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Construction and Analysis of Samurai Sudoku

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***Abstract*—**Samurai Sudoku consists of five Sudoku square designs each having nine treatments in each row (column or sub-block) only once such the five Sudoku designs overlaps. Two or more Samurai designs can be joint together to give an extended Samurai design. In addition, two Samurai designs, each containing five Sudoku square designs, are mutually orthogonal (Graeco). If we superimpose two Samurai designs and obtained a pair of Latin and Greek letters in each row (column or sub-block) of the five Sudoku designs only once, then we have Graeco Samurai design. In this paper, simple method of constructing Samurai designs and mutually orthogonal Samurai design are proposed. In addition, linear models and methods of data analysis for the designs are proposed.

***Keywords*—**Samurai design, Graeco samurai design, sudoku design, row or column swap.

I. Introduction

SUDOKU square design consists of treatments that are arranged in a square array such that each row, column or sub-square of the design contains each of the treatments only once [1]. The standard Sudoku square of order 9 entails, a 9 × 9 array of numbers 1 through 9, such that every row, every column and every 3 × 3 sub-block contains each number exactly once. Sudoku squares of any other order $k = pq$ are also similarly defined as an $k \times k$ array of numbers 1 through k such that every row, every column and every $p \times q$ internal block contains each number exactly once, see [2] and [3]. The basic properties of $k \times k$ Sudoku squares with construction procedure are discussed by [1]. In [1] and [4], Sudoku Square is made into a new design and was applied to field experiments. This design can make a layout of k treatments with k replications and control the three-way soil-environmental variation was discussed by [5]. A part from the construction procedure of Sudoku squares also presented the mathematical model and statistical method of analyzing data from a Sudoku square design [1]. They also compared the Latin square design with the Sudoku square design and stated that Sudoku adds a term of box effect in the source of variation, and that the soil-environment variation can be better controlled when the Sudoku square is used in a field experiment. It should show a smaller error, and more precise tests for treatment means. Particularly, when a field experiment is conducted on the area with lumpy soil variation, the Sudoku square design should be recommended [1]. In another approach, [6] discussed the construction of a class of Sudoku designs of order m^2. They presented four different models for analyzing the data obtained from such design. They also gave an illustration of analysis and application these designs in various fields of agriculture. A paper presented by [7] discussed the orthogonal Sudoku square. Two Sudoku squares of order m^2 are said to be mutually orthogonal Sudoku squares. If we superimpose the two Sudoku squares then one may get each pair of numbers only once as discussed in [7]-[9].

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Construction of Sudoku and mutually orthogonal Sudoku squares has been considered by [2], [6]-[10]. These researchers used integers at different modules in the construction of Sudoku squares. For example, [6] outlined the following steps for construction of Sudoku designs of order m^2 sequentially:

1. Write the m^2 numbers from 1 to m^2 in a matrix form sequentially starting from row 1 to row m.
2. Write the m columns obtained in step 1, one by one to get a column of order m^2.
3. Column 2 can be obtained from column 1 by adding 1 to each of its elements and reduce to mod m^2 if it exceeds the value m^2. Proceed in the similar way to complete all the columns.

In construction of mutually orthogonal Sudoku squares, [9] considered m^2 (= $2n + 1$) to be the number of symbols coded by (1, 2, 3,…,,m^2) and is denoted by the set S = (1,2,3,…,m^2). A pair of orthogonal Latin squares is obtained by generating the two initial rows mod(m^2), where m^2 and m are odd numbers, see [9].

One of the problems of these methods is that Latin letters cannot be used directly to represent treatments, and instead codes or numbers 1 to m^2 are used. If treatments are in Latin letter, addition of 1 to each elements (treatments) is impossible, since Latin letters are not real numbers or integers. In addition, only odd order Sudoku design can be constructed with this procedure, while if k = p×q, where p < q, then this procedure cannot be used to construct Sudoku designs.

The application of Kronecker product (direct product) to two Latin squares $A_{n\times n}$ and $B_{m\times m}$ to obtain a Latin square $A \otimes B$ of order n × m is considered by [13]–[17]. Instead of using Kronecker product to Latin squares, [16] and [17] work with two Sudoku Latin squares of order ab and cd and then apply the row and column permutation to obtain a mutually orthogonal Sudoku Latin square of order abcd. The problem of this method is that some mutually orthogonal Sudoku Latin squares cannot be constructed, especially of order 4 and 9 (since there is no Sudoku square of order 2 or 3)

A new class of permutation matrices based on a tensor product of permutation matrices of reverse cyclic stride was proposed by [18]. A permutation matrix is a square matrix with one and only one-unit element in each row and column. In this method, permutation matrix is obtained by cyclic shifting of columns. The problems of this procedure are that matrix must have one and only one-unit element in each row and column, and rearrangement of elements of a given vector after multiplication by a permutation matrix. In addition, Latin letters cannot be used directly in this method.

Sudoku based space filling designs were studied by [11]. Reference [12] discussed joining several Sudoku squares to form what is called Samurai design. A Samurai design consists of five overlapping Sudoku grids, for which several entries are provided, and the remaining entries must be filled subject to each row, column and three-by-three sub-square containing the integers 1 to 9 precisely once [12].

This paper proposed a simple method of constructing of Sudoku design and Samurai designs by using the cyclic permutations of rows (or columns) of an initial sub-block of k treatments. Methods of data analysis from Samurai and orthogonal (Gaeco) Samurai designs are also proposed.

II. Construction

Let D be a block matrix and D_{11} a sub-block of D. Suppose that D contains n Latin letters. To construct a Sudoku design is to simply fix D_{11} in the first row and first column of D. Then to obtain D_{12} in the first raw and second column of D, perform raw swap of D_{11} in cyclic order and so on until each Latin letter occurs once in each row of the first row-block of D. Similarly, to obtain D_{21} in the second row and first column, perform column swap of D_{11} in cyclic order, and then obtain D_{22} in the second row and second column of D by performing row swap of D_{21} in cyclic order and so on until each Latin letter appears once in each row of the second row-block of D. Repeat this procedure until each Latin letter appears once in each column of the column-blocks of D. Then, the block matrix D obtained using this procedure is a Sudoku design.

Alternatively, Let A be an $n \times m$ matrix and B be a $p \times q$ matrix. The direct (or Kronecker) product of A and B (written $A \otimes B$) is defined as the $np \times mq$ matrix [19].

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \cdots & \vdots \\ a_{n1}B & \ldots & a_{nm}B \end{pmatrix}$$

Let A be an $n \times m$ matrix of ones. Suppose that B_i^r *and* B^{c_j} are sub-block matrices having the same elements with B, where $i, j = 1, 2, \cdots, m$. Then, the from Kronecker product of the matrix of ones (A) and B, we can form a bock matrix whose elements are sub-block matrices formed by raw or column swap in a cyclic order. That is,

$$A \otimes B = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \otimes B = \begin{pmatrix} B & B_1^r & B_2^r & \cdots & B_{m-1}^r \\ B^{c_1} & B_1^{c_1 r} & B_2^{c_1 r} & \cdots & B_{m-1}^{c_1 r} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ B^{c_{m-1}} & B_1^{c_{m-1} r} & B_2^{c_{m-1} r} & \cdots & B_{m-1}^{c_{m-1} r} \end{pmatrix}$$

Note that to construct a Sudoku square design using Kronecker product, A and B must be $n \times n$ matrices or n = q and m = p (i.e., $A_{q \times p}$ and $B_{p \times q}$) in the case where B is not a square matrix. To construct a mutually orthogonal Sudoku Square designs using this method, B must contain both the Latin and Greek letters and then perform the Kronecker product (modified) of A and B. Alternatively, if the elements of A are the Greek letters, and the elements of B are the Latin letters, then their Kronecker product gives all the elements of the Graeco Sudoku square, which after some arrangements can form mutually orthogonal Sudoku square or non-orthogonal square. An advantage of this method is that Latin letters, Greek letters, or integers can be used in the construction of Sudoku square designs. For example, if k = 9, p = q = 3, we have $k = 3^2$ and;

$$B = \begin{pmatrix} C & D & E \\ F & G & H \\ I & J & K \end{pmatrix}_{3\times 3}.$$

Then;

$$A \otimes B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \otimes B = \left(\begin{array}{ccc:ccc:ccc} C & D & E & F & G & H & I & J & K \\ F & G & H & I & J & K & C & D & E \\ I & J & K & C & D & E & F & G & H \\ & \cdots & & & \cdots & & & \cdots & \\ G & H & F & J & K & I & D & E & C \\ J & K & I & D & E & C & G & H & F \\ D & E & C & G & H & F & J & K & I \\ & \cdots & & & \cdots & & & \cdots & \\ K & I & J & E & C & D & H & F & G \\ E & C & D & H & F & G & K & I & J \\ H & F & G & K & I & J & E & C & D \end{array}\right)$$

To construct a samurai design, the row-blocks of D are swapped as in Fig. 1. An example of Samurai formed by D_{11} with 3×3 letters is presented in Fig. 2. Similarly, performing row swap of row-blocks of D and joining the grids with Fig. 1 will result in an extended Samurai design presented in Fig. 3. However, to construct Graeco Samurai design D_{11} must contains both Latin and Greek letter of which row (or column) swap of Greek letter be in the reverse cyclic order.

D_{11}	D_{12}	D_{13}		D_{21}	D_{22}	D_{23}
D_{21}	D_{22}	D_{23}		D_{33}	D_{32}	D_{31}
D_{31}	D_{32}	D_{33}	D_{32}	D_{11}	D_{12}	D_{13}
		D_{13}	D_{12}	D_{21}		
D_{21}	D_{22}	D_{23}	D_{22}	D_{31}	D_{32}	D_{33}
D_{31}	D_{32}	D_{33}		D_{11}	D_{12}	D_{13}
D_{11}	D_{12}	D_{13}		D_{21}	D_{22}	D_{23}

Fig. 1 Swap of row-blocks

Similarly, construction of other forms of Samurai designs can be made by using the above procedure. For example, if D_{11} is a 4×2 initial sub-block, then we have the Samurai in Fig. 2.

D_{11}	D_{12}	D_{13}	D_{14}			D_{13}	D_{14}	D_{11}	D_{12}
D_{21}	D_{22}	D_{23}	D_{24}	D_{21}	D_{22}	D_{23}	D_{24}	D_{21}	D_{22}
D_{12}	D_{13}	D_{14}	D_{11}	D_{12}	D_{13}	D_{14}	D_{11}	D_{12}	D_{13}
D_{21}	D_{22}	D_{23}	D_{24}			D_{24}	D_{21}	D_{22}	D_{23}

Fig. 2 Samurai for Sudoku with 4×2 sub-blocks

Example 1. Suppose that:

$$D_{11} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

Swapping row or columns of D_{11} gives the following Samurai design presented in Fig. 2. The resulted Samurai of Fig. 3 can be easily extended to give an extended Samurai design presented in Fig. 4.

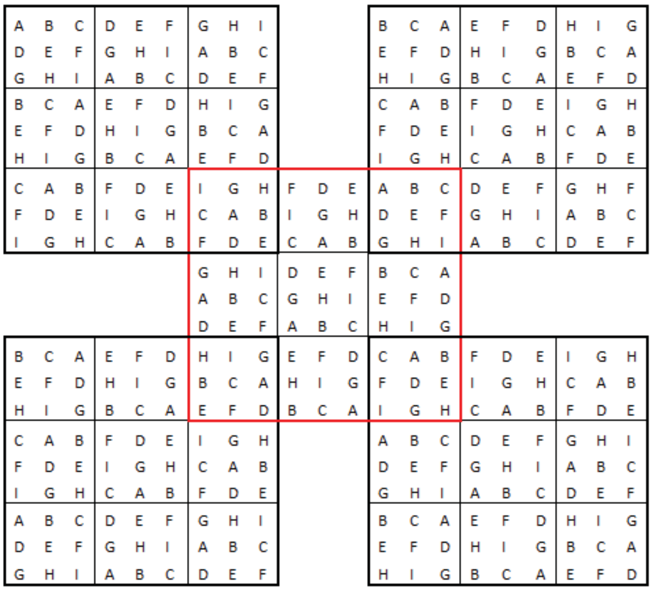

Fig. 3 Samurai design

The following Extended Samurai Design is also formed with the above initial sub-block D_{11}:

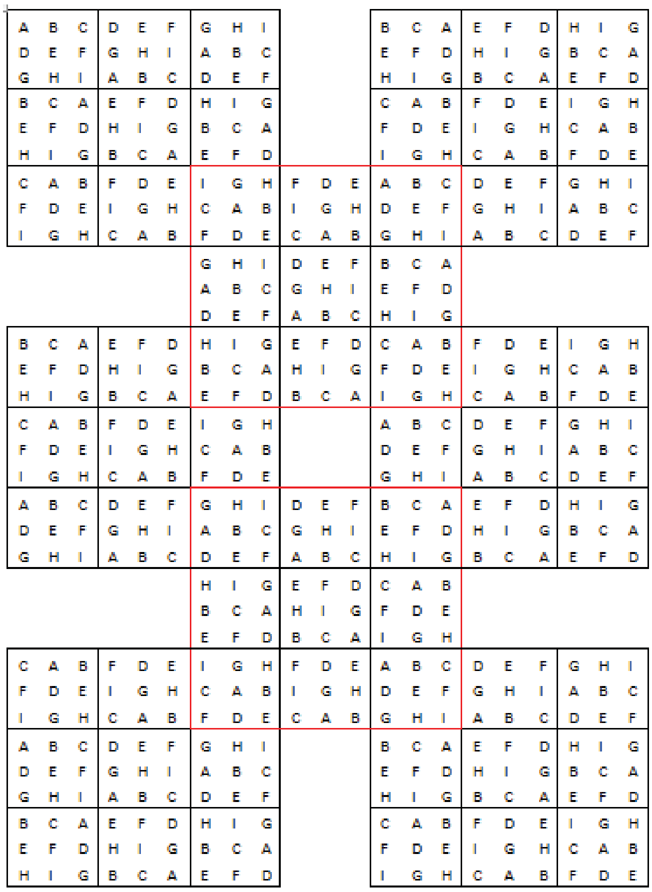

Fig. 4 Extended Samurai Sudoku Design

Example 2. Suppose that:

$$D_{11} = \begin{pmatrix} A\alpha & D\theta & G\varepsilon \\ B\beta & E\phi & H\sigma \\ C\gamma & F\mu & I\lambda \end{pmatrix}$$

Performing row (or column) swapping in cyclic order for D_{11} as described above gives the Graeco Samurai design presented in Fig. 5.

Similarly, other forms of Samurai and Graeco Samurai designs can be constructed using this procedure. For example, if $D_{11} = \begin{pmatrix} H & L \\ I & M \\ J & N \\ K & O \end{pmatrix}$, we have Samurai design presented in Fig. 6. Similarly, we have Graeco Samurai design presented in Fig. 7 if

$$D_{11} = \begin{pmatrix} A\alpha & E\phi \\ B\beta & F\mu \\ C\gamma & G\varepsilon \\ D\theta & H\lambda \end{pmatrix}$$

III. Analysis

Suppose that a Samurai design consists of g Sudoku square designs each of order k such that c sub-squares of the Sudoku designs overlapped. Then, the number of sub-squares of the samurai designs r = $g \times k$ – cs, where s is the number of jointly Samurai designs, i.e., s = 1 for single Samurai design, s = 2 for two joint Samurai designs etc. The linear model proposed for Samurai square designs is as:

$$y_{(ij)lmx} = \mu + \theta_x + \alpha_i + \beta_{jx} + \delta_{lx} + \gamma_{mx} + \varepsilon_{(ij)lmx} \begin{cases} i = 1,2,\dots,k \\ j = 1,2,\dots,w \\ l = 1,2,\dots,k \\ m = 1,2,\dots,k \\ x = 1,2,\dots,g \end{cases}$$

where $y_{(ij)lm}$ is an observed value of the plot in the lth row and mth column, subjected to the ith treatment, jth box of the xth Sudoku design; μ is the grand mean, α_i, β_j, δ_l, γ_m, θ_x, are the main effects of the ith treatment, jth box, lth row, mth column, xth Sudoku design, respectively, ε_{ijm} is the random error.

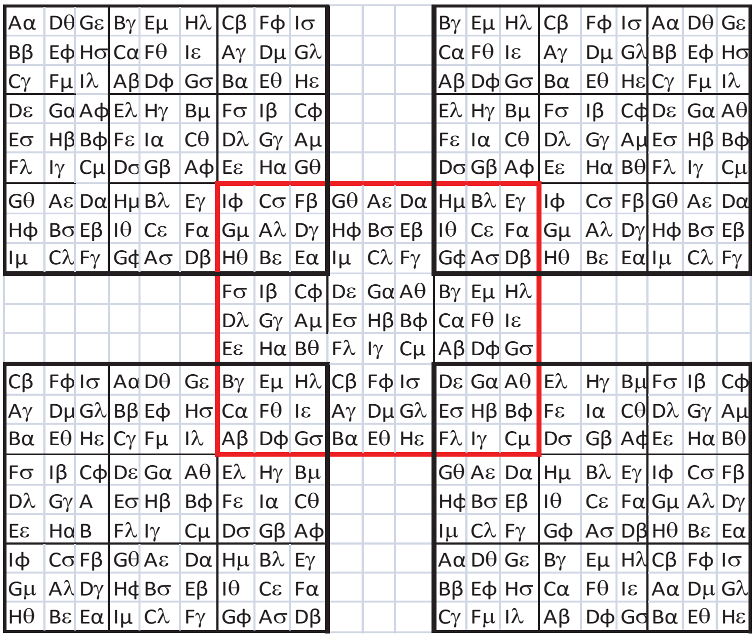

Fig. 5 Graeco Samurai design

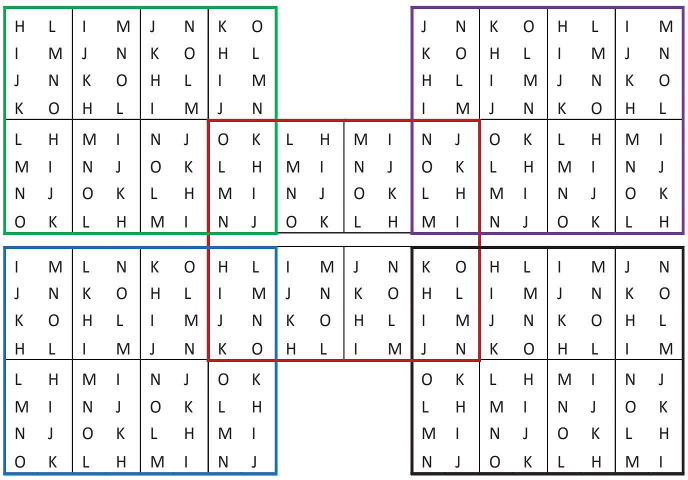

Fig. 6 Samurai design of 8x8 Sudoku design

Aα	Eϕ	Bθ	Fλ	Cε	Gγ	Dμ	Hβ					Cγ	Gε	Dβ	Hμ	Aϕ	Eα	Bλ	Fθ
Bβ	Fμ	Cα	Gϕ	Dλ	Hθ	Aε	Eγ					Dθ	Hλ	Aγ	Eε	Bμ	Fβ	Cϕ	Gα
Cγ	Gε	Dβ	Hμ	Aϕ	Eα	Bλ	Fθ					Aα	Eϕ	Bθ	Fλ	Cε	Gγ	Dμ	Hβ
Dθ	Hλ	Aγ	Eε	Bμ	Fβ	Cϕ	Gα					Bβ	Fμ	Cα	Gϕ	Dλ	Hθ	Aε	Eγ
Eλ	Aθ	Fε	Bγ	Gβ	Cμ	Hα	Dϕ	Eθ	Aλ	Fε	Bγ	Gμ	Cβ	Hϕ	Dα	Eθ	Aλ	Fγ	Bε
Fϕ	Bα	Gλ	Cθ	Hγ	Dε	Eβ	Aμ	Fα	Bϕ	Gλ	Cθ	Hε	Dγ	Eμ	Aβ	Fα	Bϕ	Gθθ	Cλ
Gμ	Cβ	Hϕ	Dα	Eθ	Aλ	Fγ	Bε	Gβ	Cμ	Hϕ	Dα	Eλ	Aθ	Fε	Bγ	Gβ	Cμ	Hα	Dϕ
Hε	Dγ	Eμ	Aβ	Fα	Bϕ	Gθ	Cλ	Hγ	Dε	Eμ	Aβ	Fϕ	Bα	Gλ	Cθ	Hγ	Dε	Eβ	Aμ
Bθ	Fλ	Cγ	Gε	Dμ	Hβ	Aϕ	Eα	Bλ	Fθ	Cγ	Gε	Dβ	Hμ	Aα	Eϕ	Bλ	Fθ	Cε	Gγ
Cα	Gϕ	Dθ	Hλ	Aε	Eγ	Bμ	Fβ	Cϕ	Gα	Dθ	Hλ	Aγ	Eε	Bβ	Fμ	Cϕ	Gα	Dλ	Hθ
Dβ	Hμ	Aα	Eϕ	Bλ	Fθ	Cε	Gγ	Dμ	Hβ	Aα	Eϕ	Bθ	Fλ	Cγ	Gε	Dμ	Hβ	Aϕ	Eα
Aγ	Eε	Bβ	Fμ	Cϕ	Gα	Dλ	Hθ	Aε	Eγ	Bβ	Fμ	Cα	Gϕ	Dθ	Hλ	Aε	Eγ	Bμ	Fβ
Fϕ	Bα	Gλ	Cθ	Hγ	Dε	Eβ	Aμ					Hϕ	Dα	Eλ	Aθ	Fγ	Bε	Gβ	Cμ
Gμ	Cβ	Hϕ	Dα	Eθ	Aλ	Fγ	Bε					Eμ	Aβ	Fϕ	Bα	Gθ	Cλ	Hγ	Dε
Hε	Dγ	Eμ	Aβ	Fα	Bϕ	Gθ	Cλ					Fε	Bγ	Gμ	Cβ	Hα	Dϕ	Eθ	Aλ
Eλ	Aθ	Fε	Bγ	Gβ	Cμ	Hα	Dϕ					Gλ	Cθ	Hε	Dγ	Eβ	Aμ	Fα	Bϕ

Fig. 7 Graeco Samurai with 8x8 Sudoku

TABLE I
ANOVA OUTLINE FOR SAMURAI DESIGNS OF DATA FROM G SUDOKU SQUARE DESIGNS OF ORDER 9

Source	df	ss
Sudoku squares	g – 1	SS_{ss}
Treatments	k – 1	SSt
Rows	g(k – 1)	SSr
Columns	g(k – 1)	SSc
Sub-squares	gk – cs – 1	SSs
Error	gk(k – 3) – k + g + cs +2	SSe
Total	gk^2- 1	

Where SS_{ss} is the total sum of squares for Sudoku designs, SSt is the total sum of squares for treatments, SSr is the total sum of squares for rows, SSc is the total sum of squares for columns, SSs is the total sum of squares for sub-blocks and SSe is the total error sum of squares.

The following linear model is proposed for Graeco Samurai design:

$$y_{(ij)lmx} = \mu + \theta_x + \alpha_i + \beta_{jx} + \delta_{lx} + \gamma_{mx} + \phi_p + \varepsilon_{(ij)lmx} \begin{cases} i = 1,2,...,k \\ j = 1,2,...,w \\ l = 1,2,...,k \\ m = 1,2,...,k \\ x = 1,2,\ldots,g \\ p = 1,2,\ldots,k \end{cases}$$

where $y_{(ij)lm}$ is an observed value of the plot in the *l*th row and *m*th column, subjected to the *i*th treatment, *j*th box of the xth Sudoku design, μ is the grand mean, α_i, β_j, δ_l, γ_m, θ_x, are the main effects of the *i*th Latin letter, pth Greek letter, *j*th box, *l*th row, *m*th column, xth Sudoku design, respectively, ε_{ijm} is the random error.

TABLE II
ANOVA OUTLINE FOR GRAECO SAMURAI DESIGNS OF DATA FROM G SUDOKU SQUARE DESIGNS ORDER 9

Source	df	ss
Sudoku squares	g – 1	SSEE
Latin letters	k – 1	SSL
Greek letters	k – 1	SSG
Rows	g(k – 1)	SSr
Columns	g(k – 1)	SSc
Sub-squares	gk – cs – 1	SSs
Error	gk(k – 3)+ g – 2k + cs +3	SSe
Total	gk^2- 1	

Where SS_{ss} is the total sum of squares for Sudoku designs, SSL is the total sum of squares for Latin letters, SSG is the total sum of squares for Greek letters, SSr is the total sum of squares for rows, SSc is the total sum of squares for columns, SSs is the total sum of squares for sub-blocks and SSe is the total error sum of squares.

IV. CONCLUSION

In this paper, simple methods of constructing Samurai designs and orthogonal (Graeco) Samurai have been developed by cyclic permutation of row (or column) of matrix. Analyses of the designs were also discussed.

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