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Tree Sign Patterns of Small Order that Allow an Eventually Positive Matrix

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Abstract—A sign pattern is a matrix whose entries belong to the set $\{+, -, 0\}$. An n-by-n sign pattern $\mathcal{A}$ is said to allow an eventually positive matrix if there exist some real matrices A with the same sign pattern as $\mathcal{A}$ and a positive integer k_0 such that $A^k > 0$ for all $k \geq k_0$. It is well known that identifying and classifying the n-by-n sign patterns that allow an eventually positive matrix are posed as two open problems. In this article, the tree sign patterns of small order that allow an eventually positive matrix are classified completely.

Keywords—Eventually positive matrix, sign pattern, tree.

I. Introduction

LET $\mathcal{A} = [\alpha_{ij}]$ be a sign pattern matrix. An n-by-n real matrix A with the same sign pattern as $\mathcal{A}$ is called a realization of $\mathcal{A}$. The set of all realizations of sign pattern $\mathcal{A}$ is called the qualitative class of $\mathcal{A}$ and is denoted by $Q(\mathcal{A})$. A subpattern of $\mathcal{A} = [\alpha_{ij}]$ is an n-by-n sign pattern $\mathcal{B} = [\beta_{ij}]$ such that $\beta_{ij} = 0$ whenever $\alpha_{ij} = 0$. If $\mathcal{B} \neq \mathcal{A}$, then $\mathcal{B}$ is a proper subpattern of $\mathcal{A}$. If $\mathcal{B}$ is a subpattern of $\mathcal{A}$, then $\mathcal{A}$ is said to be a superpattern of $\mathcal{B}$. A pattern $\mathcal{A}$ is reducible if there is a permutation matrix $\mathcal{P}$ such that

$$\mathcal{P}^T \mathcal{A} \mathcal{P} = \begin{bmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix},$$

where $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are square matrices of order at least one. A pattern is irreducible if it is not reducible; see, e.g. [1]

A sign pattern matrix $\mathcal{A}$ is said to require a certain property P referring to real matrices if every real matrix $A \in Q(\mathcal{A})$ has the property P and allow P or be potentially P if there is some $A \in Q(\mathcal{A})$ that has property P.

An n-by-n sign pattern $\mathcal{A}$ is said to allow an eventually positive matrix or be potentially eventually positive (PEP), if there exists some $A \in Q(\mathcal{A})$ such that A is eventually positive; see, e.g., [2] and [3]. An n-by-n sign pattern $\mathcal{A}$ is said to be a minimal potentially eventually positive sign pattern (MPEP sign pattern) if $\mathcal{A}$ is PEP and no proper subpattern of $\mathcal{A}$ is PEP; see, e.g. [4] and [5]. Sign patterns that allow an eventually positive matrix have been studied first in [3], where a sufficient condition and some necessary conditions for a sign pattern to be potentially eventually positive have been established. However, the identification of necessary and sufficient conditions for an n-by-n sign pattern ($n \geq 4$) to be potentially eventually positive remains open. Also open is the classification of sign patterns that are potentially eventually positive.

In the last few years, there has been an increasing interest in potentially eventual properties of sign patterns; see, e.g., [4], [5], [6], [7] and references therein. For example, the minimal potentially eventually positive tridiagonal sign patterns have been identified and all potentially eventually positive tridiagonal sign patterns have been classified in [6].

In this article, we focus on the tree sign patterns of small order. Our work is organized as follows. In Section II, some preliminary results are established. The potentially eventually positive tree sign patterns of small order are classified in Section III. Concluding remarks are addressed in Section IV.

II. Preliminary Results

We begin this section with introducing some necessary graph theoretical concepts which can be seen from [1], [2] and the references therein.

A square sign pattern $\mathcal{A} = [\alpha_{ij}]$ is combinatorially symmetric if $\alpha_{ij} \neq 0$ whenever $\alpha_{ji} \neq 0$. Let $G(\mathcal{A})$ be the graph of order n with vertices 1, 2, ..., n and an edge $\{i, j\}$ joining vertices i and j if and only if $i \neq j$ and $\alpha_{ij} \neq 0$. We call $G(\mathcal{A})$ the graph of the pattern $\mathcal{A}$. A combinatorially symmetric sign pattern matrix $\mathcal{A}$ is called a tree sign pattern if $G(\mathcal{A})$ is a tree. Similarly, path (or tridiagonal) sign patterns can be defined.

A sign pattern $\mathcal{A} = [\alpha_{ij}]$ has signed digraph $\Gamma(\mathcal{A})$ with vertex set $\{1, 2, \cdots, n\}$ and a positive (respectively, negative) arc from i to j if and only if α_{ij} is positive (respectively, negative). A (directed) simple cycle of length k is a sequence of k arcs $(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_1)$ such that the vertices $i_1, \cdots, i_k$ are distinct. Recall that a digraph $D = (V, E)$ is primitive if it is strongly connected and the greatest common divisor of the lengths of its cycles is 1. It is well known that a digraph D is primitive if and only if there exists a natural number k such that for all $V_i \in V$, $V_j \in V$, there is a walk of length k from V_i to V_j. A nonnegative sign pattern $\mathcal{A}$ is primitive if its signed digraph $\Gamma(\mathcal{A})$ is primitive; see, e.g. [3] for more details.

For a sign pattern $\mathcal{A} = [\alpha_{ij}]$, we define the positive part of $\mathcal{A}$ to be $\mathcal{A}^+ = [\alpha_{ij}^+]$, where $\alpha_{ij}^+ = +$ for $\alpha_{ij} = +$, otherwise $\alpha_{ij}^+ = 0$. The negative part of $\mathcal{A}$ can be defined similarly. In [3], it has been shown that if sign pattern $\mathcal{A}^+$ is primitive, then $\mathcal{A}$ is PEP. Here, we cite some necessary conditions for an n-by-n sign pattern to be potentially eventually positive in [3] as Lemmas 1 to 5 in order to state our work clearly.

Lemma 1. *If the n-by-n sign pattern $\mathcal{A}$ is PEP, then every*

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superpattern of $\mathcal{A}$ is PEP.

Lemma 2. *If the n-by-n sign pattern $\mathcal{A}$ is PEP, then the sign pattern $\hat{\mathcal{A}}$ obtained from sign pattern $\mathcal{A}$ by changing all 0 and $-$ diagonal entries to $+$ is also PEP.*

Lemma 3. *If the n-by-n sign pattern $\mathcal{A}$ is PEP, then there is an eventually positive matrix $A \in Q(\mathcal{A})$ such that*

(1) $\rho(A) = 1$.

(2) $A\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the $n \times 1$ all ones vector.

If $n \geq 2$, the sum of all the off-diagonal entries of A is positive.

We denote a sign pattern consisting entirely of positive (respectively, negative) entries by $[+]$ (respectively, $[-]$). For block sign patterns, we have the following two lemmas.

Lemma 4. *If $\mathcal{A}$ is the checkerboard block sign pattern*

$$\begin{bmatrix} [+] & [-] & [+] & \cdots \\ [-] & [+] & [-] & \cdots \\ [+] & [-] & [+] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with square diagonal blocks. Then $-\mathcal{A}$ is not PEP, and if $\mathcal{A}$ has a negative entry, then $\mathcal{A}$ is not PEP.

Lemma 5. *Let the n-by-n sign pattern*

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix},$$

where $\mathcal{A}_{11}$, $\mathcal{A}_{22}$ are square. If $\mathcal{A}_{12} = \mathcal{A}_{12}^{-}$, $\mathcal{A}_{21} = \mathcal{A}_{21}^{+}$, then $\mathcal{A}$ and $\mathcal{A}^T$ are not PEP.

Following [6], let $\mathcal{T}_i = [\tau_{ij}]$ be the irreducible tridiagonal sign pattern with $\tau_{j,j+1} = \tau_{j+1,j} = +$ for $j = 1, 2, \ldots, n-1$ and all diagonal entries are zero except the (i,i) entry $\tau_{ii} = +$. In [6], all minimal potentially eventually positive tridiagonal sign patterns have been identified in Theorem 2. We cite it here as the following lemma.

Lemma 6. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ irreducible tridiagonal sign pattern. Then $\mathcal{A}$ is a minimal potentially eventually positive sign pattern if and only if $\mathcal{A}$ is equivalent to one of the sign patterns $\mathcal{T}_1$, $\mathcal{T}_2$, …, $\mathcal{T}_{\lceil \frac{n}{2} \rceil}$.*

III. Tree Sign Patterns of Small Order that Allow an Eventually Positive Matrix

It is known that an n-by-n real matrix A is eventually positive if and only if A^T is eventually positive, and if only if both matrices A and A^T possess the strong Perron-Forbenius property, i.e., the spectral radius $\rho(A)$ is a simple, positive and strictly dominant eigenvalue and the corresponding eigenvector is positive.

We note that the potential eventual positivity of sign patterns is preserved under permutation similar and transposition. That is, an n-by-n sign pattern $\mathcal{A}$ is PEP if and only if $\mathcal{P}^T\mathcal{A}\mathcal{P}$ or $\mathcal{P}^T\mathcal{A}^T\mathcal{P}$ is PEP, where $\mathcal{P}$ is an n-by-n permutation pattern. In this case, we say they are equivalent.

To consider the minimality of potentially eventually positive tree sign pattern $\mathcal{A}$, it is necessary to discuss the number of positive diagonal entries of potentially eventually positive tree sign pattern $\mathcal{A}$.

Now, we classify the tree sign patterns of small order that are potentially eventually positive. Proposition 1 is clear and its proof is omitted.

Proposition 1. *The 1-by-1 tree sign pattern $\mathcal{A}$ is potentially eventually positive if and only if $\mathcal{A} = [+]$.*

Proposition 2. *The 2-by-2 tree sign pattern $\mathcal{A}$ is potentially eventually positive if and only if $\mathcal{A}$ is a superpattern of*

$$\begin{bmatrix} + & + \\ + & 0 \end{bmatrix}.$$

Proof. Proposition 2 follows readily from the fact that sign pattern $\begin{bmatrix} + & + \\ + & 0 \end{bmatrix}$ is a minimal potentially eventually positive sign pattern in [4] and [6].

Proposition 3. *The 3-by-3 tree sign pattern $\mathcal{A}$ is potentially eventually positive if and only if $\mathcal{A}$ is a superpattern of*

$$\begin{bmatrix} + & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & + & 0 \\ + & + & + \\ 0 & + & 0 \end{bmatrix},$$

up to equivalence.

Proof. In fact, a 3-by-3 tree sign pattern is equivalent to a path (tridiagonal) sign pattern. By Lemma 6, we have two tridiagonal sign patterns

$$\begin{bmatrix} + & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & + & 0 \\ + & + & + \\ 0 & + & 0 \end{bmatrix}$$

which are minimal potentially eventually positive sign patterns. Consequently, Proposition 3 follows directly from Corollary 2 in [6].

Now, we turn our attention to the tree sign patterns of order 4 that allow eventual positivity.

Proposition 4. *The 4-by-4 path sign pattern $\mathcal{A}$ is potentially eventually positive if and only if $\mathcal{A}$ is a superpattern of*

$$\mathcal{A}_1 = \begin{bmatrix} + & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & 0 \end{bmatrix},$$

or

$$\mathcal{A}_2 = \begin{bmatrix} 0 & + & 0 & 0 \\ + & + & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & 0 \end{bmatrix}.$$

Proof. By Lemma 6, $\mathcal{A}_1$ and $\mathcal{A}_2$ are minimal potentially eventually positive sign patterns. Since every superpattern of a minimal potentially eventually positive sign pattern is also potentially eventually positive, path sign pattern $\mathcal{A}$ is potentially eventually positive. Conversely, suppose that the 4-by-4 path sign pattern $\mathcal{A}$ is potentially eventually positive. Then $\mathcal{A}$ is equivalent to a superpattern of $\mathcal{A}_1$ and $\mathcal{A}_2$ by Lemma 6.

Proposition 5. *Let sign patterns*

$$\mathcal{A}_3 = \begin{bmatrix} + & + & + & + \\ + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \end{bmatrix},$$

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and

$$\mathcal{A}_4 = \begin{bmatrix} 0 & + & + & + \\ + & + & 0 & 0 \\ + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \end{bmatrix}.$$

Then both $\mathcal{A}_3$ and $\mathcal{A}_4$ are minimal potentially eventually positive sign patterns.

Proof. Since sign patterns $\mathcal{A}_3$ and $\mathcal{A}_4$ are nonnegative and primitive, it follows that both $\mathcal{A}_3$ and $\mathcal{A}_4$ are potentially eventually positive. For the minimality, we show that every proper subpattern of $\mathcal{A}_3$ and $\mathcal{A}_4$ is not potentially eventually positive. By a way of contradiction, assume that there exists some $i \in \{2,3,4\}$ such that $\alpha_{1i} = 0$ or $\alpha_{i1} = 0$. Then both $\mathcal{A}_3$ and $\mathcal{A}_4$ are not irreducible. Consequently, $\mathcal{A}_3$ and $\mathcal{A}_4$ are not potentially eventually positive; a contradiction. If $\alpha_{11} = 0$ for sign pattern $\mathcal{A}_3$, then $\mathcal{A}_3$ is a proper subpattern of

$$\hat{\mathcal{A}}_3 = \begin{bmatrix} [-] & [+] \\ [+] & [-] \end{bmatrix},$$

where diagonal block matrices are of orders 1 and 3, respectively. By Lemma 1, we have $\hat{\mathcal{A}}_3$ is also potentially eventually positive. But the potentially eventually positive sign pattern $\hat{\mathcal{A}}_3$ is a checkerboard block sign pattern. It follows that Lemma 4 is contradicted. Thus, no proper subpattern of $\mathcal{A}_3$ is potentially eventually positive and $\mathcal{A}_3$ is a minimal potentially eventually positive sign pattern. Similarly, we can show that $\mathcal{A}_4$ is also a minimal potentially eventually positive sign pattern.

The following theorem identifies all 4-by-4 star sign patterns that are minimal potentially eventually positive sign patterns.

Theorem 1. *The 4-by-4 star sign pattern $\mathcal{A}$ is a minimal potentially eventually positive sign pattern if and only if $\mathcal{A}$ is equivalent to one of $\mathcal{A}_3$ and $\mathcal{A}_4$.*

Proof. The sufficiency is shown in Proposition 5.

For the necessity, suppose that the 4-by-4 star sign pattern $\mathcal{A}$ is a minimal potentially eventually positive sign pattern. We first claim that $\mathcal{A}$ has at least one positive diagonal entry. By a way of contradiction, suppose that all diagonal entries are $-$ or 0. Then $\alpha_{1i} = +$ and $\alpha_{i1} = +$, for all $i = 2,3,4$. Consequently, sign pattern $\mathcal{A}$ is a proper subpattern of

$$\hat{\mathcal{A}} = \begin{bmatrix} [-] & [+] \\ [+] & [-] \end{bmatrix},$$

But the checkerboard block sign pattern $\hat{\mathcal{A}}$ is not potentially eventually positive. Hence, Lemma 1 is contradicted.

The second claim is that $\alpha_{1i} = \alpha_{i1} = +$, for all $i = 2,3,4$. By a way of contradiction, suppose that $\alpha_{1k} =, \alpha_{k1} = -$, for some $k \in \{2,3,4\}$. Up to equivalence, it suffices to consider the following three sign patterns:

$$\mathcal{A}_5 = \left[\begin{array}{cc|cc} ? & + & + & + \\ + & ? & 0 & 0 \\ \hline - & 0 & ? & 0 \\ - & 0 & 0 & ? \end{array}\right],$$

$$\mathcal{A}_6 = \left[\begin{array}{ccc|c} ? & + & + & + \\ + & ? & 0 & 0 \\ + & 0 & ? & 0 \\ \hline - & 0 & 0 & ? \end{array}\right],$$

and

$$\mathcal{A}_7 = \left[\begin{array}{c|ccc} ? & + & + & + \\ \hline + & ? & 0 & 0 \\ + & 0 & ? & 0 \\ - & 0 & 0 & ? \end{array}\right].$$

By Lemma 5, it can be shown easily that sign patterns $\mathcal{A}_5$, $\mathcal{A}_6$ and $\mathcal{A}_7$ are not potentially eventually positive. It follows that $\alpha_{1i} = \alpha_{i1} = +$, for all $i = 2,3,4$.

Finally, we claim that if the 4-by-4 star sign pattern $\mathcal{A}$ is a minimal potentially eventually positive sign pattern, then $\mathcal{A}$ has exactly one positive diagonal entry. Otherwise, $\mathcal{A}$ is not a minimal potentially eventually positive sign pattern and Proposition 5 is contradicted. It follows that sign pattern $\mathcal{A}$ is equivalent to one of $\mathcal{A}_3$ and $\mathcal{A}_4$.

Theorem 2. *The 4-by-4 tree sign pattern $\mathcal{A}$ is potentially eventually positive if and only if $\mathcal{A}$ is equivalent to a superpattern of one of*

$$\mathcal{A}_1 = \begin{bmatrix} + & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & 0 \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} 0 & + & 0 & 0 \\ + & + & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & 0 \end{bmatrix},$$

$$\mathcal{A}_3 = \begin{bmatrix} + & + & + & + \\ + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \end{bmatrix},$$

and

$$\mathcal{A}_4 = \begin{bmatrix} 0 & + & + & + \\ + & + & 0 & 0 \\ + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \end{bmatrix}.$$

Proof. The sufficiency follows from Theorem 1 and Proposition 4.

For the necessity, it suffices to consider the path sign patterns and star sign patterns of order 4, respectively. If the 4-by-4 tree sign pattern $\mathcal{A}$ is a path sign pattern and is potentially eventually positive, then by Proposition 4, $\mathcal{A}$ is a superpattern of one of $\mathcal{A}_1$ and $\mathcal{A}_2$. If the 4-by-4 tree sign pattern $\mathcal{A}$ is a star sign pattern and is potentially eventually positive, then by Theorem 1, $\mathcal{A}$ is a superpattern of one of $\mathcal{A}_3$ and $\mathcal{A}_4$.

Recall that an arbitrary n-by-n sign pattern $\mathcal{A}$ is said to require an eventually positive matrix, if every matrix $A \in Q(\mathcal{A})$ is eventually positive; see e.g., [8]. It is obvious that an arbitrary sign pattern $\mathcal{A}$ requires eventual positivity, then $\mathcal{A}$ is potentially eventually positive. But the converse is not true. We end this section by drawing an interesting conclusion about minimal potentially eventually positive tree sign patterns of small order and tree sign patterns that require eventual positivity.

Corollary 1. *Let $\mathcal{A}$ be an $n \times n$ $(n \leq 4)$ tree sign pattern with $2n-1$ nonzero entries. Then the following are equivalent:*

1) *$\mathcal{A}$ is a minimal potentially eventually positive sign pattern;*
2) *$\mathcal{A}$ is nonnegative and primitive;*
3) *$\mathcal{A}$ requires an eventual positive matrix.*

Proof. That 1) implies 2) follows from the fact that the minimal potentially eventually positive tree sign patterns are nonnegative and primitive. That 2) implies 3) follows from Theorem 2.3 in [8]. Finally, we show that 3) implies 1). suppose that $\mathcal{A}$ requires an eventual positive matrix. Then $\mathcal{A}$ is potentially eventually positive. By Propositions 1, 2, 3, 4 and Theorem 2, $\mathcal{A}$ is equivalent to a superpattern of some minimal potentially eventually positive sign patterns. Since $\mathcal{A}$ has exactly $2n-1$ nonzero entries, $\mathcal{A}$ is a minimal potentially eventually positive sign pattern.

IV. Conclusion

By considering the minimality of a potentially eventually positive sign patterns, the tree sign patterns of small order n (≤ 4) that allow an eventually positive matrix are classified as the superpatterns of four specific minimal potentially eventually positive tree sign patterns. It seems that the minimal potentially eventually positive sign patterns is essential for identifying and classifying the potentially eventually positive sign patterns. In the future work, classifying the potentially eventually positive tree sign patterns of larger order is necessary.

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