

Spherical Spectrum Properties of Quaternionic Operators

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Abstract—In this paper, the similarity invariant and the upper semi-continuity of spherical spectrum, and the spherical spectrum properties for infinite direct sums of quaternionic operators are characterized, respectively. As an application of some results established, a concrete example about the computation of the spherical spectrum of a compact quaternionic operator with form of infinite direct sums of quaternionic matrices is also given.

Keywords—Spherical spectrum, Quaternionic operator, Upper semi-continuity, Direct sum of operators.

I. INTRODUCTION

LET \mathbb{H} denote the field of quaternions, which contains all elements of the form $q = q_0 + q_1i + q_2j + q_3k$, where q_0, q_1, q_2 and q_3 are real numbers and i, j, k satisfy:

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

Therefore, we see that the multiplication is noncommutative in \mathbb{H} . For $q = q_0 + q_1i + q_2j + q_3k \in \mathbb{H}$, write

$$\bar{q} = q_0 - q_1i - q_2j - q_3k,$$

$$|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

Quaternions have become increasingly useful in theory and applications. For example, the research of quantum mechanics based on quaternion analysis, quaternion's applications in control systems and computer science. For more material on quaternions and its applications, see S. L. Adler [1], L. P. Horwitz and L. C. Biedenharn [2], and L. Rodman [3].

As in complex functional analysis, the uniform boundedness principle, the open map theorem, the closed graph theorem and the Riesz representation theorem are also valid for quaternionic operators. However, it is not clear whether we could extend the definition of spectrum of a bounded linear operator in a complex Hilbert to a bounded quaternionic operator. The notion of spherical spectrum of an operator acting on quaternionic Hilbert spaces has been introduced only few years ago [4] in the more general context of operators on Banach modules. R. Ghiloni, V. Moretti and A. Perotti [5] investigated the continuous functional calculus for quaternionic operator and also discussed the spherical spectrum and its general properties. M. Fashandi [6] gave some properties of spherical spectrum of compact operators on quaternionic Hilbert spaces. Ghiloni, V. Moretti and A. Perotti [7] also characterized the spherical spectrum for compact normal quaternionic operators.

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By [4] and [5], we know that the spherical spectrum of quaternionic operator is a non-empty compact subset of \mathbb{H} , the spherical spectrum radius of a quaternionic operator is not bigger than norm of such quaternionic operator. But the definition of spherical spectrum is quite different from that of spectrum in complex Hilbert spaces, it is an interesting question that if there are other properties of spherical spectrum of quaternionic operators are similar to the spectral properties of bounded operators acting on complex Hilbert spaces. The goal of this paper is to determine the similarity invariant and the upper semi-continuity of spherical spectrum for quaternionic operators, and the spherical spectrum of infinite direct sums of quaternionic operators acting on quaternionic Hilbert spaces, we obtain Theorem 1 and 2, which are analogous to some results in [8] and [9], furthermore, as an application of Theorem 2, we give an example which is the computation of the spherical spectrum of a compact quaternionic operator with form of infinite direct sums of 2×2 quaternionic matrices.

II. PRELIMINARIES

In this section, we give some notions, definitions and properties which are needed in this paper.

Let H denote the linear vector space over \mathbb{H} with right scalar multiplication, H is called a right quaternionic inner product space if there is a quaternionic scalar product $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{H}$ satisfying the following three properties:

- (1) $\langle f, g \rangle = \langle g, f \rangle$ if $f, g \in H$;
- (2) if $f \in H$, then $\langle f, f \rangle \geq 0$ and $f = 0$ if $\langle f, f \rangle = 0$;
- (3) $\langle f, gp + hq \rangle = \langle f, g \rangle p + \langle f, h \rangle q$ if $p, q \in \mathbb{H}$ and $f, g, h \in H$.

The quaternionic norm of an element $f \in H$ is defined as $\|f\| = \sqrt{\langle f, f \rangle}$. If H is complete with respect to its natural distance $d(f, g) = \|f - g\|$, we call it is a right quaternionic Hilbert space. Similarly, the notion of a left quaternionic Hilbert space has also been defined in [4], here we only focus on right quaternionic Hilbert spaces.

Definition 1. Let H be a right quaternionic Hilbert space. A right quaternionic operator is defined as a map $T : H \rightarrow H$ such that:

$$T(fp + gq) = (Tf)p + (Tg)q$$

for all $f, g \in H$ and $p, q \in \mathbb{H}$.

A right quaternionic operator T is bounded if there exists $K \geq 0$ such that $\|Tf\| \leq K\|f\|$ for all $f \in H$. $\|T\| = \sup\{\|Tf\| : f \in H \text{ and } \|f\| = 1\}$ is said to be the norm of T . Write $B(H)$ as the set of all bounded quaternionic operators.

Similar to [4], without confusion, throughout this paper, the right quaternionic Hilbert space and the right quaternionic

operator are also called as the quaternionic Hilbert space and quaternionic operator, respectively.

For $T \in B(H)$, let $\ker(T)$ and $\text{ran}(T)$ denote the kernel and the range of T , respectively. If $q \in \mathbb{H}$, we write

$$\Delta_q(T) = T^2 - T(q + \bar{q}) + I|q|^2.$$

Since the map $T - Iq$ is not even a right quaternionic operator, Colombo [4] have provided a replacement $\Delta_q(T)$ for the operator $T - Iq$ and generalized equivalently the notion of spectrum of a right quaternionic operator, which is called as spherical spectrum of T , see following definition 2.

Definition 2 ([4]). Let $T \in B(H)$, the spherical resolvent set of T is the set of $\rho_S(T)$ of $q \in \mathbb{H}$ such that:

- (1) $\text{Ker}(\Delta_q(T)) = \{0\}$;
- (2) $\text{Ran}(\Delta_q(T))$ is dense in H ;
- (3) $\Delta_q(T)^{-1} : \text{Ran}(\Delta_q(T)) \rightarrow D(T^2)$ is bounded.

The spherical spectrum $\sigma_S(T)$ of T is defined by

$$\sigma_S(T) = \mathbb{H} \setminus \rho_S(T).$$

R. Ghiloni, V. Moretti and A. Perotti [5] pointed that the spherical spectrum of T can be decomposed into three disjoint circular (i.e. invariant by conjugation) subsets, that is following Definition 3.

Definition 3 ([5]). Let $T \in B(H)$, then

$$\sigma_S(T) = \sigma_{pS}(T) \cup \sigma_{rS}(T) \cup \sigma_{cS}(T),$$

where

- (a) $\sigma_{pS}(T)$ is the spherical point spectrum of T :
 $\sigma_{pS}(T) = \{q \in \mathbb{H} : \text{Ker}(\Delta_q(T)) \neq \{0\}\}$;
- (b) $\sigma_{rS}(T)$ is the spherical residual spectrum of T :
 $\sigma_{rS}(T) = \{q \in \mathbb{H} : \text{Ker}(\Delta_q(T)) = \{0\} \text{ and } \overline{\text{Ran}(\Delta_q(T))} \neq H\}$;
- (c) $\sigma_{cS}(T)$ is the spherical continuous spectrum of T :
 $\sigma_{cS}(T) = \{q \in \mathbb{H} : \text{Ker}(\Delta_q(T)) = \{0\} \text{ and } \overline{\text{Ran}(\Delta_q(T))} = H, \Delta_q(T)^{-1} \notin B(H)\}$.

Let H_n be a quaternionic Hilbert space, $T_n \in B(H_n)$, $H = \bigoplus_{n=1}^{\infty} H_n$, $T = \bigoplus_{n=1}^{\infty} T_n$, that is

$$H = \{f = (f_n) : f_n \in H_n, \|f\|_H^2 = \sum_{n=1}^{\infty} \|f_n\|_{H_n}^2 < +\infty\}$$

and $Tf = (T_n f_n) \in H$.

In what follows, we list Lemma 1, which proof is similar to that of the case of a complex linear bounded operator on complex Hilbert space, here we omit its proof.

Lemma 1. Let $\{T_n\}_{n=1}^{\infty}$ be a uniformly bounded family of compact quaternionic operators and $T = \bigoplus_{n=1}^{\infty} T_n \in B(H)$, then T is a compact quaternionic operator if and only if $\lim_{n \rightarrow \infty} \|T_n\| = 0$.

III. CERTAIN PROPERTIES OF SPHERICAL SPECTRUM OF QUATERNIONIC OPERATORS

In this section, we show that the spherical spectrum for quaternionic operator is similarity invariant and upper semi-continuous, and also describes the spherical spectrum of infinite direct sums of quaternionic operators acting on quaternionic Hilbert spaces.

Proposition 1. Let $T \in B(H)$, then $\sigma_S(T) = \sigma_S(X^{-1}TX)$ for any invertible operator $X \in B(H)$.

Proof. Note that $(q + \bar{q})$ and $|q|^2$ are real numbers, then for any invertible operator $X \in B(H)$

$$\begin{aligned} \Delta_q(X^{-1}TX) &= X^{-1}T^2X - X^{-1}TX(q + \bar{q}) + I|q|^2 \\ &= X^{-1}\Delta_q(T)X, \end{aligned}$$

by Definition 3 (a), (b) and (c), we have

$$\sigma_{pS}(T) = \sigma_{pS}(X^{-1}TX), \quad \sigma_{rS}(T) = \sigma_{rS}(X^{-1}TX)$$

and $\sigma_{cS}(T) = \sigma_{cS}(X^{-1}TX)$. Hence

$$\sigma_S(T) = \sigma_S(X^{-1}TX)$$

for any invertible operator $X \in B(H)$. The proof follows.

In the following, we prove the upper semi-continuity of quaternionic operators, we first give Lemma 2. To be brief, if $f_n, f \in H$ and $\lim_{n \rightarrow \infty} \|f_n - f\| = 0$, we write as $f_n \rightarrow f$. In addition, if $T_n, T \in B(H)$ and $\lim_{n \rightarrow \infty} \|T_n - T\| = 0$, we also write as $T_n \rightarrow T$.

Lemma 2. Let $T_n, T \in B(H)$ and $q_n, q \in \mathbb{H}$. If $\Delta_{q_n}(T_n) \rightarrow \Delta_q(T)$, then

- (1). If $\text{Ker}(\Delta_q(T)) = \{0\}$ then $\text{Ker}(\Delta_{q_n}(T_n)) = \{0\}$;
- (2). If $\text{Ran}(\Delta_q(T))$ is dense in H , then there exists a natural number N such that $\text{Ran}(\Delta_{q_n}(T_n))$ is dense in H if $n \geq N$;
- (3). If $q \in \rho_S(T)$, then there exists a natural number N such that $\Delta_{q_n}(T_n)^{-1} : \text{Ran}(\Delta_{q_n}(T_n)) \rightarrow D(T_n^2)$ is bounded if $n \geq N$.

Proof. (1). Let $f \in \text{Ker}(\Delta_{q_n}(T_n))$, then

$$\begin{aligned} \|\Delta_q(T)f\| &= \|\Delta_{q_n}(T_n)f - \Delta_q(T)f\| \\ &\leq \|\Delta_{q_n}(T_n) - \Delta_q(T)\| \|f\|. \end{aligned}$$

Since $\text{Ker}(\Delta_q(T)) = \{0\}$ and $\Delta_{q_n}(T_n) \rightarrow \Delta_q(T)$, we have $f = 0$. Thus $\text{Ker}(\Delta_{q_n}(T_n)) = \{0\}$.

(2). Let $f \in H$, note that $\text{Ran}(\Delta_q(T))$ is dense in H , for arbitrary $\varepsilon > 0$, then there exists $g_\varepsilon \in H$ such that

$$\Delta_q(T)g_\varepsilon = f'_\varepsilon \text{ and } \|f'_\varepsilon - f\| < \varepsilon/2.$$

Let $\Delta_{q_n}(T_n)g_\varepsilon = f''_\varepsilon$, then

$$\begin{aligned} \|f - f''_\varepsilon\| &\leq \|f - f'_\varepsilon\| + \|f'_\varepsilon - f''_\varepsilon\| \\ &\leq \|f - f'_\varepsilon\| + \|\Delta_q(T) - \Delta_{q_n}(T_n)\| \|g_\varepsilon\|. \end{aligned}$$

Since $\Delta_{q_n}(T_n) \rightarrow \Delta_q(T)$, let $n \rightarrow \infty$, we can imply that there exists a natural number N such that $\|f - f''_\varepsilon\| < \varepsilon$ if $n \geq N$. Whence there exists a natural number N such that $\text{Ran}(\Delta_{q_n}(T_n))$ is dense in H if $n \geq N$.

(3). By Lemma's assumption and Lemma 2 (1), then

$$\Delta_{q_n}(T_n)^{-1} : \text{Ran}(\Delta_{q_n}(T_n)) \rightarrow D(T_n^2)$$

is well defined. Since $q \in \rho_S(T)$, one has that there exists a constant $c > 0$ such that $c\|h\| \leq \|\Delta_q(T)h\|$.

Take $h = (\Delta_{q_n}(T_n))^{-1}f$, then

$$\begin{aligned} c\|\Delta_{q_n}(T_n)^{-1}f\| &\leq \|(\Delta_q(T) - \Delta_{q_n}(T_n))\| \|\Delta_{q_n}(T_n)^{-1}f\| \\ &\quad + \|\Delta_{q_n}(T_n)\Delta_{q_n}(T_n)^{-1}\| \|f\|. \end{aligned}$$

According to $\Delta_{q_n}(T_n) \rightarrow \Delta_q(T)$, thus there exists a natural number N such that

$$\|\Delta_q(T) - \Delta_{q_n}(T_n)\| < c/2,$$

for $n \geq N$. By above arguments, for $n \geq N$,

$$\frac{1}{2}c\|\Delta_{q_n}(T_n)^{-1}f\| \leq \|f\|.$$

Theorem 1. $\sigma_S(T)$ is upper semi-continuous.

Proof. Let $T_n \in B(H)$ and $T_n \rightarrow T$, let $q_n \in \sigma_S(T_n)$, $q \in \mathbb{H}$ and $\lim_{n \rightarrow \infty} |q_n - q| = 0$. Since

$$\begin{aligned} & \|\Delta_{q_n}(T_n) - \Delta_q(T)\| \\ & \leq \|T_n - T\|\|T_n + T\| + \|T_n - T\|\|q_n\| \\ & + |q_n - q|\|T\| + \|T_n - T\|\|q_n\| \\ & + |\bar{q}_n - \bar{q}|\|T\| + I(|q_n| - |q|)(|q_n| + |q|), \end{aligned}$$

it follows that $\lim_{n \rightarrow \infty} \|\Delta_{q_n}(T_n) - \Delta_q(T)\| = 0$. Note that $q_n \in \sigma_S(T_n)$, by Lemma 2, then $q \in \sigma_S(T)$. The proof is completed.

The following Theorem 2 describes the spherical spectrum of infinite direct sums of quaternionic operators acting on quaternionic Hilbert spaces.

Theorem 2. Let $\{T_n\}_{n=1}^{\infty}$ be a uniformly bounded family of quaternionic operators and $T = \bigoplus_{n=1}^{\infty} T_n \in B(H)$, then

- (1) $\sigma_{pS}(T) = \bigcup_{n=1}^{\infty} \sigma_{pS}(T_n)$,
- (2) $\sigma_S(T) = [\bigcup_{n=1}^{\infty} \sigma_S(T_n)] \cup \sigma$, where $\sigma = \{\lambda \notin \bigcup_{n=1}^{\infty} \sigma_S(T_n) : \{\|\Delta_q(T_n)^{-1}\|\}_{n=1}^{\infty} \text{ is not uniformly bounded}\}$,
- (3) $\sigma_{rS}(T) = [\bigcup_{n=1}^{\infty} \sigma_{rS}(T_n)] \cap (\sigma_{pS}(T))^c$,
- (4) $\sigma_{cS}(T) = [\bigcup_{n=1}^{\infty} \sigma_{cS}(T_n)] \cap (\sigma_{pS}(T))^c \cap [\bigcup_{n=1}^{\infty} \sigma_{rS}(T_n)]^c$.

Proof. (1). Since $T = \bigoplus_{n=1}^{\infty} T_n$,

$$\begin{aligned} \Delta_q(T) &= T^2 - T(q + \bar{q}) + I|q|^2, \\ \Delta_q(T_n) &= T_n^2 - T_n(q + \bar{q}) + I|q|^2, \end{aligned}$$

by simple computation, we have

$$\Delta_q(T) = \bigoplus_{n=1}^{\infty} \Delta_q(T_n). \quad (1)$$

By the equality (1), we have

$$\sigma_{pS}(T) \subset \bigcup_{n=1}^{\infty} \sigma_{pS}(T_n).$$

If $q \in \bigcup_{n=1}^{\infty} \sigma_{pS}(T_n)$, then there exists a natural number n_0 such that $q \in \sigma_{pS}(T_{n_0})$. Again apply the equality (1), then

$$\bigcup_{n=1}^{\infty} \sigma_{pS}(T_n) \subset \sigma_{pS}(T).$$

(2). By (1), if $q \in (\bigcup_{n=1}^{\infty} \sigma_S(T_n)) \setminus (\bigcup_{n=1}^{\infty} \sigma_{pS}(T_n))$, then there exists a natural number n_1 such that

$$\overline{\text{Ran}(\Delta_q(T_{n_1}))} \neq H_{n_1},$$

or

$$\overline{\text{Ran}(\Delta_q(T_{n_1}))} = H_{n_1}, \quad (\Delta_q(T_{n_1}))^{-1} \notin B(H_{n_1}).$$

When $\overline{\text{Ran}(\Delta_q(T_{n_1}))} \neq H_{n_1}$, then

$$\overline{\text{Ran}(\Delta_q(T))} \neq H.$$

Hence $q \in \sigma_S(T)$. When

$$\overline{\text{Ran}(\Delta_q(T_{n_1}))} = H_{n_1}, \quad (\Delta_q(T_{n_1}))^{-1} \notin B(H_{n_1}),$$

since $\Delta_q(T)^{-1} = \bigoplus_{n=1}^{\infty} \Delta_q(T_n)^{-1}$, we have $\Delta_q(T)^{-1} \notin B(H)$. Hence

$$\bigcup_{n=1}^{\infty} \sigma_S(T_n) \subset \sigma_S(T).$$

If $q \in \sigma$, then

$$\Delta_q(T)^{-1} : \text{Ran}(\Delta_q(T)) \rightarrow D(T^2)$$

is not bounded. Thus $\sigma \subset \sigma_{cS}(T)$, $\sigma \subset \sigma_S(T)$ and

$$[\bigcup_{n=1}^{\infty} \sigma_S(T_n)] \cup \sigma \subset \sigma_S(T).$$

Conversely, if $q \in \sigma_S(T) \setminus \sigma_{pS}(T)$, by Lemma 2 (1), then there exist natural numbers m_1, m_2 such that

$$\overline{\text{Ran}(\Delta_q(T_{m_1}))} \neq H_{m_1} \quad (2)$$

or

$$\overline{\text{Ran}(\Delta_q(T_{m_2}))} = H_{m_2}, \quad (\Delta_q(T_{m_2}))^{-1} \notin B(H_{m_2}). \quad (3)$$

When the inequality (2) is valid, then $q \in \sigma_S(T_{m_1})$.

When the equality (3) is valid, then $q \in \sigma$. So

$$\sigma_S(T) \subset [\bigcup_{n=1}^{\infty} \sigma_S(T_n)] \cup \sigma.$$

Hence

$$[\bigcup_{n=1}^{\infty} \sigma_S(T_n)] \cup \sigma = \sigma_S(T).$$

(3). If $q \in \sigma_{rS}(T)$, then

$$\ker(\Delta_q(T)) = \{0\} \text{ and } \overline{\text{Ran}(\Delta_q(T))} \neq H.$$

Hence $q \in (\sigma_{pS}(T))^c$ and there exists a natural number n_0 such that

$$\overline{\text{Ran}(\Delta_q(T_{n_0}))} \neq H_{n_0}. \quad (4)$$

By the inequality (4), we have $q \in \bigcup_{n=1}^{\infty} \sigma_{rS}(T_n)$. Thus

$$\sigma_{rS}(T) \subset [\bigcup_{n=1}^{\infty} \sigma_{rS}(T_n)] \cap (\sigma_{pS}(T))^c.$$

By the Definition 3 (a), then

$$[\bigcup_{n=1}^{\infty} \sigma_{rS}(T_n)] \cap (\sigma_{pS}(T))^c \subset \sigma_{rS}(T)$$

is clear valid. Hence

$$\sigma_{rS}(T) = [\bigcup_{n=1}^{\infty} \sigma_{rS}(T_n)] \cap (\sigma_{pS}(T))^c.$$

(4). The proof is analogous to that of (3), Here we omit it. Thus, the proof of Theorem 2 is completed.

Remark 1. Theorem 1 and 2 are analogous to some results in [8] and [9] for complex Hilbert spaces.

Corollary 1. Let $\{T_n\}_{n=1}^{\infty}$ be a uniformly bounded family of compact quaternionic operators, $T = \bigoplus_{n=1}^{\infty} T_n \in B(H)$. If T is a compact quaternionic operator, then

$$\sigma_S(T) = \sigma_{pS}(T) \cup \{0\}.$$

Proof. For every n , apply [6, Corollary 2] to $T_n \in B(H_n)$, again apply Theorem 2, then

$$\sigma_S(T) = \sigma_{pS}(T) \cup \sigma.$$

where σ is same as that of Theorem 2 (2). Since T is a compact quaternionic operator, by Lemma 1, we have

$$\lim_{n \rightarrow \infty} \|T_n\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|\Delta_0(T_n)\| = 0.$$

If $0 \notin \sigma_{pS}(T)$, then $\Delta_0(T_n)$ is invertible, moreover

$$\lim_{n \rightarrow \infty} \|(\Delta_0(T_n))^{-1}\| = \infty.$$

Hence $0 \in \sigma$. By above arguments and [6, Corollary 2], then $\sigma_S(T) = \sigma_{pS}(T) \cup \{0\}$. The proof is completed.

As an application of Theorem 2 and Corollary 1, we give the following example to illustrate the computation of the spherical spectrum of the direct sums of quaternionic matrices.

Example. Let $A_n = \frac{1}{n}A$, $T = \bigoplus_{n=1}^{\infty} A_n$, where A is a quaternionic matrix with matrix representation

$$A = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}.$$

Note that $\|A_n\| \rightarrow 0$, $n \rightarrow \infty$, by Lemma 1, then T is a bounded compact quaternionic operator. Since

$$\begin{aligned} \Delta_q(A_n) &= A_n^2 - 2q_0A_n + |q|^2I \\ &= \frac{1}{n^2}A^2 - \frac{2}{n}q_0A + |q|^2I \\ &= \frac{1}{n^2}[A^2 - 2nq_0A + n^2|q|^2I] \\ &= \frac{1}{n^2} \begin{bmatrix} 1 - 2nq_0 + n^2|q|^2 & 0 \\ 0 & -1 - 2nq_0i + n^2|q|^2 \end{bmatrix}, \end{aligned}$$

we can imply that if $q \in \sigma_{pS}(A_n)$, then

$$1 - 2nq_0 + n^2|q|^2 = 0 \quad (5)$$

or

$$-1 - 2nq_0i + n^2|q|^2 = 0. \quad (6)$$

By the equality (5), then $1 - 2nq_0 + n^2|q|^2$

$$\begin{aligned} &= 1 - 2nq_0 + n^2q_0^2 + n^2q_1^2 + n^2q_2^2 + n^2q_3^2 \\ &= (1 - nq_0)^2 + n^2q_1^2 + n^2q_2^2 + n^2q_3^2 \\ &= 0 \end{aligned}$$

Hence,

$$q_0 = \frac{1}{n}, \quad q_1 = q_2 = q_3 = 0.$$

By the equality (6), note that $-1 + n^2|q|^2$ is real, then $q_0 = 0$ and $n^2|q|^2 = 1$, that is

$$q_0 = 0, \quad q_1^2 + q_2^2 + q_3^2 = \frac{1}{n^2}.$$

By Theorem 2 (1), $\sigma_{pS}(T) =$

$$\bigcup_{n=1}^{\infty} \left\{ \frac{1}{n} \right\} \cup \left\{ q : q \in \mathbb{H}, q_0 = 0, q_1^2 + q_2^2 + q_3^2 = \frac{1}{n^2} \right\}. \quad (7)$$

When $q \notin \sigma_{pS}(T)$, again use Theorem 2 (1), then

$$1 - 2nq_0 + n^2|q|^2 \neq 0$$

and

$$-1 - 2nq_0i + n^2|q|^2 \neq 0.$$

By a property of quaternion, write

$$\begin{aligned} a_{nq} &= \frac{1}{|1 - 2nq_0 + n^2|q|^2|} (1 - 2nq_0 + n^2|q|^2), \\ b_{nq} &= \frac{1}{|-1 - 2nq_0i + n^2|q|^2|} (-1 - 2nq_0i + n^2|q|^2). \end{aligned}$$

Let

$$B_n = n^2 \begin{bmatrix} a_{nq} & 0 \\ 0 & b_{nq} \end{bmatrix}, \quad (8)$$

Then

$$\Delta_q(A_n)B_n = B_n\Delta_q(A_n) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (9)$$

By the equalities (8) and (9), then

$$\sigma_{rS}(T) = \emptyset \text{ and } \sigma_{cS}(T) = \emptyset.$$

Note that

$$\begin{aligned} \Delta_q(A_n)^{-1} &= B_n, \\ \|B_n\| &= \max\{n^2|a_{nq}|, n^2|b_{nq}|\}, \\ |a_{nq}| &= \frac{1}{|1 - 2nq_0 + n^2|q|^2|}, \\ |b_{nq}| &= \frac{1}{|-1 - 2nq_0i + n^2|q|^2|}, \end{aligned}$$

by simple computation, if $q \neq 0$, we can imply that

$$\lim_{n \rightarrow \infty} n^2|a_{nq}| = \frac{1}{|q|^2}, \quad \lim_{n \rightarrow \infty} n^2|b_{nq}| = \frac{1}{|q|^2}.$$

If $q = 0$, then

$$\lim_{n \rightarrow \infty} n^2|a_{nq}| = \infty, \quad \lim_{n \rightarrow \infty} n^2|b_{nq}| = \infty.$$

Hence $\lim_{n \rightarrow \infty} \|B_n\| = \infty$.

By above arguments, we have

$$\sigma_S(T) = \sigma_{pS}(T) \cup \{0\}, \quad (10)$$

where $\sigma_{pS}(T)$ is the same as the equality (7).

Remark 2. In this example, if we consider T as a bounded operator in complex Hilbert spaces, then its spectrum is

$$\{0\} \cup \left(\bigcup_{n=1}^{\infty} \left\{ \frac{1}{n}, \frac{i}{n} \right\} \right). \quad (11)$$

Combine the equality (11) with (7) and (10), thus the spherical spectrum of the compact operator T is quite different from the spectrum in complex Hilbert spaces.

Acknowledgement. This work was supported by Natural Science Foundation of Shandong province in China (Grant No. BS2013SF014).

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