A Generalization of Planar Pascal’s Triangle to Polynomial Expansion and Connection with Sierpinski Patterns

Wajdi Mohamed Ratemi

Abstract—The very well-known stacked sets of numbers referred to as Pascal’s triangle present the coefficients of the binomial expansion of the form \((x+y)^n\). This paper presents an approach (the Staircase Horizontal Vertical, SHV-method) to the generalization of planar Pascal’s triangle for polynomial expansion of the form \((x+y+z+w+r+\ldots)^{\lambda}\). The presented generalization of Pascal’s triangle is different from other generalizations of Pascal’s triangles given in the literature. The coefficients of the generalized Pascal’s triangles, presented in this work, are generated by inspection, using embedded Pascal’s triangles. The coefficients of I-variables expansion are generated by horizontally laying out the Pascal’s elements of (I-1) variables expansion, in a staircase manner, and multiplying them with the relevant columns of vertically laid out classical Pascal’s elements, hence avoiding factorial calculations for generating the coefficients of the polynomial expansion. Furthermore, the classical Pascal’s triangle has some pattern built into it regarding its odd and even numbers. Such pattern is known as the Sierpinski’s triangle. In this study, a presentation of Sierpinski-like patterns of the generalized Pascal’s triangles is given. Applications related to those coefficients of the binomial expansion (Pascal’s triangle), or polynomial expansion (generalized Pascal’s triangles) can be in areas of combinatorics, and probabilities.

Keywords—Generalized Pascal’s triangle, Pascal’s triangle, polynomial expansion, Sierpinski’s triangle, staircase horizontal vertical method.

I. INTRODUCTION

PASCAL’S triangle which may, at first, looks like a pile of stacked numbers, it carries many secrets and patterns. One of such secrets is its representation of the coefficients of the binomial expansion. In earlier work, the author introduced some polynomial expansion of the form \((\omega + \lambda)\), which, compactly, presented as \(\sum_{k=0}^{\infty} \omega^{k} \lambda^{k-1} \lambda \ldots \lambda\), and is named the Guelph expansion [1]. The term \(\sum_{k=0}^{\infty} \omega^{k} \lambda^{k-1} \lambda \ldots \lambda\) denotes the sum of the products of each and every possible combination of k elements of the set \(\lambda, \lambda, \ldots, \lambda\), and is the binomial coefficient. Such sum of combinations of roots, also, represents the coefficients, \(a_{n-k}\), of the expansion:

\[a_{n} \omega^{n} + a_{n-1} \omega^{n-1} + a_{n-2} \omega^{n-2} + \ldots + a_{0} \omega^{0} = \sum_{k=0}^{n} a_{n-k} \omega^{n-k} .\]

Therefore, \(a_{n-k} = \sum_{k=0}^{n} \lambda^{k-1} \lambda \ldots \lambda\) can be used to evaluate the coefficients of a polynomial knowing its roots \(\lambda_{1}, \lambda_{2}, \lambda_{3}, \ldots, \lambda_{n}\).

The binomial coefficient \(T_{nk}\) (for \(n=0, 1, 2, 3, \ldots, k=0, 1, 2, 3, \ldots, n\)) can generate what is known to the most of the western world as Pascal’s triangle [3], in China as Yang Hui’s triangle, and in Iran as Khayyam’s triangle [4].

Numerous studies are available in the literature about the Pascal’s triangles, and binomial expansion [5]-[9], as well as polynomials expansion and their coefficients [10]-[12]. Generalizations of Pascal’s triangles both planar and geometrical have also been discussed by many authors [13]-[15]. In geometrical representation, the generalization of Pascal’s triangle is made for trinomials expansion (Pascal’s pyramid), and hyper pyramids for higher orders of expansions.

The author approached a generalization of binomial expansion to polynomial expansion in the form of:

\[\sum_{k=0}^{n} T_{nk} \omega^{n-k} \lambda^{k} \ldots \lambda = (\omega + \lambda)^{n} \]

The coefficients of this formalization of the polynomial expansion can be generated by inspection using what is called the Embedded Pascal’s Triangles (EPTs) inspection method. In such inspection method, one can generate the coefficients of I-variables expansion by multiplying horizontally laid out classical Pascal elements with the corresponding coefficients of the (I-1) variables expansion [16]. An efficient algorithm is available to generate the coefficients of those polynomials expansion, as well as the monomials accompanying such expansion which are in lexicographic order [17].

The EPTs-based generalization of Pascal’s triangles is different from the generalization, given in the literature [18], of Pascal’s triangles of \(s^{th}\) orders (\(s^{th}\) arithmetic triangle) of polynomials of the form:

\[(1 + x + x^{2} + \ldots + x^{s-1})^{n} = \sum_{m=0}^{\frac{n}{s}} \binom{r}{m}_{s} x^{m} , s \geq 2\]

where, \(\binom{r}{m}_{s}\) are the generalized binomial coefficients of order \(s\). The expansion \((1 + x + x^{2} + \ldots + x^{s-1})^{n}\) is, actually, a

W. M. Ratemi is with the Nuclear Engineering Department, University of Tripoli, Tripoli, Libya (e-mail: wm_ratemi@hotmail.com).
The horizontally laid out elements of the classical Pascal’s triangle carry a meaning as the coefficients of the binomial expansion raised to the power of n, where n=0, 1, 2, 3, … . Moreover, they represent the number of non-combined roots, single combined roots, two combined roots, three combined roots, …., k-combined roots of a polynomial of degree n of the form \( \sum_{k=0}^{n} a_{n-k} x^{n-k} = 0 \). Similarly, the vertically laid out elements of Pascal’s triangle carry a meaning, simply; each column represents, successively, the number of the coefficients of monomials expansion \( x^n \), binomial expansion \((x + y)^n \), trinomial expansion \((x + y + z)^n \), tetrannomial expansion \((x + y + z + w)^n \), ……, etc., for n=0, 1, 2, 3, … .

Those sets of numbers are called Waterloo numbers, and the exact values of these coefficients of the considered expansion (called the attached values of Waterloo numbers) are generated by the EPTs inspection method [19]. Those Waterloo numbers can be generated by a form of a polynomial, called Tripoli’s polynomial which have two interesting properties, namely: the roots of a degree m Tripoli polynomial are at \(-1, -2, -3, …., -m\), and the sum of its coefficients is \( m+1 \) [20].

In this paper, a new inspection method to generate planar Pascal’s triangles of order 0, 1, 2, 3, 4, …., N is introduced. The introduced new inspection method, in this paper, is an improvement of the EPTs method with the advantage of being much straight forward and simple. The 0-order represents the expansion of zero-nomial; \((0)^n \) expansion which is simply 1, the 1st order represents monomials expansion \((x)^n \), the 2nd order corresponds to the classical Pascal’s triangle representing the coefficients of the binomial expansion; \((x + y)^n \), whereas order 3 of Pascal’s triangle represents the coefficients of the trinomial expansion \((x + y + z)^n \), and so forth. In other words, one blows up 1 to many planar Pascal’s triangles using the proposed SHV-method to be explained in the Section II of this paper.

The pattern of the odd and even numbers in Pascal’s triangle of order 2 (classical Pascal’s triangle) is known as the Sierpinski’s triangle. In this work, patterns of the odd and even numbers are evolved for the higher orders of the SHV-Sierpinski’s triangle. In this work, patterns of the odd and even numbers are evolved for the higher orders of the SHV-Sierpinski’s triangle. In this section, the Staircase Horizontal Vertical (SHV) method which is deduced, solely, from the elements of Pascal’s triangle is derived from the EPTs inspection method, and is a much easier straight forward method. Such method starts with 1 and explodes it, successively, to Pascal’s triangles of different orders; starting from 0-variables expansion of the form \((0)^n \), and proceed to 1-variable expansion of the form \((x)^n \), 2-variables expansion of the form \((x + y)^n \), 3-variables expansion of the form \((x + y + z)^n \), … until I-variables expansion of the form \((x + y + z + w + r + …)^n \).

A. Pascal’s Triangle and Waterloo Matrix

The elements of Pascal’s triangle are generated by the binomial coefficients

\[
T_{nk} = \binom{n}{k} = \frac{n!}{(n-k)k!}, \quad k = 0, 1, 2, 3, …, n
\]

are given in Table I.

<table>
<thead>
<tr>
<th>TABLE I PASCAL’S TRIANGLE</th>
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<tbody>
<tr>
<td>n/k</td>
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<td>-----</td>
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<td>0</td>
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<td>1</td>
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<tr>
<td>2</td>
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<td>3</td>
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<tr>
<td>4</td>
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<td>……</td>
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</table>

If one pulls up each column so as to start from the row corresponding to n=0, then the result will be what is called the Waterloo matrix which presents the Waterloo numbers [7]. The elements of this matrix can be calculated using (2):

\[
W_{nk} = \binom{n+k}{k} = \frac{(n+k)!}{n!k!}
\]

Table II presents the Waterloo matrix.

<table>
<thead>
<tr>
<th>TABLE II THE WATERLOO MATRIX</th>
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</thead>
<tbody>
<tr>
<td>n/k</td>
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<td>0</td>
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<td>4</td>
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<td>5</td>
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<tr>
<td>……</td>
</tr>
</tbody>
</table>
B. Generating Pascal’s Triangles of Different Orders Using SHV-Method

Starting with the expansion of 0-variables this corresponds to the expansion of \((0)^n\) which has a value of 1 for \(n=0\), hence Pascal’s triangle of order 0 is 1, (see Fig. 1 (a)). To generate the next monomial expansion of the form\((x)^n\), one draws one step and lay out the value of 1 corresponding to \(n=0\), and to generate the other elements for \(n=1, 2, 3, \ldots\), one multiplies those laid out elements for each step by the elements of the previous monomial expansion in a staircase manner of step size \(k\), one lays out, horizontally, all of the 1’s of the previous step and lay out the value of 1 corresponding to \(n=0\), and to complete for the rest of the elements, one multiplies those laid out elements for each step by the elements of the Waterloo matrix in a consecutive manner, i.e.;

\[
W_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \end{bmatrix}, W_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \cdots \end{bmatrix}, \ldots
\]

Hence, one gets the classical Pascal’s triangle which of order 2, (see Fig. 1 (c)). Now, to generate the coefficients of monomials of the form \((x+y)^n\), one draws a staircase of varying step width, i.e.; one element width, two elements width, three elements width, etc. Such width size of the staircase steps corresponds to the elements of the 2\(^{nd}\) column of the Waterloo matrix \(W_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \cdots \end{bmatrix}\). One next lays out the previously generated binomial coefficients under the steps of the new staircase (bolded numbers in Fig. 1 (d)). Finally, to complete the rest of the elements, one multiplies those laid out elements for each step by the elements of the columns of the Waterloo matrix in a consecutive manner, i.e.; \(W_1, W_2, W_3, W_4, \ldots\). The full coefficients of the trinomial expansion corresponding to planar Pascal’s triangle of order 3 are depicted in Fig. 1 (d). To get further expansions, one follows the same algorithm.

C. Polynomial Expansion

The SHV-method has been presented in Section II to generate the coefficients of different orders of Pascal’s triangles. Those coefficients belong to the corresponding polynomial expansions. The monomials of the respected expansion can be generated in a lexicographic order.

\[
\text{Table III presents the required coefficients and monomials for the expansion with } n=4. \text{ The full expansion is given by (5):}
\]

\[
(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
\]

Similarly, (3) reduces to (6) for trinomial expansion:

\[
(x + y + z)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k z^{k'}, k' = 0, 1, 2, \ldots, k;
\]

\[
k'' = 0, 1, 2, \ldots, k';\ldots\text{etc.}
\]

The monomials shown in (3) are in lexicographic order; hence, they can be used to generate the related expansion. Equation (3) reduces to the well-known formula for binomial expansion given in (4):

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^k
\]

\[
\text{Table III} \quad \text{BINOmIAL EXPANSION OF } (x + y)^4
\]

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td>0,1,2,...,n</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

Can be read directly from the corresponding row of Pascal’s triangle of order 2.

\(a\)Monomials are in lexicographic order.

Table IV presents the coefficients, and monomials of the following trinomial expansion:

\[
(x + y + z)^3 = x^3 + 3x^2y + 3x^2z + 3xy^2 + 6xyz + 3xz^2 + y^3 + 3yz^2 + 3y^2z + z^3.
\]
D. Graphical Distributions of the Elements of Pascal’s Triangles

A graphical presentation of the distribution of the elements of the different orders of Pascal’s triangles as a function of the degree of expansion n, and the index \([k] = W_{n,k}\) (Waterloo number) representing the number of the coefficients in the corresponding expansion is demonstrated in this section. Referring to Table III which represents the binomial expansion for \(n=4\), one notices that \(k=0, 1, 2, 3, 4\), hence there are 5 coefficients for such expansion. This number \((W_{n,k} = W_{4,1})\) can be read directly from the Waterloo matrix for \(n=4\), and \(k=1\) (2nd column; corresponding to a binomial expansion). On the other hand, for the trinomial expansion raised to power \(n\), \(k=0, 1, 2, 3, …, n\), and \(k'=0, 1, 2, 3, …k\). Referring to Table IV for the trinomial expansion where \(n=3\), the number of coefficients are 10. This number can be read directly from the Waterloo matrix which corresponds to \(W_{3,2}\) (\(n=3\), \(k=2\): 3rd column representing trinomials).

<table>
<thead>
<tr>
<th>Table IV</th>
<th>Binomial Expansion of ((x+y)^n)</th>
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</thead>
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<tr>
<td>(n)</td>
<td>(k)</td>
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<td>0</td>
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<td>1</td>
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<td>1</td>
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<tr>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

\(k = 0, 1, 2, ..., n\), and \(k' = 0, 1, 2, ..., k\).

The distribution of the binomial expansion up to the expansion of \((x+y)^{19}\) is shown in Fig. 2 (the number of coefficients corresponding to \(n=19\) is \([k] = W_{n,k} = W_{19,1} = 20\)). The distribution approaches the normal distribution as \(n\) (the degree of the expansion) gets larger. Similarly, the distribution for trinomial expansion up to \((x+y+z)^5\) is shown in Fig. 3 (the number of coefficients corresponding to \(n=5\) is \([k] = W_{n,k} = W_{5,2} = 21\)).

III. THE SIERPINSKI’S LIKE PATTERNS

It is quite interesting that when shading odd, and even numbers of the classical Pascal’s triangle, a pattern is evolved and is known as the Sierpinski’s triangle [21]. Fig. 4 (a) presents such pattern for a right angled Pascal’s triangle. In a similar manner, one can apply such odd-even shading to generate patterns for higher orders of planar Pascal’s triangles. Figs. 4 (b)-(d) present such patterns for Pascal’s triangles of orders 3, 4, and 5, respectively.
(a) World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:10, No:2, 2016

(b) ISNI:0000000401950263

(c) International Scholarly and Scientific Research & Innovation 10(2) 2016
Fig. 4 Odd-even patterns of Pascal’s triangles; (a) pattern for order 2: \((x + y)^2\) (Sierpinski’s triangle), (b) pattern for order 3: \((x + y + z)^3\), (c) pattern for order 4: \((x + y + z + w)^4\), and (d) pattern of order 5: \((x + y + z + w + r)^5\).

REFERENCES


Wajdi M. Ratemi was born in Tripoli, Libya in 1954. Ratemi earned his B.Sc. in nuclear engineering from University of Wisconsin, Madison, Wisconsin, USA in 1977, his M.Sc. in nuclear engineering from the same University in 1982, and his Doctor of Engineering from Kyoto University, Kyoto, Japan in 1992.


Prof. Ratemi chaired the department of nuclear engineering (1997-2000), as well as the graduate study department of engineering management (2004-2008) of University of Tripoli. Also, Prof. Ratemi is a member of the board of directors of the Libyan atomic energy establishment since 2010.