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The Exploitation of Balancing an Inverted Pendulum System Using Sliding Mode Control

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***Abstract*—**The inverted pendulum system is a classic control problem that is used in universities around the world. It is a suitable process to test prototype controllers due to its high non-linearities and lack of stability. The inverted pendulum represents a challenging control problem, which continually moves toward an uncontrolled state. This paper presents the possibility of balancing an inverted pendulum system using sliding mode control (SMC). The goal is to determine which control strategy delivers better performance with respect to pendulum's angle and cart's position. Therefore, proportional-integral-derivative (PID) is used for comparison. Results have proven SMC control produced better response compared to PID control in both normal and noisy systems.

***Keywords*—**Inverted pendulum (IP) proportional-integral-derivative (PID), sliding mode control (SMC).

I. Introduction

An inverted pendulum is a pendulum which has its mass above its pivot point. It is often implemented with the pivot point mounted on a cart that can move horizontally and may be called a cart and pole. Whereas a normal pendulum is stable when hanging downwards, an inverted pendulum is a pendulum which has its mass above pivot point mounted on a cart that can move horizontally and may be called a cart and pole. Whereas a normal pendulum is stable when hanging downwards, an inverted pendulum is inherently unstable, and must be actively balanced in order to remain upright, either by applying a torque at the pivot point or by moving the pivot point horizontally as part of a feedback system. The inverted pendulum is a classic problem in dynamics and control theory and widely used as benchmark for testing control algorithms (PID controllers, neural networks, fuzzy control, genetic algorithms, etc.). Variations on this problem include multiple links, allowing the motion of the cart to be commanded while maintaining the pendulum, and balancing the cart-pendulum system on a see-saw. The inverted pendulum is related to rocket or missile guidance, where thrust is actuated at the bottom of a tall vehicle. The understanding of a similar problem is built in the technology of segway, a self-balancing transportation device. The largest implemented use is on huge lifting cranes on shipyards. When moving the shipping containers back and forth, the cranes move the box accordingly so that it never swings or sways [1]. This research presents a simulation study of a sliding mode control to keep the balance of the inverted pendulum system.

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II. Proposed System

The system consists of an inverted pole with mass, m, hinged by an angle θ from vertical axis on a cart with mass, M, which is free to move in the x direction as shown in Fig. 1. A force, F is required to push the cart horizontally. The dynamical equations of the system will be derived. The inverted pendulum system that will be used in this thesis have the following specifications given in Table I [2].

TABLE I
SYSTEM PARAMETERS

parameter	Values
M	0.5 kg
m	0.2 kg
b	0.1 N/m/sec
L	0.3 m
I	0.006 kg.m^2
g	9.8 m/s^2

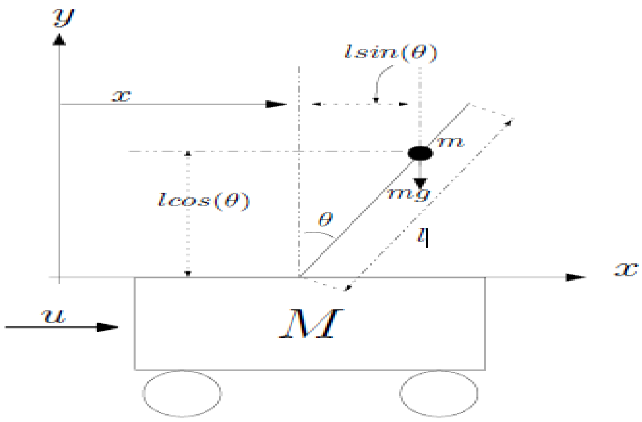


Fig. 1 The inverted pendulum force analysis

The following equation represents the equation of motion. The velocity has two components, one due to the motion of the cart ($\dot{X}$) and the other due to the angular motion of pendulum.

- The horizontal position: $X + L\sin\theta$
- The vertical position: $L\cos\theta$

The total kinetic energy is:

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m[(\frac{\delta}{\delta t}(X + L\sin\theta))^2 + (\frac{\delta}{\delta t}(L\cos\theta))^2] \quad (1)$$

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m[(\dot{X} + L\dot{\theta}\cos\theta))^2 + (-L\dot{\theta}\sin\theta)^2] \quad (2)$$

The potential energy is:

$$V = V0 + mgL\cos\theta \quad (3)$$

$V0$ is the potential energy for $\theta = 90$. The Lagrange function:

$$L = T - V = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m[(\dot{X} + L\dot{\theta}\cos\theta)^2 + (-L\dot{\theta}\sin\theta)^2] - V0 - mgL\cos\theta \quad (4)$$

For X equation:

$$\frac{\delta}{\delta t}[\frac{\delta L}{\delta \dot{X}}] - \frac{\delta L}{\delta X} = F_1 \quad (5)$$

For θ equation:

$$\frac{\delta}{\delta t}[\frac{\delta L}{\delta \dot{\theta}}] - \frac{\delta L}{\delta \theta} = F_1 \quad (6)$$

The equations related to X:

$$\frac{\delta}{\delta t}[\frac{\delta L}{\delta \dot{X}}] = M\ddot{X} + m(\ddot{X} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta) \quad (7)$$

$$(M+m)\ddot{X} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta = U \quad (8)$$

$$\frac{\delta L}{\delta \dot{\theta}} = m[(\dot{X} + L\dot{\theta}\cos\theta)L\cos\theta + L^2\dot{\theta}\sin^2\theta] \quad (9)$$

$$\frac{\delta L}{\delta \dot{\theta}} = mL\dot{X}\cos\theta + m\overset{2}{L}\dot{\theta} \quad (10)$$

$$\frac{\delta L}{\delta \theta} = m[(\dot{X} + L\dot{\theta}\cos\theta)(-L\dot{\theta}\sin\theta + L^2\dot{\theta}^2\cos\theta] + mg \quad (11)$$

$$\frac{\delta L}{\delta \theta} = -mL\dot{\theta}\dot{X}\sin\theta - mgL\sin\theta \quad (12)$$

$$\frac{\delta}{\delta t}[\frac{\delta L}{\delta \dot{\theta}}] = m\ddot{X}L\cos\theta - L\dot{\theta}\dot{X}\sin\theta + mL^2\ddot{\theta} \quad (13)$$

$$mL\ddot{X}\cos\theta + mL^2\ddot{\theta} - mgL\sin\theta = 0 \quad (14)$$

$$(I + mL^2)\ddot{\theta} + mL\ddot{X}\cos\theta + mgL\sin\theta = 0 \quad (15)$$

$$(M+m)\ddot{X} + b\dot{X} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta = U \quad (16)$$

The Linearized of equations: $cos(\theta) = -1$, $sin(\theta) = 0$, and $(d(\theta)/dt)\hat{}2 = 0$. To obtain the transfer function of the linearized system equations analytically, we must first take the Laplace transform of the system.

$$(I + mL^2)\phi(s)s^2 - mgL\,\phi(s) = mLX(s)s^2 \quad (17)$$

$$(M+m)X(s)s^2 + bX(s)s - mL\phi(s)s^2 = U(s) \quad (18)$$

where U is the input.

Since we will be looking at the angle ø as the output of interest, and solving (17) for *X(s):*

$$X(s) = \left[\frac{(I + mL^2)}{mL} - \frac{g}{s^2}\right]\phi(s) \quad (19)$$

Then, substituting into (18):

$$(M+m)\left[\frac{(I+mL^2)}{mL} - \frac{g}{s^2}\right]\phi(s)s^2 + b\left[\frac{(I+mL^2)}{mL} - \frac{g}{s^2}\right]\phi(s)s - mL\phi(s)s^2 = U(s) \quad (20)$$

The transfer function of this system for the pendulum's angle is:

$$\frac{\phi(s)}{U(s)} = \frac{\frac{mL}{q}s^2}{s^4 + \frac{b(I+mL^2)}{q}s^3 - \frac{(M+m)mgL}{q}s^2 - \frac{bmgL}{q}s} \quad (21)$$

The transfer function of this system for the pendulum's position is:

$$\frac{X(s)}{U(s)} = \frac{\frac{I+mL^2}{q}s^2 - \frac{mgL}{q}}{s^4 + \frac{b(I+mL^2)}{q}s^3 - \frac{(M+m)mgL}{q}s^2 - \frac{bmgL}{q}s} \quad (22)$$

where:

$$q = [(M+m)(I+mL^2) - (mL)^2] \quad (23)$$

The linearized system equations can also be represented in state-space form. The four states represent the position, velocity of the cart, the angle, and angular velocity of the pendulum. The output *y(t)* contains both the position of the cart and the angle of the pendulum we will assume that we can only measure the angle of the pendulum, and this is the output we want to regulate.

State space in general form is:

$$\dot{X} = AX + BU \quad (24)$$

$$\dot{Y} = CX + DU \quad (25)$$

Let:

$X_1 = X$, $X_3 = \theta$, $X_2 = \dot{X} = X_1$, $X_4 = \dot{\theta} = X_3$, $\dot{X}_1 = X_2$, $\dot{X}_3 = X_4$

From (3), (6), and from the linearization:

$$(M+m)\dot{X}_2 + bX_2 - mL\dot{X}_4 = U \quad (26)$$

$$(I+mL^2)\dot{X}_4 - mgLX_3 - mL\dot{X}_2 = 0 \quad (27)$$

From (26), (27):

$$\dot{X}_2 = \frac{(I+mL^2)}{mL}\dot{X}_4 - gX_3$$

$$\dot{X}_4 = \frac{mLU}{(I+mL^2)(M+m)-m^2L^2} - \frac{bmLX_2}{(I+mL^2)(M+m)-m^2L^2} + \frac{(M+m)gmLX_3}{(I+mL^2)(M+m)-m^2L^2} \quad (28)$$

$$P = I(M+m) + MmL^2 \quad (29)$$

$$\dot{X}_4 = -\frac{bmLX_2}{P} + \frac{(M+m)gmLX_3}{P} + \frac{mLU}{P} \quad (30)$$

$$\dot{X}_2 = \frac{-b(I+mL^2)}{P}X_2 + \frac{m^2L^2g}{P}X_3 + \frac{(I+mL^2)U}{P} \quad (31)$$

$$Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \quad (32)$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+mL^2)b}{P} & \frac{m^2gL^2}{P} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mLb}{P} & \frac{mgL(M+m)}{P} & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+mL^2}{P} \\ 0 \\ \frac{mL}{P} \end{bmatrix} U \quad (33)$$

III. Sliding Mode Control (SMC)

The way SMC deals with uncertainty is to drive the plants state trajectory into a sliding surface and maintain the error trajectory on this surface for all subsequent times. The advantage of SMC is that the controlled system becomes insensitive to system disturbances [4]. The sliding surface is defined such that the state tracking error converges to zero with input reference. The idea of sliding to stable manifold (reaching phase), then slide to equilibrium (sliding phase). Consider a second order system:

$$\dot{X}_1 = X_2 \quad (34)$$

$$\dot{X}_2 = f(X) + g(X)u \quad (35)$$

Assumptions:

- $f(X)$ and $g(X)$ are nonlinear functions. $g(X) > 0$.
- $f(X)$ and $g(X)$ need not to be continuous.

We want to design a state feed back to stabilize the origin, the motion satisfies the differential equation.

$$\dot{X}_1 = -aX_1$$

X_1 is stable if:

$$\dot{X}_1 = -aX_1 \ , a \rangle 0.$$

Define coordinate with respect to the stable manifold.

$$S = X_2 + aX_1 \quad (36)$$

so that:

$$\dot{X}_1 = X_2 = -aX_1 + S \quad (37)$$

is stable if $S = 0$.

The time derivative of S is:

$$\dot{S} = \dot{X}_2 + a\dot{X}_1 = f(X) + g(X)u + aX_2 \quad (38)$$

To evaluate stability, evaluate the Lyapunov candidate.

$$v = \frac{1}{2}S^2 \quad (39)$$

$$\dot{v} = \dot{S}S = S[f(X) + g(X)u + aX_2] \quad (40)$$

$\dot{v}$ is negative definite if:

$$f(X) + g(X)u + aX_2 = \begin{Bmatrix} \prec 0 \ for. S \succ 0 \\ = 0 \ for .S = 0 \\ \succ 0 \ for \ S \prec 0 \end{Bmatrix}$$

Stability is insured if:

$$u = \begin{Bmatrix} \prec B(X) \ for. S \succ 0 \\ = B(X) \ for .S = 0 \\ \succ B(X) \ for \ S \prec 0 \end{Bmatrix}$$

$$B(X) = -\frac{f(X) + a(X_2)}{g(X)} \quad (41)$$

By using the control law, and chatter may be reduced by replacing the signum function with a sigmoid (smooth) function. This essentially creates a boundary layer around the sliding surface. In this case, the control law is [5]:

$$u = B(X) - KSign(S) \quad (42)$$

where $K > 0$. From (7) and (15):

$$(M+m)\ddot{X} + b\dot{X} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta = U \quad (43)$$

$$(I+mL^2)\ddot{\theta} + mgL\sin\theta = -mL\ddot{X}\cos\theta \quad (44)$$

From (42):

$$\ddot{X} = \frac{U + mL\dot{\theta}^2 \sin\theta - mL\ddot{\theta}\cos\theta - b\dot{X}}{M+m} \quad (45)$$

and substitute in (43):

$$\ddot{\theta} = \frac{mgL\sin\theta(M+m) + mL\cos\theta U + m^2L^2\dot{\theta}^2\sin\theta\cos\theta - mLb\cos\theta\dot{X}}{m^2L^2\cos^2\theta - (M+m)(I+mL^2)} \quad (46)$$

and from (45) and substitute in (44):

$$\ddot{X} = \frac{b\dot{X}(I+mL^2) - m^2L^2 g\sin\theta\cos\theta - mL\sin\theta\dot{\theta}^2(I+mL^2) - U(I+mL^2)}{m^2L^2\cos^2\theta - (M+m)(I+mL^2)} \quad (47)$$

From state space equations:

$$X_1 = X$$
$$X_2 = \dot{X}$$
$$X_3 = \theta$$
$$X_4 = \dot{\theta}$$

The equations become:

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = \frac{bX_2(I+mL^2) - m^2L^2 g\sin(X_3)\cos(X_3) - mL\sin(X_3)X_4^2(I+mL^2) - U(I+mL^2)}{m^2L^2\cos^2(X_3) - (M+m)(I+mL^2)} \quad (48)$$

$$\dot{X}_3 = X_4$$

$$\dot{X}_4 = \frac{mgL\sin(X_3)(M+m) + mL\cos(X_3)U + m^2L^2X_4^2\sin(X_3)\cos(X_3) - mLb\cos(X_3)X_2}{m^2L^2\cos^2(X_3) - (M+m)(I+mL^2)} \quad (49)$$

Identifying $f(X)$ and $g(X)$.

$$f(X) = \frac{mgL\sin(X_3)(M+m) + m^2L^2X_4^2\sin(X_3)\cos(X_3) - mLb\cos(X_3)X_2}{m^2L^2\cos^2(X_3) - (M+m)(I+mL^2)} \quad (50)$$

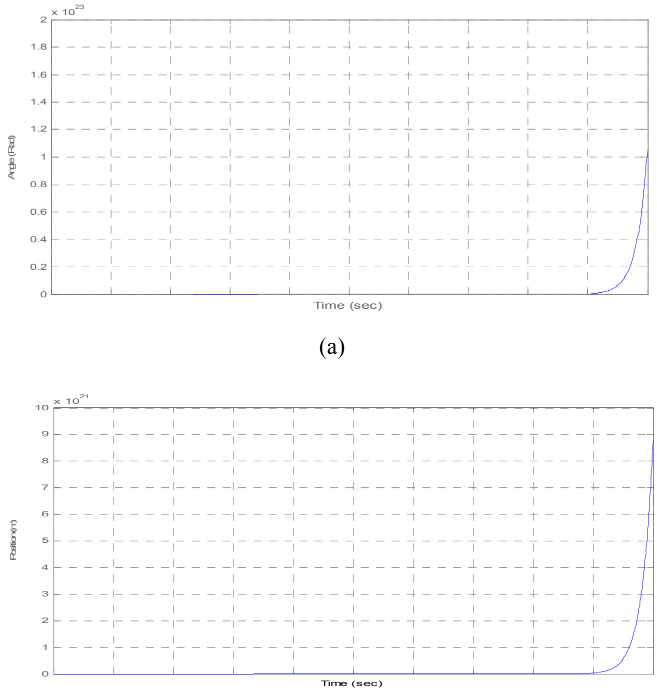
$$g(X) = \frac{mL\cos(X_3)}{m^2L^2\cos^2(X_3) - (M+m)(I+mL^2)} \quad (51)$$

and K is chosen to:

$$K = K_1 + K_2S^2, K_1K_2 \rangle 0$$

IV. Results

The mathematical model in the open loop system and equations using the transfer function and state-space of the inverted pendulum have been determined. All these equations were implemented using Matlab M-file code. From Fig. 2, it can be noticed that the inverted pendulum system is not stable without a controller. The curves of the pendulum's angle and cart's position were approached infinity as the time increases. Therefore, some controllers need to be designed in order to stabilize the system, as shown in Figs. 2 (a) and (b), respectively.



(a)

(b)

Fig. 2 Open loop Inverted Pendulum Response (a) Pendulum's Angle (b) Cart's Position

A. PID Controller Results

The implementation of PID control method is done by adjusting the value of gain Kp, Ki and Kd in order to get the best impulse response of the system [2].

Using suitable values of gains Kp =240, Ki =378, Kd =38, the pendulum's angle is satisfactorily achieved as shown in Fig. 3. From Fig. 4, it can be seen that the cart moves in the negative direction with a constant velocity. Thus, although the PID controller stabilizes the angle of the pendulum, this control method would not be feasible to be implemented on an actual physical system that was presented of cart's angle and pendulum's position as the outputs of the system [3].

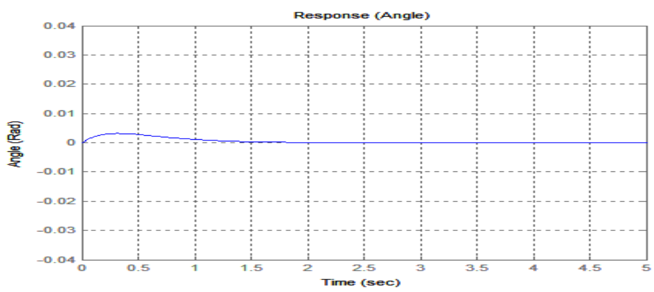


Fig. 3 Response for Pendulum 's angle of PID

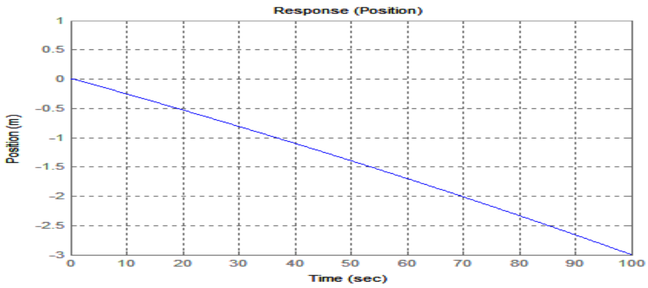


Fig. 4 Response for cart's position of PID

B. SMC Results

The results as follows: Recall that the control objective is to keep the pendulum in its upright position and the cart in the specified position. Using the state space and the controller of sliding mode control for the position and angle were achieved, as can be seen from Figs. 5 and 6.

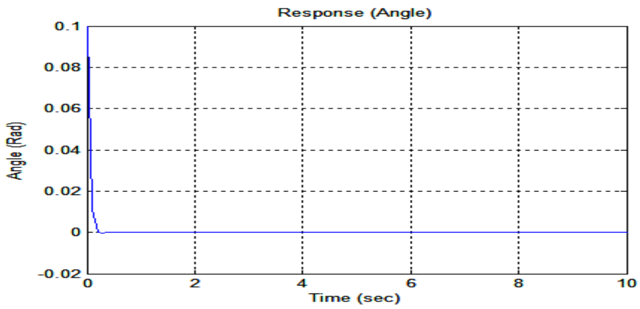


Fig. 5 Response for Pendulum's Angle of SMC

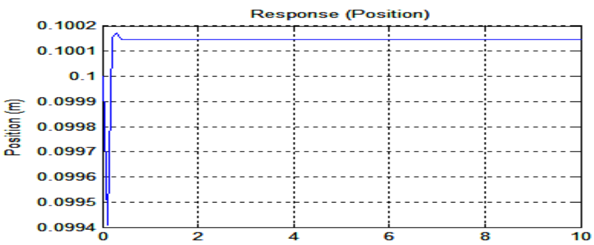


Fig. 6 Response for cart's Position of SMC

V. Results Analysis

A. Pendulum's Angle

All the characteristics values of the responses are summarized in Table II.

TABLE II
Comparison of Output Response of Pendulum's Angle

Characteristic	Controller	
	PID	SMC
Rising Time (tr) Sec	0.2	0.05
Peak Time (tp) Sec	0.43	0.2
Settling Time (ts) Sec	1.44	0.3
Over Shoot Percentage (%OS)	2.65	0.9

Based on Table II, the sliding mode control has settling time of 0.3 seconds while the PID controller has a larger settling time of 1.44 seconds. However, for the maximum overshoot range, sliding mode control has a better range of 0.9. Therefore, while the PID controller has a larger overshoot percentage of 2.65, the sliding mode control (since) has outperformed the PID satisfying the design criteria needed for pendulum angle. From both controllers' characteristics, it can be said that the sliding mode control controller has the ability to response quickly compared to the PID controller, as shown in Fig. 7.

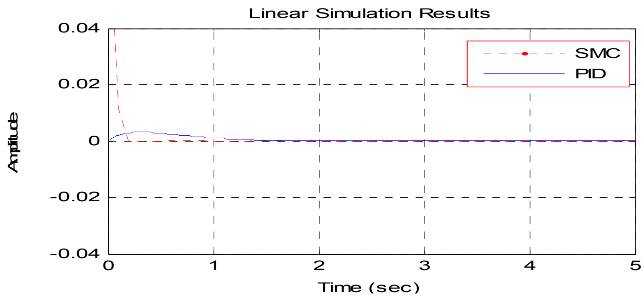


Fig. 7 Comparison for both controllers of cart's angle

B. Pendulum's Position

All responses of cart's position are illustrated in Table III. In Table III, it can be clearly seen the difference of the characteristics between responses. The sliding mode control has a smaller value of settling time, (Ts) of 0.55, seconds, while PID controller has a larger value of settling time, (Ts) of 11.1 seconds. SMC has also a smaller rising time (Tr) of 0.4 seconds. For the overshoot percentage (%OS), sliding mode control has also a smaller overshoot. Therefore, it can be said that the SMC controller has the ability to response quickly compared to the PID controller, as shown in Fig. 8.

TABLE III
Comparison of Output Response of Cart's Position

Characteristic	Controller	
	PID	SMC
Rising Time (tr) Sec	10	0.4
Peak Time (tp) Sec	9.61	0.18
Settling Time (ts) Sec	11.1	0.55
Over Shoot Percentage (%OS)	0.52	0.05

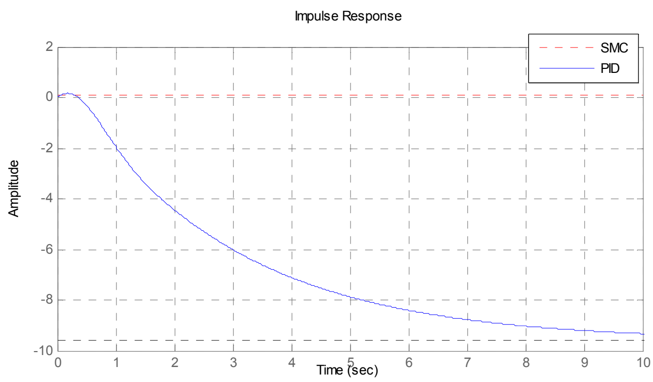


Fig. 8 Comparison of output response of cart's position

VI. Adding Noise to the System for Both Controllers

All previous simulations are assuming no disturbance and no measurement noise.

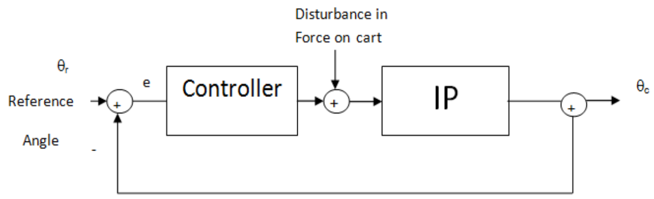


Fig. 9 Simulation system with noise

The disturbance was added to the input system, as shown in Fig. 9. Gaussian noise was introduced to the system with zero mean and standard deviation 1.5. The noise added to the angle is Gaussian, $\mu = 0$ and $\sigma = 1.5$ rad, and the results of PID and SMC are shown in Figs. 10 and 11, respectively.

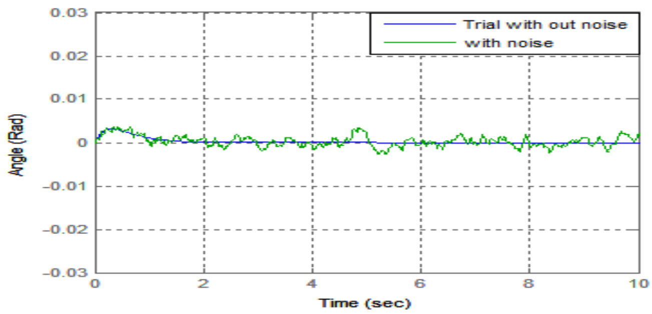


Fig. 10 Angle simulation results with/ without noise of PID

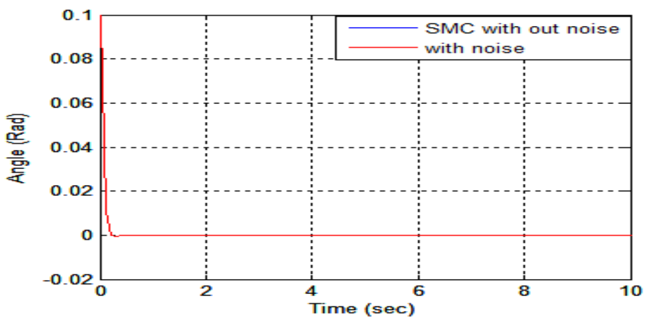


Fig. 11 Angle simulation results with/without noise of SMC

From Figs. 10 and 11, the noise is added to the angle, with a Gaussian noise, the sliding mode control is not affected with noise. However, the system with PID controller, the noise has moved the system from stable operation to unstable.

As shown in Figs. 12 and 13, the noise added is Gaussian, $\mu = 0$, $\sigma = 1.5$ m. When using Gaussian noise, the sliding mode control is not affected with noise. However, the system with PID controller, the noise has moved the system from stable operation to unstable.

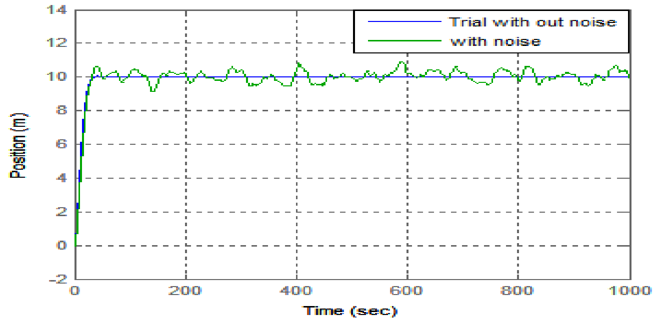


Fig. 12 Position simulation results with/without noise of PID

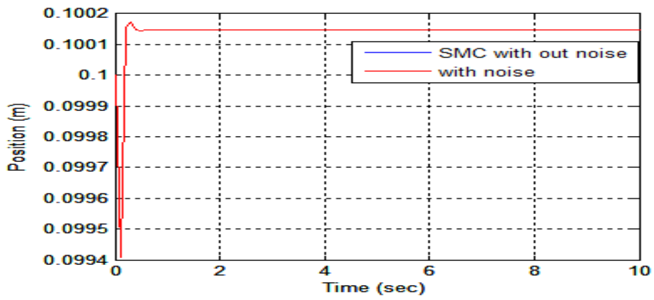


Fig. 13 Position simulation results with /without noise of SMC

VII. Conclusions

From the early results and analysis, it can be concluded that the PID conventional controller is capable of controlling the inverted pendulum's angle and the cart's position. The Sliding mode control for inverted pendulum system has shown a better response. SMC results have also proven to be a powerful research tool that can reduce time, and noise compared with the performance of the PID controller because the SMC strives the system being driven towards the sliding surface, and drives a close loop system to a stable state. The control input will rapidly switch for system states close to the surface.

In SMC, it is not possible to bring the trajectory on switching surface in one step, we try to move our trajectory slowly in k steps. Therefore, as a future work, K can be tuned by genetic algorithm to remove any chattering. In addition, fuzzy- genetic control can also be taken into account as further work for controlling the inverted pendulum system.

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