Neuron Dynamics of Single-Compartment Traub Model for Hardware Implementations

J. C. Moctezuma, V. Breña-Medina, Jose Luis Nunez-Yanez and Joseph P. McGeehan

Abstract**—**In this work we make a bifurcation analysis for a single compartment representation of Traub model, one of the most important conductance-based models. The analysis focus in two principal parameters: current and leakage conductance. Study of stable and unstable solutions are explored; also Hop-bifurcation and frequency interpretation when current varies is examined. This study allows having control of neuron dynamics and neuron response when these parameters change. Analysis like this is particularly important for several applications such as: tuning parameters in learning process, neuron excitability tests, measure bursting properties of the neuron, etc. Finally, a hardware implementation results were developed to corroborate these results.

Keywords**—**Traub model, Pinsky-Rinzel model, Hopf bifurcation, single-compartment models, Bifurcation analysis, neuron modeling.

I. Introduction

THE primary goal of conductance-based models is try to incorporate as much cellular detail as is possible in order to have realistic neuron models. One of the most important features of these models is that they are biophysical compatible and hence neuroscientists, biologists, psychologists can, at certain level, study the properties and co-relate directly parameters with their biological counterparts.

Besides the well know Hodgkin-Huxley (H-H) model; there are several conductance-based models which highlight different features in the neuron dynamics. One of the most used models, which take into account information about calcium ion channel Ca^{2+}, is Traub Model [1]; where the conductance g_{Ca} is high-voltage activated type. Calcium dynamics is another important element in the chemical and electrical behavior in the neuron. This model can reproduce some burst patterns that H-H model does not exist [2].

The original work of Traub et al. consists in of a 19-compartment scheme for a CA3 hippocampal pyramidal neuron. However compute all these compartments to represent a single cell require big computational resources. In order to simplify this model, Pinsky and Rizel (P-R) [3] elaborated a 2-compartment model equivalent to original Traub, maintaining the essential ionic currents dynamics and capable of producing similar phenomena to the Traub model. The two-compartment model includes two parts: a soma-like, which has the Na^+ and K^+ activated currents; and a distal dendrite-like, where Ca^{2+} activated and Ca^{2+} dependent of potassium currents are considered.

The present research takes in consideration only the soma-like part of the model. Since our primary investigation is about biophysically compatible neuro-simulators for hardware implementations. It is important to keep good performance in order to have biological compatibility and reproduce results from real nervous systems.

In addition, it is of particular interest of this research to study the neuron dynamics behavior when two parameters change: applied current (I_e) and leakage conductance (g_l). Applied current is important since we want to have control about when neuron fires or produce bursting/periodic spikes. This is useful for some kind of experiments such as tuning parameters in learning process, neuron excitability tests, measure bursting properties of the neuron, etc.[4]-[6].

Bifurcations mechanisms involved are involved in the generation of action potentials (spikes) by neurons. Hence a bifurcation study can determined the neuro-computational properties and dynamics of the cells [7], [8].

Using common parameter values of Traub model at resting state, it produces periodic spikes, even without any current applied; then it is fundamental to maintain neuron at equilibrium voltage when neuron it is at resting state and without compromising the dynamics of the neuron. This can be done adjusting the value of leakage conductance. Another interesting aspect about leakage conductance, which mainly consists of chloride ions, is that it is related with some ion-channel diseases and can be used in feedback control schemes to lead new electrical stimulation systems [5]-[9]. This issue will be topic of discussion for future work.

We intend that our neuro-simulator platform [10] can simulate either single or two-compartment (soma+dendrite) type of neurons. Here we study the single compartment neuron dynamics through bifurcation techniques in order to analyze the excitability properties when a particular current is injected to the neuron.

II. Mathematics of Traub Model

Equation (1) represent the soma-like based on the original work of Pinsky-Rinzel [3]. Where C_m is the capacitance for the cell membrane ($\mu F/cm^2$); V represent the membrane potential (mV);$\bar{g}_i$ and E_i are the maximal conductance and the equilibrium potential of the ionic specie i respectively (mS/cm^2 and mV); I_e stands for injected current ($\mu A/cm^2$); m, h and n are the unitless activation/inactivation gating variables having the form of the first kinetic formula. These variables

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depends of the functions α(V) and β(V), which describe the transition rates between open and closed states of the channels.

Like other conductance-based models, the gating variables for Traub Model has specific representations for α(V) and β(V) measured experimentally, Table I shows these values for a resting potential of -60 mV.

$$V' = \left[-\bar{g}_{Na} m_{\infty}^{2} h(V - E_{Na}) - \bar{g}_{K_{DR}} n(V - E_{K_{DR}}) - \bar{g}_{L}(V - E_{L}) + I_e\right]/C_m$$

$$m' = \alpha_m(V)(1 - m) - \beta_m(V)m \quad (1)$$

$$n' = \alpha_n(V)(1 - n) - \beta_n(V)n$$

$$h' = \alpha_h(V)(1 - h) - \beta_h(V)h$$

TABLE I
SPECIFIC VALUES OF GATING VARIABLES IN TRAUB MODEL

Gating variable	α(V)	β(V)
m	$\frac{-0.32(V+46.9)}{\exp\left[\frac{-(V+46.9)}{4}\right]-1}$	$\frac{0.28(V+19.9)}{\exp\left[\frac{(V+19.9)}{5}\right]-1}$
h	$0.128\exp\left[\frac{-(V+43)}{18}\right]$	$\frac{4}{\exp\left[\frac{-(V+20)}{5}\right]+1}$
n	$\frac{-0.016(V+24.9)}{\exp\left[\frac{-(V+24.9)}{5}\right]-1}$	$0.25\exp\left[\frac{-(V+40)}{40}\right]$

III. DYNAMICS NEURON ANALYSIS AND RESULTS

The first step to analyze neuron dynamics is finding the possible equilibrium points given in (1). An equilibrium point is defined as the set (V_e, m_e, h_e, n_e) where there is no more change in the behavior of each variable; i.e. the derivative with respect to time is zero. So right parts of (1) are zero, and then we have the new set of steady state equations for the system:

$$\begin{gathered} \bar{g}_{Na} m_{\infty}^{2} h(V - E_{Na}) + \bar{g}_{K_{DR}} n(V - E_{K_{DR}}) = -\bar{g}_{L}(V - E_{L}) \\ m = \alpha_m(V)/[\alpha_m(V) + \beta_m(V)] \\ n = \alpha_n(V)/[\alpha_n(V) + \beta_n(V)] \\ h = \alpha_h(V)/[\alpha_h(V) + \beta_h(V)] \end{gathered} \quad (2)$$

For the particular equation involving the three ion channel currents, and because we are interested in the behavior starting with a zero current applied (I_e=0). We can find the equilibrium points with a simulation of voltage-clamped technique which obey the equation:$I_{Na} + I_{K\_DR} = -I_L$.

We have found two set of equilibrium points (A and B)where these two currents intersect; with this equilibrium points, we can find the equilibrium set (V_e, m_e, h_e, n_e), which correspond to the values where the set of in (2) becomes zero.

It has to be mentioned that point A is not properly an intersection point between currents, but it is approximated crossing spot and hence equilibrium point; this is due to instability in default point for leakage conductance ($g_l = 0.3$). However when this conductance is slightly shifted, the stability is achieved; we will discuss this in Section III.

In order to analyze the stability for this equilibrium points and since the Jacobian matrix $J(X_e)$ at the equilibrium point X_e is used in the linearization of original non-linear system; the Jacobian matrix is calculated, where *fv, fm, fh,* and *fn* have the form of the right side in (1).

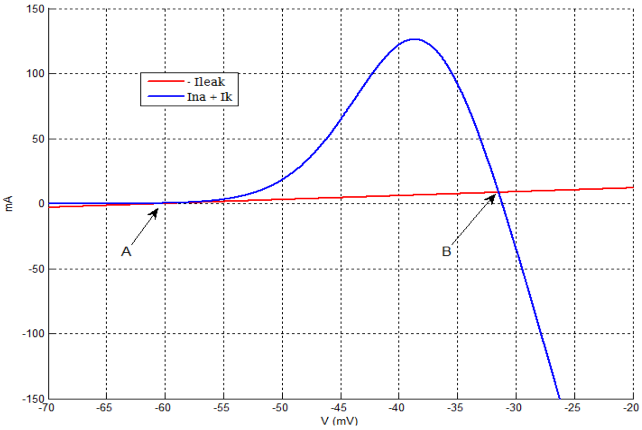


Fig. 1 Relation between the three currents with respect to voltage clamped simulation

TABLE II
EQUILIBRIUM POINTS FOR SINGLE COMPARTMENT TRAUB MODEL

Equilibrium point	V_e	m_e	h_e	n_e
A	-58.649	0.01902	0.99428	0.00158
B	-31.462	0.58412	0.1552	0.16071

$$J = \begin{bmatrix} \frac{dfv}{dv} & \frac{dfv}{dm} & \frac{dfv}{dh} & \frac{dfv}{dn} \\ \frac{dfm}{dv} & \frac{dfm}{dm} & 0 & 0 \\ \frac{dfh}{dv} & 0 & \frac{dfh}{dh} & 0 \\ \frac{dfn}{dv} & 0 & 0 & \frac{dfn}{dn} \end{bmatrix}$$

The Eigen-values of equilibrium points A and B are solved and interpret next:

$$\Psi_A = -10.6314; -0.0132; -0.4818; -0.3078$$
$$\Psi_B = -3.075 + 7.206i; -3.075 - 7.206i; 1.427; -0.291$$

The equilibrium point A, with its four negative values is asymptotically stable node; i.e. membrane potential will remain at its rest state unless some excitation pulse generate an action potential, then membrane voltage will return to the equilibrium state again. According to point B, because the presence of positive value it is unstable, moreover, since it has a couple of conjugate eigen-values, which makes the appearance of *Hopf bifurcation*, i.e. stable or unstable limit cycles [11], [12]. As we now, limit cycles is the graphical representation of bursting and periodic spiking for a neuron. So, if we want to study the effects when parameters such I_e or g_l change, then we need to focus on point B. So equilibrium and initial condition in point B is used to start the bifurcation analysis using the tool XPPAUT.

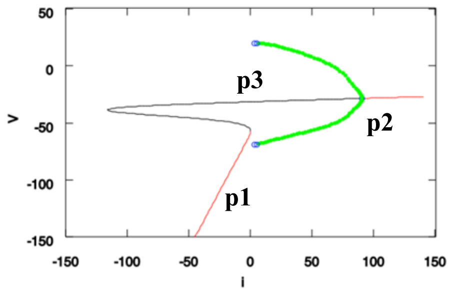


Fig. 2 Bifurcation diagram for membrane voltage V (mV) when applied current I(mA) varies. Red, black, green and blue lines/spots indicate: stable fixed points, unstable fixed points, stable periodic orbits and unstable orbits respectively

There are three main points it is worth to focus. First, P1 (V≈-58.64, Ie≈0) is in fact, the equilibrium point A and it is also a bifurcation point, if voltage and current becomes more negative, the membrane voltage tends to go a specific stable fixed point and gets higher (more negative) as current is more negative. However, if membrane voltage slightly increases (becomes less negative) then it goes to an unstable fixed points zone, and it start to oscillate. This is not good, especially if we want that neuron stay at its resting value when 0 mA current is applied, instead of start oscillating. This will be fixed in Section IV.

P2 (V≈-28.6, Ie≈90) is where a Hopf bifurcation appears and stable periodic orbits increase in amplitude as current tends to zero, i.e. when current starts to increase from zero the neuron start to fires periodically with spikes amplitudes near of 15 mV, whereas current keeps increasing, the neuron maintains firing periodically but with less amplitude and more frequency, eventually when current applied reach 90 mA, then membrane voltage tends to a fixed stable as we can see the right red line after P2. However is the value of 65 mA where the membrane voltage output can still consider as action potentials, i.e. spikes amplitude is bigger than -20 mV threshold.

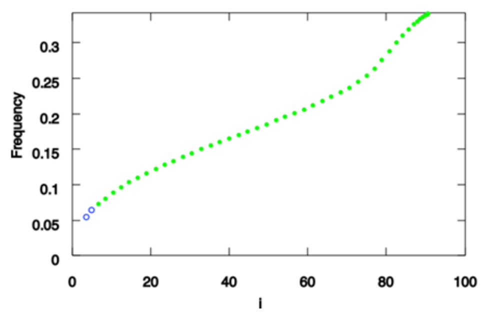


Fig. 3 Frequency diagram of periodic spikes. Current I and frequency is given in mA and KHz units respectively

A frequency study is also shown in Fig. 3, where we can conclude that for an applied current between 0 and 65 mA, the spike frequency response range is [50 341] Hz.

P3 is a very small region where negative current near zero generates sporadic single spikes, which are consider as unstable orbits marked as blue spots in Fig. 2. In fact P1 and P3 have the same current value, but with different membrane voltage steady states.

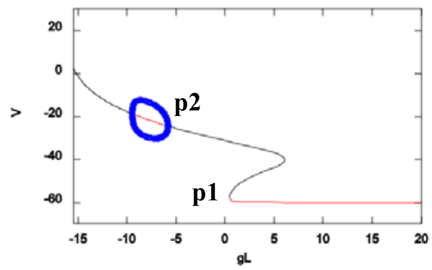


Fig. 4 Bifurcation diagram for membrane voltage V (mV) when leakage conductance gL (mS/cm^2) varies. Red, black, green and blue lines/spots indicate: stable fixed points, unstable fixed points, stable periodic orbits and unstable orbits respectively

Regarding leakage conductance parameter, a bifurcation diagram is shown in Fig. 4 when applied current is zero. The important point to notice is P1, where there is a bifurcation point at $g_l = 0.4522$mS/cm^2, values of leakage conductance smaller than this value makes the neuron unstable, producing action potentials even if the neuron has no stimulus. If this kind of model is intended to be used in neural networks with learning process, this is not a desired behavior. So it is necessary to choose the proper value of leakage conductance where we can guarantee the neuron will stay at stable value when there is no current applied, this is done by choosing a value bigger than 0.4522 mS/cm^2, where as we can see in Fig. 4, the neuron will remain at resting value of 60 mV approximately (red line). The equilibrium point A was achieved with a leakage conductance of 0.5 mS/cm^2.

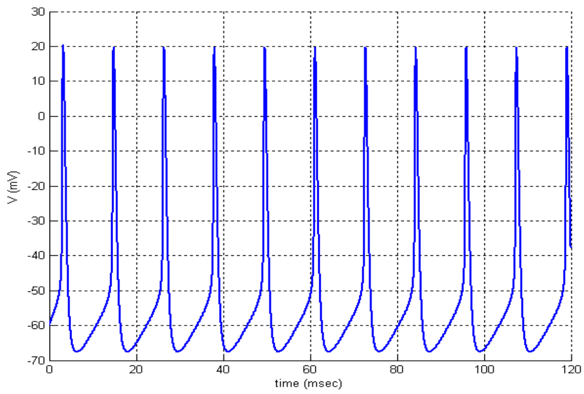
In point P2, when $g_l = -5.8$ a *Hopf bifurcation* rises with unstable orbits (blue spots); nevertheless it is a mathematical bifurcation point, from a biophysical point of view does not have relevance due to negative values for leakage conductance.

The complete list of parameters used for this study is: $\bar{g}_{Na} = 30$ mS/cm^2;$\bar{g}_K = 15$ mS/cm^2; $\bar{g}_l = 0.5$ mS/cm^2; $E_{Na} = 40$ mV; $E_k = -75$ mV; $E_l = -60$ mV; $C_m = 3$ µF/cm^2. These values are consistent with the original work in [1]-[3].

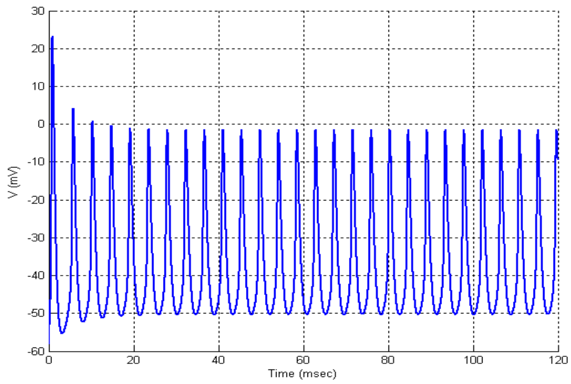
IV. Hardware Implementation Test

Finally the single compartment Traub model was tested in a FPGA device. Following a scheme of state machines, floating-point arithmetic units and BRAMs for store internal results, a soma-like neuroprocessor was developed. This implementation is part of the project "*Efficient and biophysical accurate neuroprocessors*". Because this neuroprocessors are implemented in digital programmable devices, (1) need to be solved by numerical methods. Exponential Euler method offers good tradeoff between system stability and computational resources for implementation. Because the development of such neuro processors is not the principal discussion on this paper, more details about this platform can be found in [2]-[10].

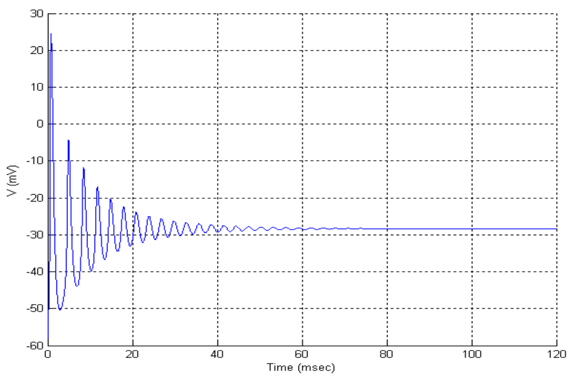
Several tests were run on this platform and Fig. 5 presents the results for three different applied currents. The frequency of spikes is consistent with results obtained by current bifurcation. And when a current bigger than 90 mA is injected; a short transient response occurs and then membrane voltage goes to a steady state of 28 mV.



(a)



(b)



(c)

Fig. 5 Periodic spikes hardware implementation results for different applied current. a) 10 mA, b) 65 mA and c) 95 mA

V. Conclusions and Future Work

A mathematical analysis for neuron dynamics was developed. The Traub model used is one of the most important conductance-based models and a single compartment representation was analyzed. The leakage conductance value was tuned in order the neuron remains at fixed value when it is at resting state. This parameter is the best option to change if does not want to compromise the dynamic of the original model.

Through bifurcation analysis, it was detected the stable and unstable solutions for this model. A Hopf bifurcation was discovered at the point I = 90 mA, given to the current range [0 90] mA a set of stable periodic orbits with different action potentials amplitudes. The frequencies range for this periodic orbits are from 50 to 341 Hz. All this information is relevant since it is the first analysis done for a single compartment Traub model and it is well established quantitative current values where we can have knowledge of what kind of response from the neuron we can expect; this is particularly important if we want to use this neuro processors in a neural network and incorporate a learning process; it is necessary to know which values of current will make the neuron fires.

As future work we are planning to make a similar analysis for the complete two-compartment Traub model, taking in consideration the dendrite calcium ion channel information. Also using the same bifurcation analysis found relevant information related with ion-channel diseases in order to implement neural control models which helps in this particular issues.

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