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Numerical Inverse Laplace Transform Using Chebyshev Polynomial

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***Abstract*—**In this paper, numerical approximate Laplace transform inversion algorithm based on Chebyshev polynomial of second kind is developed using odd cosine series. The technique has been tested for three different functions to work efficiently. The illustrations show that the new developed numerical inverse Laplace transform is very much close to the classical analytic inverse Laplace transform.

***Keywords*—**Chebyshev polynomial, Numerical inverse Laplace transform, Odd cosine series.

I. Introduction

A. The Laplace Transform

THE theoretical foundation to a class of integral transforms is due to French mathematician Pierre Simon de Laplace (1829-1949) who made use of integral in his work on probability theory [11, p. 112], [13, p. 162]. Laplace transform is the most popular integral transform in which a function $f(t)$ of one variable t transforms into function $F(s)$ of another domain s (s may be complex). Mathematically, the Laplace transform of a function $f(t)$, denoted by $L[f(t)]$ is defined as

$$F(s) = L[(f(t)] = \int_0^\infty e^{-st} f(t)dt, \tag{1}$$

provided integral exists. e^{-st} is called the kernel of the transformation [11, p.114].

Definition 1. A function $f(t)$ is said to be piecewise (sectional) continuous on the interval $[a,b]$ if it is bounded and has at most finitely many discontinuities in that interval.

Definition 2. Let $f(t)$ be piecewise continuous function defined for all positive t in the range $(0,\infty)$ with the property that there exists a real number γ_0 such that

$$\lim_{t\to\infty} |f(t)| e^{-\gamma t} = \begin{cases} 0, \gamma > \gamma_0 \\ \pm\infty, \gamma < \gamma_0 \end{cases}$$

Then, $f(t)$ is said to be of exponential order γ_0 as $t \to \infty$. For $\gamma = \gamma_0$ the above limit may or may not be satisfied [13, p.166].

Theorem 1 (Existence Theorem). A necessary condition for convergence of integral defined by (1) is that $\mathrm{Re}(s) > \gamma_0$, where constant γ_0 is exponential order of $f(t)$.

The technique of Laplace transformation plays a significant role in mathematical applications in basic & social sciences and engineering. It is one of the most powerful techniques for solving differential and integral equations. Laplace transform is an ideal tool for solving initial value problems and plays a key role in modern approach to analysis and design of engineering systems such as electrical circuits and mechanical vibrations [6], [7].

The popularization of Laplace transform was due to the work of English electrical engineer Oliver Heaviside (1850-1925) when he applied a similar approach without proof and lack of mathematical rigor to ordinary differential equation with constant coefficients [11, p. 112]. His approach was widely accepted and spread in the fields of improper integral, asymptotic series and transform theory. It, indeed, took many years to recognize that Laplace provided a theoretical formulation of Heaviside work almost a century before. It was recognized that Laplace transform provided a more systematic alternative approach to ordinary differential equations than by Heaviside.

B. Inverse Laplace Transform

If $L[f(t)] = F(s)$, then the inverse Laplace transform of function $f(t)$ which maps the Laplace transform of a function back to the original function is defined as

$$L^{-1}(F(s)) = f(t)\,. \tag{2}$$

Analytic inversion of Laplace transform is in fact related to the theory of complex variables. The classical and simplest inversion formula is generally called Bromwich integral or Bromwich Mellin Contour integral or sometimes Heaviside inversion formula [1], [6].

Theorem 2 (The Bromwich Inversion Theorem) [1]. Let $f(t)$ possess a continuous derivative and $|f(t)| < ke^{\gamma t}$, where k and γ are the positive constants. Define (1). Then inversion of $F(s)$ is given by the integral

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$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds,$$

$\text{Re}(s) = c$ is so chosen that the poles (singularities) lie to the left on the line $\text{Re}(s) = c$. Usually $F(s)$ is an analytic function on the half plane $\text{Re}(s) > \gamma_0$. $F(s)$ has some singularity on the line $\text{Re}(s) = \gamma_0$.

Theorem 3 (Lerch's Theorem) [1]. If (1) is satisfied by a continuous function $f(t)$ for $\text{Re}(s) > \gamma_0$. then there is no other continuous function which satisfies the given Laplace transform. That is, inverse transform is unique.

C. Numerical Inverse Laplace Transform

Sometimes it is very arduous or even impossible to invert into original time-domain function from its frequency-domain characterization due to difficulty in finding the poles and residues of $F(s)$. Sometimes time-domain function may not be defined analytically, but rather through graphics, experimental measurements, sections or in discrete form. Examples may be of systems with nonlinear frequency dependence. To overcome these situations numerical techniques for finding the inverse Laplace transform was introduced in the sixties by Bellman, Kalaba and Lockett in 1966 instead of analytical expression [10]. Numerical inverse technique has gained the importance due to its various applications. The numerical inverse Laplace transform has been successfully applied in analyzing electromagnetic transients such as uniform, non-uniform and excited transmission lines, transformer and machine windings, underground water power transmission and submarine cables, groundwater solute report and flow problems [8].

A large number of methods for numerically inverting the Laplace transform are available in [1], [3], [4], [6]:

- Methods based on Gaussian quadrature.
- Methods which expand $f(t)$ in exponential function.
- Methods based on Fourier series.
- Methods based on rational approximation.

Legendre function, Gauss Legendre quadrature rule, Pade approximant, Laguerre series and Fourier series method are certain methods feasible in literature [1], [2], [4], [5] for numerically inverting the Laplace transform. In [12], Sheng et al. have successfully tested Invlap, Gavsteh and numerical inverse Laplace transform (NILT) algorithms for finding inverse Laplace transforms for fractional order differential equations.

II. Numerical Inversion Based on Chebyshev Polynomial

Here we follow parallel technique of Papoulis [5] as introduced in [9], whereby making the substitution

$$\sin\theta = e^{-\sigma t}, \sigma > 0. \tag{3}$$

The interval (0, ∞) transformed into $\left(0, \frac{\pi}{2}\right)$ and $f(t)$ becomes

$$f\left(-\frac{1}{\sigma}\log(\sin\theta)\right) = g(\theta). \tag{4}$$

Now (1) takes the form

$$\sigma F(s) = \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{s}{\sigma}-1}\cos\theta\, g(\theta)\,d\theta. \tag{5}$$

By setting $s = (2k+1)\sigma$, $k = 0, 1, 2\ldots$, we have

$$\sigma F((2k+1)\sigma) = \int_0^{\frac{\pi}{2}} (\sin\theta)^{2k}\cos\theta\, g(\theta)\,d\theta. \tag{6}$$

Here we assume that $g\left(\frac{\pi}{2}\right) = f(0) = 0$. In case this is not satisfied then arrange it by subtracting a suitable function from $g(\theta)$. The function $g(\theta)$ can be expanded in $\left(0, \frac{\pi}{2}\right)$ as the odd cosine series

$$g(\theta) = \sum_{k=0}^{\infty} \alpha_k \cos(2k+1)\theta \tag{7}$$

and is valid in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus with $k = 0,1,2\ldots$ we get

$$\frac{4}{\pi}\sigma F(\sigma) = \alpha_0. \tag{8}$$

$$2^2(-1)^1\frac{4}{\pi}\sigma F(3\sigma) = \alpha_1 - \alpha_0.$$

$$\frac{4}{\pi}(-1)^k 2^{2k}\sigma F((2k+1)\sigma) =$$

$$\left\{(-1)^{k-1}\left[\binom{2k}{k-1} - \binom{2k}{k}\right]\alpha_0 \ldots + (-1)^{r-1}\left[\binom{2k}{r-1} - \binom{2k}{r}\right]\alpha_{k-r} + \ldots + \alpha_k\right\}.$$

Thus, α_k can be obtained from (8) by the forward substitution and hence $g(\theta)$ can be obtained from (7).

Define $U_{k-1}(x) = \frac{\cos k\theta}{\cos\theta}$, $x = \sin\theta$, where $U_k(x)$ the Chebyshev polynomial of second kind of degree k is. Thereby using $\cos\theta = \left(1 - e^{-2\sigma t}\right)^{1/2}$, we express

$$f(x) = \left(1 - e^{-2\sigma t}\right)^{1/2}\sum_{k=0}^{\infty}\alpha_k U_{2k}(e^{-\sigma t}). \tag{9}$$

III. Numerical Implementation

We consider here three different functions to exhibit the accuracy of the method.

Example 1. Consider the function

$$F(s)=\frac{1}{s\sqrt{s+a}}$$

having the exact solution $f(t)=\frac{erf\sqrt{at}}{\sqrt{a}}$ [14].

By employing the method, firstly we check the condition $f(0)=0$ by the initial value theorem. In this case

$$f(0)=\lim_{s\to\infty} sF(s)=\lim_{s\to\infty} s\frac{1}{s\sqrt{s+a}}=0$$

As it is satisfying the assumption so the technique is applicable here. Thus, the solution $f(t)$ can be obtained.

Table I represents the coefficients obtained from relation (8). In Table II, we have compared the numerical approximate solution with exact solution of inverse Laplace transform.

TABLE I
CALCULATION OF COEFFICIENTS IN THE EXPANSION OF $f(t)$

k	α_k
0	0.902592444
1	0.05325596
2	0.018235124
3	0.00884848
4	0.005063258
5	0.003252933
6	0.002302184
7	0.001841575
8	0.001760011
9	0.001911068
10	0.001982293

TABLE II
COMPARISONS OF EXACT SOLUTION AND PRESENT APPROXIMATE SOLUTION

t	Exact solution	Present approximate solution	Absolute error
0	0	2.20076E-17	2.20076E-17
0.1	0.345279154	0.345704621	4.25468E-04
0.2	0.472910743	0.472470767	4.39976E-04
0.3	0.561421967	0.562037648	6.15681E-04
0.4	0.628906628	0.627904922	1.00171E-03
0.5	0.682689187	0.683347691	6.58504E-04
0.6	0.726678106	0.728711115	2.03301E-03
0.7	0.763276268	0.763550699	2.74431E-04
0.8	0.794096679	0.792753088	1.34359E-03
0.9	0.820287413	0.820028582	2.58831E-04
1	0.842700735	0.845385366	2.68463E-03

Example 2. Consider the function $F(s)=e^{-a\sqrt{s}}$ with the exact inverse Laplace transform as $f(t)=\frac{a}{2\sqrt{\pi t^3}}e^{\frac{-a^2}{4t}}$ [14]. Firstly, we will check the condition, using L'Hopital's rule

$$\lim_{s\to\infty} sF(s)=\lim_{s\to\infty} se^{-a\sqrt{s}}=\lim_{s\to\infty}\frac{s}{e^{a\sqrt{s}}}=0$$

Hence, the given function is satisfying the assumption. Thus the Numerical Laplace transform of given function is found.

Table III represents the coefficients obtained from relation (9) and in Table IV we have compared the numerical approximate solution with exact inverse by setting $a=2$.

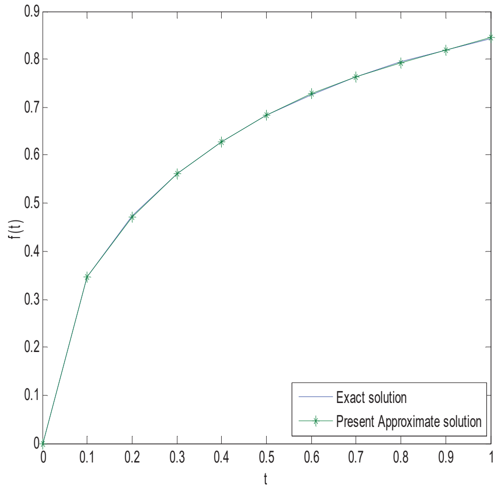

Fig. 1 Comparison of Exact and Approximate Inverse Laplace Transform in Example 1

TABLE III
CALCULATION OF COEFFICIENTS IN THE EXPANSION

k	α_k
0	0.172327795
1	0.0129098
2	-0.07321951
3	-0.030870628
4	0.00512977
5	-0.004579998
6	-0.00855954
7	-0.002343518
8	-0.002370305
9	-0.003177849
10	-0.001894941
11	-0.001744373
12	-0.001754549
13	-0.001376952
14	-0.001293252
15	-0.001196989
16	-0.001024496
17	-0.000855646
18	-0.000417877

TABLE IV
COMPARISONS OF PRESENT APPROXIMATE INVERSE WITH EXACT LAPLACE INVERSE

t	Exact solution	Numerical inverse	Absolute error
0.1	0.00080999	0.000931632	1.21641E-04
0.2	0.04250183	0.042668359	1.66526E-04
0.3	0.12248839	0.122658138	1.69753E-04
0.4	0.18306228	0.182715254	3.47028E-04
0.5	0.21596387	0.216392144	4.28278E-04
0.6	0.22928416	0.228920001	3.64159E-04
0.7	0.23086458	0.230780557	8.40270E-05
0.8	0.22590299	0.226653927	7.50936E-04
0.9	0.21752635	0.217120049	4.06302E-04
1	0.20755375	0.206624265	9.29484E-04

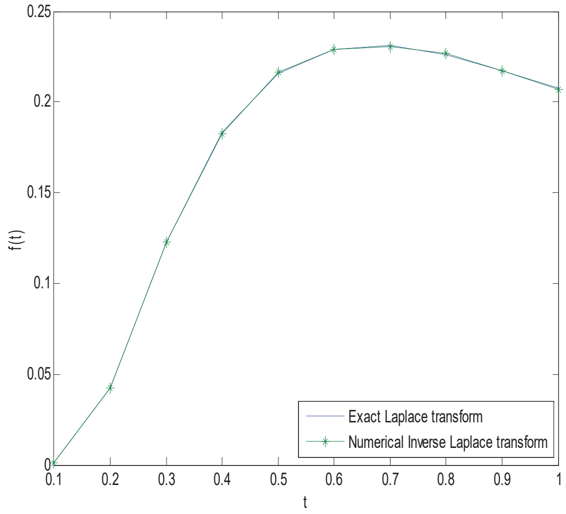

Fig. 2 Comparison of Exact and Approximate Inverse Laplace Transform in Example 2

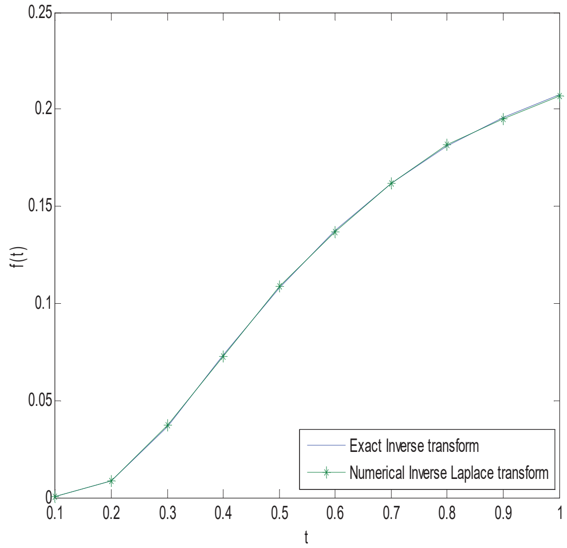

Fig. 3 Comparison of Exact and Approximate Inverse Laplace Transform in Example 3

Example 3. Consider the function

$$F(s) = \frac{e^{-a\sqrt{s}}}{\sqrt{s}}$$

with the exact inverse Laplace transform given by

$$f(t) = \frac{e^{\frac{-a^2}{4t}}}{\sqrt{\pi t}} \text{ [14].}$$

To satisfy the condition of the aforementioned technique, by using initial value theorem and using L'Hopital rule we attain

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} se^{-a\sqrt{s}} = \lim_{s\to\infty} \frac{\sqrt{s}}{e^{a\sqrt{s}}} = 0$$

The given function is satisfying the restriction. Therefore, the Numerical Laplace transform of above function is obtained and is presented through Fig. 3 by considering $a = 2$.

IV. Conclusion

The numerical approximate solutions in examples exhibit the applicability of technique to find the inverse Laplace transform of functions. Solution expressed in terms of odd cosine series is based on Chebyshev polynomial. The result shows that the method is efficient and easy to implement.

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