by Moiola and Chen [7], the existence of Hopf bifurcation parameter is determined and shown that Hopf bifurcation occurs when the bifurcation parameter exceeds a critical value. In Section III, some numerical simulations are carried out to verify the correctness of theoretical analysis result. Finally, some conclusions are included in Section IV.

II. THE EXISTENCE OF HOPF BIFURCATION

It is obvious that system (4) has a unique positive equilibrium $E(x_0, x_0, x_0)$, where

$$x_0 = \frac{\alpha}{\alpha + 2\gamma}.$$

We can rewrite the nonlinear system (4) as a matrix form

$$\frac{dx(t)}{dt} = Ax(t) + H(x), \qquad (5)$$

where

$$x = (x_1(t), x_2(t), x_3(t))^T,$$

$$A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix},$$

$$H(x) = - \begin{pmatrix} \alpha x_1 x_1(t-\tau) + \gamma x_1 x_2 + \gamma x_1 x_3 \\ \alpha x_2 x_2(t-\tau) + \gamma x_1 x_2 + \gamma x_2 x_3 \\ \alpha x_3 x_3(t-\tau) + \gamma x_1 x_3 + \gamma x_2 x_3 \end{pmatrix}.$$

Choosing the coefficient α as a bifurcation and introducing a "state-feedback control" $u = g[y(t-\tau); \alpha]$, where

$$y(t) = (y_1(t), y_2(t), y_3(t))^T,$$

we obtain a linear system with a non-linear feedback as follows

$$\begin{cases} \frac{dx}{dt} = Ax + Bu, \\ y = -Cx, \\ u = g[y(t-\tau); \alpha], \end{cases} \qquad (6)$$

where

$$\begin{aligned} B &= C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ u &= g[y(t-\tau), \alpha] \\ &= - \begin{pmatrix} \alpha y_1 y_1(t-\tau) + \gamma y_1 y_2 + \gamma y_1 y_3 \\ \alpha y_2 y_2(t-\tau) + \gamma y_1 y_2 + \gamma y_2 y_3 \\ \alpha y_3 y_3(t-\tau) + \gamma y_1 y_3 + \gamma y_2 y_3 \end{pmatrix}. \end{aligned}$$

Next, taking Laplace transform on (6), we obtain the standard transfer matrix of the linear part of the system

$$G(s; \tau) = C[sI - A]^{-1}B.$$

Then

$$G(s; \alpha) = \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix}, \qquad (7)$$

If this feedback system is linearized about the equilibrium

$$y = -C(x_0, x_0, x_0)^T,$$

then the Jacobian of $u = g[y(t-\tau); \alpha]$ is given by

$$\begin{aligned} J(\alpha) &= \frac{\partial g}{\partial y}\Big|_{y=\tilde{y}=-C(x_0, x_0, x_0)^T} \\ &= \begin{pmatrix} a & \gamma x_0 & \gamma x_0 \\ \gamma x_0 & a & \gamma x_0 \\ \gamma x_0 & \gamma x_0 & a \end{pmatrix}. \end{aligned}$$

where

$$a = \alpha + 2\gamma x_0 + \alpha x_0 e^{-s\tau}.$$

Let

$$\begin{aligned} &h(\lambda, s; \alpha) \\ &= \det |\lambda I - G(s; \alpha) J(\alpha)| \\ &= \left(\lambda - \frac{a}{\alpha}\right)^3 - 2\left(\frac{\gamma x_0}{\alpha}\right)^3 \\ &\quad -3\left(\lambda - \frac{a}{\alpha}\right)\left(\frac{\gamma x_0}{\alpha}\right)^2. \end{aligned}$$

Applying the generalized Nyquist stability criterion with $s = i\omega$, we obtain the following results.

Lemma 1 [7] *If an eigenvalue of the corresponding Jacobian of the nonlinear system, in the time domain, assumes a purely imaginary value $i\omega_0$ at a particular $\alpha = \alpha_0$, then the corresponding eigenvalue of the constant matrix $G(i\omega_0; \alpha_0)J(\alpha_0)$ in the frequency domain must assume the value $-1 + i0$ at $\alpha = \alpha_0$.*

To apply Lemma 1, let $\hat{\lambda} = \hat{\lambda}(i\omega; \alpha)$ be the eigenvalue of $G(i\omega; \alpha)J(\alpha)$ that satisfies

$$\hat{\lambda}(i\omega_0; \alpha_0) = -1 + 0i.$$

Then

$$h(-1, i\omega_0; \alpha_0) = 0.$$

That is

$$b_1 e^{-i\omega_0\tau} + b_2 e^{-2i\omega_0\tau} + b_3 e^{-3i\omega_0\tau} + b_4 = 0, \qquad (8)$$

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:9, No:2, 2015

where

$$b_1 = 3\alpha_0 x_0(\gamma x_0)^2 - 12\alpha_0 x_0(\alpha_0 + \gamma x_0)^2, \quad (9)$$

$$b_2 = -6(\alpha_0 x_0)^2(\alpha_0 + \gamma x_0), \quad (10)$$

$$b_3 = -(\alpha_0 x_0)^3, \quad (11)$$

$$b_4 = -[8(\alpha_0 + \gamma x_0)^3 + 2(\alpha_0 x_0)^3$$
$$- 6(\alpha_0 + \gamma x_0)(\gamma x_0)^2]. \quad (12)$$

Multiplying $e^{i\omega_0\tau}$ on both side of (8), we get

$$b_1 + b_2 e^{-i\omega_0\tau} + b_3 e^{-2i\omega_0\tau} + b_4 e^{i\omega_0\tau} = 0, \quad (13)$$

Separating the real and imaginary parts and rearranging, we obtain

$$\begin{cases} (b_2 + b_4)\cos\omega_0\tau + b_2\sin 2\omega_0\tau = -b_1, \\ (b_2 + b_4)\sin\omega_0\tau + b_3\sin 2\omega_0\tau = 0. \end{cases} \quad (14)$$

It follows from (13) that

$$\begin{cases} \cos\omega_0\tau = -\dfrac{b_2 + b_4}{2b_3}, \\ \sin\omega_0\tau = -\dfrac{b_1 b_3 + (b_2 + b - 4)^2}{2b_2(b_2 + b_4)}. \end{cases} \quad (15)$$

Then we derive

$$\left[\frac{b_2 + b_4}{2b_3}\right]^2 + \left[\frac{b_1 b_3 + (b_2 + b_4)^2}{2b_2(b_2 + b_4)}\right]^2 = 1. \quad (16)$$

If the coefficients of the system (4) are given, it is easy to use computer to calculate the roots of (16) (say α_0). Then we have the following results.

Theorem 2 (Existence of Hopf bifurcation parameter) *Let $b_i(i = 1, 2, 3, 4)$ are defined by (9), (10),(11),(12), respectively. For system (4), if α_0 is positive real roots of (16), then Hopf bifurcation point of system (4) is α_0.*

III. NUMERICAL EXAMPLES

In this section, we present some numerical results of system (4) to verify the analytical predictions obtained in the previous section. Let us consider the following system

$$\begin{cases} \dot{x}_1(t) = x_1[\alpha - \alpha x_1(t-2) \\ \qquad\quad - 3x_2(t) - 3x_3(t)], \\ \dot{x}_2(t) = x_2[\alpha - \alpha x_2(t-2) \\ \qquad\quad - 3x_1(t) - 3x_3(t)], \\ \dot{x}_3(t) = x_3[\alpha - \alpha x_3(t-2) \\ \qquad\quad - 3x_1(t) - 3x_2(t)]. \end{cases} \quad (17)$$

From Theorem 2, we can obtain $\alpha_0 \approx 5.2$. If we choose the parameter $\alpha = 4.3$, then system (17) has a positive equilibrium $E(x_0, x_0, x_0) \approx (0.2020, 0.5711, 1.0417)$ and satisfies the conditions indicated in Theorem 2. The positive equilibrium $E \approx (0.4175, 0.4175, 0.4175)$ is asymptotically stable for $\alpha < \alpha_0 \approx 5.2$. Fig. 1 shows that the positive equilibrium $E \approx (0.4175, 0.4175, 0.4175)$ is asymptotically stable when $\alpha = 4.3 < \alpha_0 \approx 5.2$. When α passes through the critical value α_0, the positive equilibrium loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the positive equilibrium $E \approx (0.5383, 0.5383, 0.5383)$. Fig. 2 shows that a family of periodic solutions bifurcate from

the positive equilibrium $E \approx (0.5383, 0.5383, 0.5383)$ when $\alpha = 7 > \alpha_0 \approx 5.2$.
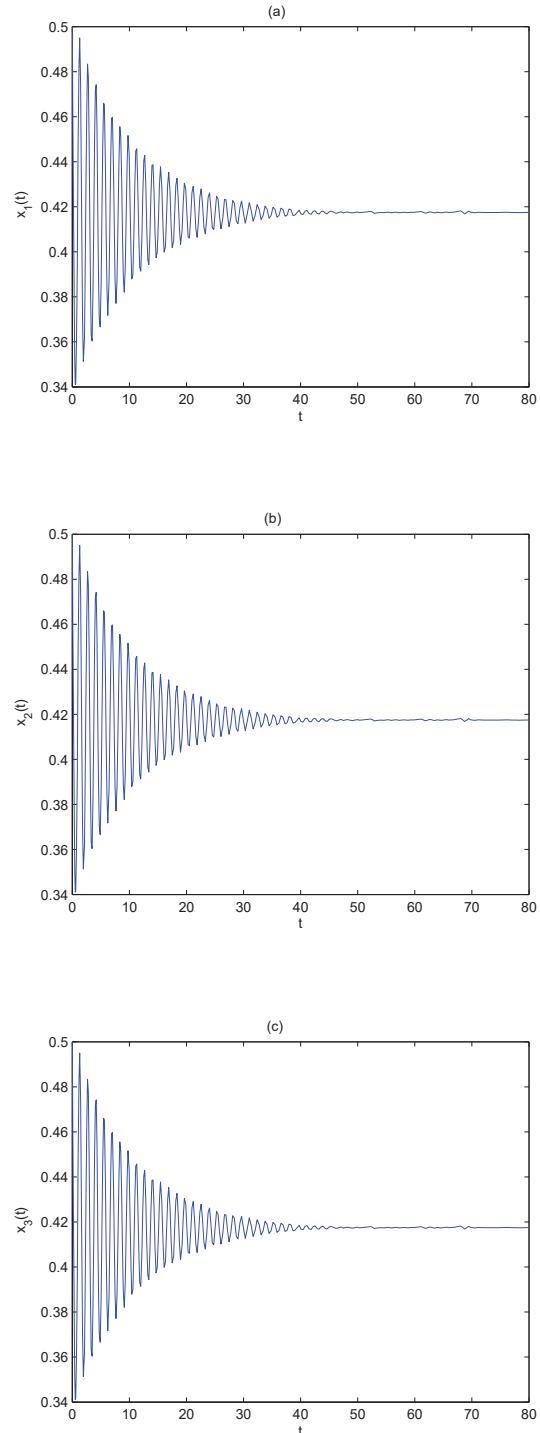

Fig. 1 Behavior and phase portrait of system (17) with $\tau = 2$ and $\alpha = 4.3 < \alpha_0 \approx 5.2$. The positive equilibrium $E \approx (0.4175, 0.4175, 0.4175)$ is asymptotically stable. The initial value is (0.5,0.5,0.5)

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
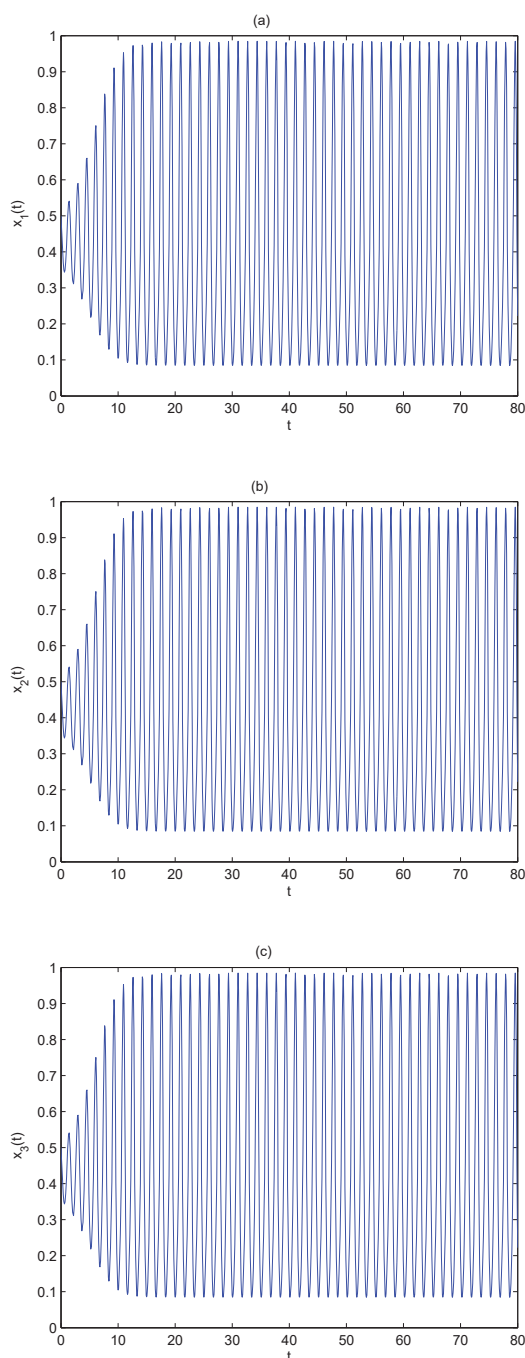Vol:9, No:2, 2015

Fig. 2 Behavior and phase portrait of system (17) with $\tau = 2$ and $\alpha = 7 < \alpha_0 \approx 5.2$. Hopf bifurcation occurs from the positive equilibrium $E \approx (0.5383, 0.5383, 0.5383)$. The initial value is (0.5,0.5,0.5)

IV. CONCLUSIONS

In this paper, by choosing the coefficient α as a bifurcating parameter, we investigated a delayed competitive web-site system in frequency domain approach. It is found that a Hopf bifurcation occurs when the bifurcating parameter α passes through a critical value. Considering computational complexity, the direction and the stability of the bifurcating periodic orbits for system (4) have not been investigated. It

is beyond the scope of the present paper and will be further investigated elsewhere in the near future.

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