Abstract—The aim of this paper is to introduce the notion of intuitionistic fuzzy positive implicative ideals with thresholds \((\lambda,\mu)\) of BCI-algebras and to investigate its properties and characterizations.

Keywords—BCI-algebra, intuitionistic fuzzy set, intuitionistic fuzzy ideal with thresholds \((\lambda,\mu)\), intuitionistic fuzzy positive implicative ideal with thresholds \((\lambda,\mu)\).

I. INTRODUCTION

A BCI-algebra is an important class of logical algebra and was introduced by Iséki [1], [2]. K. Atanassov [3] introduced the concept of intuitionistic fuzzy sets. In 2003, K. Hur [4] applied the concept to the theory of rings, and introduced the concepts of intuitionistic fuzzy subgroups and subrings. M. Jiang and X.L. Xin [5] later introduced the concepts of intuitionistic fuzzy subalgebras (ideals); some meaningful results are obtained. In [6], [7], we have given the concepts of intuitionistic fuzzy subalgebras (ideals) with thresholds \((\lambda,\mu)\) and intuitionistic fuzzy implicative ideals with thresholds \((\lambda,\mu)\) of BCI-algebras, in this paper, we introduce the notion of intuitionistic fuzzy positive implicative ideals with thresholds \((\lambda,\mu)\) of BCI-algebras and give several properties and characterizations of it.

II. PRELIMINARIES

An algebra \((X,\ast,0)\) of type \((2,0)\) is called a BCI-algebra if it satisfies the following axioms:

1. \((BCI-1)\) \(x\ast y\ast y = x\ast y\ast x\ast y\ast y\ast y\ast x = 0\),
2. \((BCI-2)\) \((x\ast(x\ast y))\ast y = y\),
3. \((BCI-3)\) \(x\ast x = 0\),
4. \((BCI-4)\) \(x\ast y = 0\) and \(y\ast x = 0\) imply \(x = y\),
5. \((BCI-5)\) \(x\ast(x\ast y)\ast y = 0\).

for all \(x, y, z \in X\). In a BCI-algebra \(X\), we can define a partial ordering \(\leq\) by putting \(x \leq y\) if and only if \(x\ast y = 0\).

In any BCI-algebra \(X\), the following hold:

1. \((x\ast y)\ast z = (x\ast z)\ast y\),
2. \(x\ast 0 = x\),
3. \(0\ast(x\ast y) = (0\ast x)\ast(0\ast y)\).

III. INTUITIONISTIC FUZZY POSITIVE IMPLICATIVE IDEALS WITH THRESHOLDS \((\lambda,\mu)\)

In this paper, \(X\) always means a BCI-algebra unless otherwise specified.

A nonempty subset \(K\) of \(X\) is called an ideal of \(X\) if \((I_1): 0 \in K, (I_2): x\ast y \in K \quad \text{and} \quad y \in K \implies x \in K\).

A nonempty subset \(K\) of \(X\) is called a positive implicative ideal of \(X\) if it satisfies \((I_1)\) and \((I_3): (x\ast z)\ast(y\ast z) \in K \quad \text{and} \quad y \in K \implies x\ast z \in K\).

Definition 1. [3] Let \(S\) be any set. An intuitionistic fuzzy subset \(A\) of \(S\) is an object of the following form

\[ A = \{(x, \mu_A(x), \nu_A(x)) : x \in S\} \quad \text{where} \quad \mu_A : S \rightarrow [0,1] \]

and \(\nu_A : S \rightarrow [0,1]\) define the degree of membership and the degree of non-membership of the element \(x \in S\) respectively and for every \(x \in S\), \(0 \leq \mu_A(x) + \nu_A(x) \leq 1\).

Denote \((I) = \{(a,b); a,b \in [0,1]\}\).

Definition 2. Let \(A = \{(x, \mu_A(x), \nu_A(x)) : x \in S\}\) be an intuitionistic fuzzy set in a set \(S\). For \((a,b) \in (I)\), the set \(A_{(a,b)} = \{x \in S : \mu_A(x) \geq a, \nu_A(x) \leq b\}\) is called a cut set of \(A\).

Definition 3. [6] Let \(\lambda, \mu \in [0,1]\) and \(\lambda < \mu\).

An intuitionistic fuzzy set \(A\) in \(X\) is said to be an intuitionistic fuzzy ideal with thresholds \((\lambda,\mu)\) of \(X\) if the following are satisfied:

\[ (IF_1) \mu_A(0) \vee \lambda \geq \mu_A(x) \wedge \mu, \]
\[ (IF_2) \nu_A(0) \wedge \mu \leq \nu_A(x) \vee \lambda, \]
\[ (IF_3) \mu_A(x) \vee \lambda \geq \mu_A(x\ast y) \wedge \mu_A(y) \wedge \mu, \]
\[ (IF_4) \nu_A(x) \wedge \lambda \leq \nu_A(x\ast y) \vee \nu_A(y) \vee \lambda, \]

for all \(x, y \in X\).

Proposition 1. [6] Let \(A\) be an intuitionistic fuzzy ideal with thresholds \((\lambda,\mu)\) of \(X\). If \(x \leq y\) holds in \(X\), then

\[ \mu_A(x) \vee \lambda \geq \mu_A(y) \wedge \mu, \quad \nu_A(x) \wedge \lambda \leq \nu_A(y) \vee \lambda. \]
Proposition 2. [6] Let \( A \) be an intuitionistic fuzzy ideal with thresholds \( (\lambda, \mu) \) of \( X \). If the inequality \( x \ast y \leq z \) holds in \( X \), then for all \( x, y, z \in X \),

\[
\mu_\lambda(x) \land \lambda \geq \mu_\lambda(y) \land \mu_\lambda(z) \land \mu,
\]
\[
v_\lambda(x) \land \mu \leq v_\lambda(y) \lor v_\lambda(z) \lor \lambda.
\]

III. INTUITIONISTIC FUZZY POSITIVE IMPLICATIVE IDEALS WITH THRESHOLDS \( (\lambda, \mu) \) OF BCI-ALGEBRAS

Definition 4. Let \( \lambda, \mu \in (0,1] \) and \( \lambda < \mu \). An intuitionistic fuzzy set \( A \) in \( X \) is called an intuitionistic fuzzy positive implicative ideal with thresholds \( (\lambda, \mu) \) of \( X \) if it satisfies \( (IF_\lambda),(IF_\mu) \) and

\[
(IF_\lambda) \mu_\lambda(x \ast z) \lor \lambda \geq \mu_\lambda(((x \ast z) \ast z) \ast (y \ast z)) \land \mu_\lambda(y) \land \mu,
\]
\[
(IF_\mu) v_\lambda(x \ast z) \land \mu \leq v_\lambda(((x \ast z) \ast z) \ast (y \ast z)) \lor v_\lambda(y) \lor \lambda,
\]

for all \( x, y, z \in X \).

Example 1. Let \( X = \{0,1,2\} \) with Cayley table given by

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Define \( A = \{(x,\mu_\lambda(x),v_\lambda(x)) : x \in S\} \) where \( \mu_\lambda : X \rightarrow [0,1] \)
and \( v_\lambda : X \rightarrow [0,1] \) by \( \mu_\lambda(0) = 2/3, \mu_\lambda(1) = \mu_\lambda(2) = 1/3, \)
\( v_\lambda(0) = 1/4, v_\lambda(1) = v_\lambda(2) = 1/2 \). Let \( \lambda = 1/8 \) and \( \mu = 3/4 \). By routine calculations, we have that \( A \) is an intuitionistic fuzzy positive implicative ideal with thresholds \( (\lambda, \mu) \) of \( X \).

The following proposition gives a relation between intuitionistic fuzzy positive implicative ideals with thresholds \( (\lambda, \mu) \) and intuitionistic fuzzy ideals with thresholds \( (\lambda, \mu) \) of \( X \).

Proposition 3. Any intuitionistic fuzzy positive implicative ideal with thresholds \( (\lambda, \mu) \) of \( X \) is an intuitionistic fuzzy ideal with thresholds \( (\lambda, \mu) \) of \( X \), but the converse does not hold.

Proof. Assume that \( A \) is an intuitionistic fuzzy positive implicative ideal with thresholds \( (\lambda, \mu) \) of \( X \) and put \( z = 0 \) in \( (IF_\lambda) \) and \( (IF_\mu) \), we get

\[
\mu_\lambda(x) \lor \lambda \geq \mu_\lambda(x \ast y) \land \mu_\lambda(y) \land \mu,
\]
\[
v_\lambda(x) \land \mu \leq v_\lambda(x \ast y) \lor v_\lambda(y) \lor \lambda.
\]

This means that \( A \) satisfies \( (IF_\lambda) \) and \( (IF_\mu) \). Combining \( (IF_\lambda) \) and \( (IF_\mu) \), \( A \) is an intuitionistic fuzzy ideal with thresholds \( (\lambda, \mu) \) of \( X \).

To show the last half part, we see the following example.

Example 2. Let \( X = \{0,1,2\} \) with Cayley table given by

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Define \( A = \{(x,\mu_\lambda(x),v_\lambda(x)) : x \in S\} \) where \( \mu_\lambda : X \rightarrow [0,1] \) and \( v_\lambda : X \rightarrow [0,1] \) by \( \mu_\lambda(0) = 2/3, \mu_\lambda(1) = \mu_\lambda(2) = 1/3, \)
\( v_\lambda(0) = 1/4, v_\lambda(1) = v_\lambda(2) = 1/2 \). Let \( \lambda = 1/8 \) and \( \mu = 3/4 \). It is easy to verify that \( A \) is an intuitionistic fuzzy ideal with thresholds \( (\lambda, \mu) \) of \( X \). But it is not an intuitionistic fuzzy positive implicative ideal with thresholds \( (\lambda, \mu) \) of \( X \) since:

\[
\mu_\lambda(2 \ast 1) \lor \lambda < \mu_\lambda(((2 \ast 1) \ast 1) \ast (0 \ast 1)) \land \mu_\lambda(0) \land \mu.
\]

Next, we give characterizations of intuitionistic fuzzy positive implicative ideals with thresholds \( (\lambda, \mu) \) of \( X \).

Proposition 4. Let \( A \) be an intuitionistic fuzzy ideal with thresholds \( (\lambda, \mu) \) of \( X \). Then the following are equivalent:

(i) \( A \) is an intuitionistic fuzzy positive implicative ideal with thresholds \( (\lambda, \mu) \) of \( X \),
(ii) \( \mu_\lambda(((x \ast y) \ast z) \ast z) \ast (y \ast z) \land \mu_\lambda(y) \land \mu_\lambda(z) \land \mu_\lambda(x), \)
\( v_\lambda(((x \ast y) \ast z) \ast z) \ast (y \ast z) \lor v_\lambda(y) \lor v_\lambda(z) \lor \lambda, \)
for all \( x, y, z \in X \),
(iii) \( \mu_\lambda(x \ast y) \lor \lambda \geq \mu_\lambda(((x \ast y) \ast y) \ast (0 \ast y)) \land \mu_\lambda(y) \land \mu_\lambda(z) \land \mu_\lambda(x), \)
\( v_\lambda((x \ast y) \ast y) \land \mu \leq v_\lambda(((x \ast y) \ast y) \ast (0 \ast y)) \lor v_\lambda(y) \lor v_\lambda(z) \lor \lambda, \)
for all \( x, y, z \in X \).

Proof. (i) \( \Rightarrow \) (ii) Suppose that \( A \) is an intuitionistic fuzzy positive implicative ideal with thresholds \( (\lambda, \mu) \) of \( X \). Since

\[
(((x \ast y) \ast z) \ast (0 \ast z)) = (((x \ast y) \ast z) \ast z) \ast ((y \ast y) \ast z)
\]
\[
= (((x \ast z) \ast y) \ast (y \ast z)) \leq ((x \ast z) \ast (y \ast z)),
\]
by \( (IF_\lambda),(IF_\mu),(IF_\mu),(IF_\mu) \) and Proposition 1, we have

\[
\mu_\lambda((x \ast y) \ast z) \lor \lambda \geq \mu_\lambda(((x \ast y) \ast z) \ast (0 \ast z)) \land \mu_\lambda(x) \land \mu_\lambda(y) \land \mu_\lambda(z) \land \mu_\lambda(x),
\]
\[
\geq \mu_\lambda(((x \ast y) \ast z) \ast (0 \ast z)) \land \lambda \land \mu_\lambda(x) \land \mu_\lambda(y) \land \mu_\lambda(z) \land \mu_\lambda(x),
\]
\[
\geq \mu_\lambda(((x \ast z) \ast (y \ast z)) \land \mu_\lambda(x) \land \mu_\lambda(y) \land \mu_\lambda(z) \land \mu_\lambda(x),
\]
\[
\mu_\lambda(((x \ast z) \ast (y \ast z)) \land \mu_\lambda(x) \land \mu_\lambda(y) \land \mu_\lambda(z) \land \mu_\lambda(x).
\]
\[ \leq \left( v_{\alpha}\left( (x*y)z \right) \right) \cap (0*z) \cup v_{\beta}(0) \cup \lambda \cap \mu \]
\[ = \left( v_{\alpha}\left( (x*y)z \right) \right) \cap (0*z) \cup (v_{\beta}(0) \cup \lambda) \cap (\lambda \cap \mu) \]
\[ \leq \left( v_{\alpha}\left( (x*y)z \right) \right) \cap (y*z) \cup \lambda \]
\[ \cup v_{\alpha}\left( (x*z) \right) \cap (y*z) \cup \lambda = v_{\alpha}\left( (x*z) \right) \cup (y*z) \cup \lambda. \]

Hence
\[ \mu_{\alpha}\left( (x*y)z \right) \cup \lambda \geq \mu_{\alpha}\left( (x*z) \right) \cup (y*z) \cup \lambda, \]
\[ v_{\alpha}\left( (x*y)z \right) \cup \lambda \leq v_{\alpha}\left( (x*z) \right) \cup (y*z) \cup \lambda. \]

and (ii) holds.

- (ii) ⇒ (iii) Substituting 0 for \( y \) and \( y \) for \( z \) in (ii), respectively, we have (iii).
- (iii) ⇒ (i) Since

\[ \left( \left( (x*y)z \right) \cap (0*y) \right) \cap \left( (x*y)z \right) \leq (z*y) \cap (0*y) \subseteq z, \]

by Proposition 2, we obtain
\[ \mu_{\alpha}\left( (x*y)z \right) \cup \lambda \geq \mu_{\alpha}\left( (x*z) \right) \cup (y*z) \cup \lambda, \]
\[ v_{\alpha}\left( (x*y)z \right) \cup \lambda \leq v_{\alpha}\left( (x*z) \right) \cup (y*z) \cup \lambda. \]

From (iii), we have
\[ \mu_{\alpha}\left( x*y \right) \cup \lambda = \left( \mu_{\alpha}\left( x*z \right) \right) \cup \lambda, \]
\[ \geq \mu_{\alpha}\left( (x*y)z \right) \cap (0*y) \cup \lambda \cup (\mu \cup \lambda) \]
\[ \geq \mu_{\alpha}\left( (x*y)z \right) \cup (y*z) \cup \lambda, \]
\[ v_{\alpha}\left( x*y \right) \cup \lambda \leq v_{\alpha}\left( (x*y)z \right) \cup (0*y) \cup \lambda \cup (\lambda \cup \mu) \]
\[ = v_{\alpha}\left( (x*y)z \right) \cup (y*z) \cup \lambda \]
\[ \leq v_{\alpha}\left( (x*y)z \right) \cup (y*z) \cup \lambda. \]

Hence, \( A \) is an intuitionistic fuzzy positive implicative ideal with thresholds \((\lambda, \mu)\) of \( X \).

**Proposition 5.** An intuitionistic fuzzy set \( A \) of \( X \) is an intuitionistic fuzzy positive implicative ideal with thresholds \((\lambda, \mu)\) of \( X \) if and only if, for all \( \alpha, \beta \in (\lambda, \mu) \), \( A_{\alpha, \beta} \) is either empty or a positive implicative ideal of \( X \).

**Proof.** Let \( A \) be an intuitionistic fuzzy positive implicative ideal with thresholds \((\lambda, \mu)\) of \( X \) and \( A_{\alpha, \beta} \neq \emptyset \) for some \( \alpha, \beta \in (\lambda, \mu) \). It is clear that \( 0 \in A_{\alpha, \beta} \). Let \( (x*z) \cap (y*z) \in A_{\alpha, \beta} \) and \( y \in A_{\alpha, \beta} \), then
\[ \mu_{\alpha}\left( (x*z) \right) \geq \mu_{\alpha}\left( (x*z) \right) \cap (y*z) \cap \mu_{\alpha}(y), \]
\[ v_{\alpha}\left( (x*z) \right) \leq v_{\alpha}\left( (x*z) \right) \cap (y*z) \cap v_{\alpha}(y). \]

It follows from \((IF_{\alpha})\) and \((IF_{\beta})\),
\[ \mu_{\alpha}\left( (x*z) \right) \cap (y*z) \cap \mu_{\alpha}(y) \geq \mu_{\alpha}\left( (x*z) \right) \cap (y*z) \cap \mu_{\alpha}(y) \]
\[ v_{\alpha}\left( (x*z) \right) \cap (y*z) \cap v_{\alpha}(y) \leq v_{\alpha}\left( (x*z) \right) \cap (y*z) \cap v_{\alpha}(y). \]

Namely, \( \mu_{\alpha}\left( (x*z) \right) \geq \mu_{\alpha}\left( (x*z) \right) \cap (y*z) \cap \mu_{\alpha}(y) \)
\[ v_{\alpha}\left( (x*z) \right) \leq v_{\alpha}\left( (x*z) \right) \cap (y*z) \cap v_{\alpha}(y). \]

This shows that \( A_{\alpha, \beta} \) is a positive implicative ideal of \( X \). Conversely, suppose that for each \( \alpha, \beta \in (\lambda, \mu) \), \( A_{\alpha, \beta} \) is either empty or a positive implicative ideal of \( X \). For any \( x \in X \), let \( \alpha = \mu_{\alpha}\left( (x*z) \right) \cap (y*z) \cap \mu_{\alpha}(y) \)
\[ v_{\alpha}\left( (x*z) \right) \leq v_{\alpha}\left( (x*z) \right) \cap (y*z) \cap v_{\alpha}(y). \]

i.e., \( \mu_{\alpha}(0) \geq \alpha \) and \( v_{\alpha}(0) \leq \beta \). We get
\[ \mu_{\alpha}(0) \geq \mu_{\alpha}(0) \geq \mu_{\alpha}(x) \cap \mu_{\alpha}(y) \]
\[ v_{\alpha}(0) \leq v_{\alpha}(0) \leq v_{\alpha}(x) \cap v_{\alpha}(y). \]

Hence \( (x*z) \cap (y*z) \in A_{\alpha, \beta} \), for all \( x \in X \). Now we only need to show that \( A \) satisfies \((IF_{\alpha})\) and \((IF_{\beta})\). Let
\[ \alpha = \mu_{\alpha}\left( (x*z) \right) \cap (y*z) \cap \mu_{\alpha}(y), \]
\[ \beta = v_{\alpha}\left( (x*z) \right) \cap (y*z) \cap v_{\alpha}(y). \]

Then
\[ \mu_{\alpha}\left( (x*z) \right) \cap (y*z) \cap \mu_{\alpha}(y) \geq \mu_{\alpha}\left( (x*z) \right) \cap (y*z) \cap \mu_{\alpha}(y) \]
\[ v_{\alpha}(0) \leq v_{\alpha}(0) \leq v_{\alpha}(x) \cap v_{\alpha}(y). \]

Hence \( (x*z) \cap (y*z) \in A_{\alpha, \beta} \) and \( y \in A_{\alpha, \beta} \). Since \( A_{\alpha, \beta} \) is a positive implicative ideal of \( X \), thus \( x*z \in A_{\alpha, \beta} \), i.e.,
\[ \mu_{\alpha}\left( (x*z) \right) \geq \mu_{\alpha}\left( (x*z) \right) \cap (y*z) \cap \mu_{\alpha}(y) \]
\[ v_{\alpha}(x*z) \leq v_{\alpha}(x*z) \cap v_{\alpha}(y). \]

We get
\[ \mu_{\alpha}\left( (x*z) \right) \cap (y*z) \cap \mu_{\alpha}(y) \geq \mu_{\alpha}\left( (x*z) \right) \cap (y*z) \cap \mu_{\alpha}(y) \]
\[ v_{\alpha}(x*z) \leq v_{\alpha}(x*z) \cap v_{\alpha}(y). \]

Namely,
\[ \mu_{\alpha}\left( (x*z) \right) \cap (y*z) \cap \mu_{\alpha}(y) \geq \mu_{\alpha}\left( (x*z) \right) \cap (y*z) \cap \mu_{\alpha}(y) \]
\[ v_{\alpha}(x*z) \leq v_{\alpha}(x*z) \cap v_{\alpha}(y). \]

Summarizing the above arguments, \( A \) is an intuitionistic fuzzy positive implicative ideal with thresholds \((\lambda, \mu)\) of \( X \).
Proposition 6 Let $J$ be a positive implicative ideal of $X$. Then there exists an intuitionistic fuzzy positive implicative ideal $A$ with thresholds $(\lambda, \mu)$ of $X$ such that $A_{\langle\alpha, \beta\rangle} = J$ for some $\alpha, \beta \in (\lambda, \mu]$.

Proof. Define $A = \{x, \mu_A(x), \nu_A(x)\}: x \in S\}$ by
$$
\mu_A(x) = \begin{cases} 
\alpha & \text{if } x \in J, \\
\lambda & \text{if } x \notin J,
\end{cases}
$$
$$
\nu_A(x) = \begin{cases} 
\beta & \text{if } x \in J, \\
\mu & \text{if } x \notin J,
\end{cases}
$$
where $\alpha, \beta$ are two fixed numbers in $(\lambda, \mu]$. Since $J$ is a positive implicative ideal of $X$, if $(x \in J) \land (y \in J)$ then $x \land y \in J$. Hence
$$
\mu_A((x \land y) \land (y \land z)) = \mu_A((x \land y) \land (y \land z)) \land \mu_A(y) \land \mu_A(\nu_A(y) \land \lambda).
$$
This means that $A$ satisfies $(IF_1)$ and $(IF_2)$. Since $0 \in J$, $\mu_A(0) \land \lambda = \alpha \land \mu$ and $\nu_A(0) \land \mu = \beta \land \nu_A(0) \land \nu_A(0) \land \lambda$, for all $x \in X$ and so $A$ satisfies $(IF_1)$ and $(IF_2)$. Thus, $A$ is an intuitionistic fuzzy positive implicative ideal with thresholds $(\lambda, \mu)$ of $X$. It is clear that $A_{\langle\alpha, \beta\rangle} = J$.

Definition 5. Let $S$ be any set. If
$$
A = \{x, \mu_A(x), \nu_A(x)\}: x \in S\}$
$$
be any two intuitionistic fuzzy subsets of $S$, then
$$
A \cap B = \{x, \mu_{A \cap B}(x), \nu_{A \cap B}(x)\}: x \in S\}
$$
where $\mu_{A \cap B}(x) = \mu_A(x) \land \mu_B(x)$ and $\nu_{A \cap B}(x) = \nu_A(x) \lor \nu_B(x)$.

Proposition 7 Let $A$ and $B$ be two intuitionistic fuzzy positive implicative ideals with thresholds $(\lambda, \mu)$ of $X$. Then $A \cap B$ is also an intuitionistic fuzzy positive implicative ideal with thresholds $(\lambda, \mu)$ of $X$.

Proof. For all $x, y, z \in X$, by Definition 4, we have
$$
\mu_{A \cap B}(0) \lor \lambda = (\mu_A(0) \lor \mu_B(0)) \lor \lambda = (\mu_A(0) \lor \lambda) \land (\mu_B(0) \lor \lambda)
$$
$$
\geq (\mu_A(x) \land \mu_B(x)) \land (\mu_B(x) \lor \lambda) = (\mu_A(x) \land \mu_B(x)) \land (\mu_B(x) \lor \lambda),
$$
$$
\nu_{A \cap B}(0) \land \mu = (\nu_A(0) \land \nu_B(0)) \land \mu = (\nu_A(0) \land \mu) \lor (\nu_B(0) \land \mu)
$$
$$
\leq (\nu_A(0) \lor \lambda) \lor (\nu_B(0) \lor \lambda) = \nu_{A \cap B}(x) \lor \lambda,
$$

Hence $A \cap B$ is an intuitionistic fuzzy positive implicative ideal with thresholds $(\lambda, \mu)$ of $X$.

Definition 6. Let $A$ and $B$ be two intuitionistic fuzzy sets of a set $X$. The Cartesian product of $A$ and $B$ is defined by
$$
A \times B = \{x, y: (x, y) \in A \times B\}
$$
where
$$
\mu_{A \times B}(x, y) = \mu_A(x) \land \mu_B(y), \nu_{A \times B}(x, y) = \nu_A(x) \lor \nu_B(y).
$$

Proposition 8. Let $A$ and $B$ be two intuitionistic fuzzy positive implicative ideals with thresholds $(\lambda, \mu)$ of $X$. Then $A \times B$ is also an intuitionistic fuzzy positive implicative ideal with thresholds $(\lambda, \mu)$ of $X \times X$.

Proof. For all $(x, y) \in X \times X$, by Definition 4, we get
$$
\mu_{A \times B}(0) \lor \lambda = (\mu_A(0) \lor \mu_B(0)) \lor \lambda = (\mu_A(0) \lor \mu_A(0)) \land (\mu_B(0) \lor \mu_B(0))
$$
$$
\geq (\mu_A(x) \land \mu_B(x)) \land (\mu_B(x) \lor \lambda) = (\mu_A(x) \land \mu_B(x)) \land (\mu_B(x) \lor \lambda),
$$
$$
\nu_{A \times B}(0) \land \mu = (\nu_A(0) \land \nu_B(0)) \land \mu = (\nu_A(0) \land \mu) \lor (\nu_B(0) \land \mu)
$$
$$
\leq (\nu_A(0) \lor \lambda) \lor (\nu_B(0) \lor \lambda) = \nu_{A \times B}(x, y) \lor \lambda,
$$

Hence $A \times B$ is an intuitionistic fuzzy positive implicative ideal with thresholds $(\lambda, \mu)$ of $X \times X$.
For all \((x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X\), we have

\[
\mu_{AB}(x_1 \ast z_1, x_2 \ast z_2) \vee \lambda = (\mu_A(x_1 \ast z_1) \land \mu_B(x_2 \ast z_2)) \vee \lambda \\
= (\mu_A(x_1 \ast z_1) \vee \lambda) \land (\mu_B(x_2 \ast z_2) \vee \lambda) \\
\geq (\mu_A((x_1 \ast z_1) \ast z_1) \ast (y_1 \ast z_1)) \land (\mu_B(y_1 \ast z_1) \land \mu) \\
\land (\mu_B((x_2 \ast z_2) \ast z_2) \ast (y_2 \ast z_2)) \land (\mu_A(y_2 \ast z_2) \land \mu) \\
= \mu_A((x_1 \ast z_1) \ast z_1) \ast (y_1 \ast z_1) \ast (x_2 \ast z_2) \ast (y_2 \ast z_2)) \\
\land \mu_B((x_1 \ast z_1) \ast z_1) \ast (y_1 \ast z_1) \ast (x_2 \ast z_2) \ast (y_2 \ast z_2)) \\
\land \mu_A(y_1 \ast z_1) \land \mu \\
\land \mu_B(y_2 \ast z_2) \land \mu \\
\nu_{AB}(x_1 \ast z_1, x_2 \ast z_2) \land \mu = (\nu_A(x_1 \ast z_1) \lor \nu_B(x_2 \ast z_2)) \land \mu \\
= (\nu_A(x_1 \ast z_1) \land \mu) \lor (\nu_B(x_2 \ast z_2) \land \mu) \\
\geq (\nu_A((x_1 \ast z_1) \ast z_1) \ast (y_1 \ast z_1)) \lor (\nu_B(y_1 \ast z_1) \lor \lambda) \\
\lor (\nu_B((x_2 \ast z_2) \ast z_2) \ast (y_2 \ast z_2)) \lor (\nu_A(y_2 \ast z_2) \lor \lambda) \\
= \nu_A((x_1 \ast z_1) \ast z_1) \ast (y_1 \ast z_1) \lor \nu_B((x_2 \ast z_2) \ast z_2) \lor (y_2 \ast z_2)) \\
\lor \nu_A(y_1 \ast z_1) \lor \nu_B(y_2 \ast z_2) \lor \lambda \\
= \nu_A((x_1 \ast z_1) \ast z_1) \ast (y_1 \ast z_1) \lor (x_2 \ast z_2) \ast (y_2 \ast z_2)) \\
\lor \nu_B(y_1 \ast z_1) \lor \nu_B(y_2 \ast z_2) \lor \lambda,
\]

Hence \(A \times B\) is an intuitionistic fuzzy positive implicative ideal with thresholds \((\lambda, \mu)\) of \(X \times X\).

**REFERENCES**