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An Iterative Method for the Symmetric Arrowhead Solution of Matrix Equation

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***Abstract*—**In this paper, according to the classical algorithm *LSQR* for solving the least-squares problem, an iterative method is proposed for least-squares solution of constrained matrix equation. By using the Kronecker product, the matrix-form *LSQR* is presented to obtain the like-minimum norm and minimum norm solutions in a constrained matrix set for the symmetric arrowhead matrices. Finally, numerical examples are also given to investigate the performance.

***Keywords*—**Symmetric arrowhead matrix, iterative method, like-minimum norm, minimum norm, Algorithm *LSQR*.

I. Introduction

LET $R^{m\times n}$ be the set of $m\times n$ real matrices, $SAR^{n\times n}$ be the set of $n\times n$ real symmetric arrowhead matrices and I_n be the identity matrix of order *n*. For any $A\in R^{m\times n}$, A^T, $A^\dagger$, $\|A\|_F$ and $\|A\|_2$ denote the transpose, Moore-Penrose generalized inverse, Frobenius norm and Euclid norm, respectively.

For $A=\left(x_{ij}\right)\in R^{m\times n}$ and $B\in R^{n\times s}$, $A\otimes B=\left(x_{ij}B\right)\in R^{mn\times ns}$ represents the Kronecker product of *A* and *B*.

For $A=\left(x_1,\cdots,x_n\right)\in R^{n\times n}$, define $vec\left(A\right)=\left(x_1^T,x_2^T,\cdots,x_n^T\right)^T$ and $x_{\alpha:\beta,i}$ as the sub-vector consisting of the elements form αth component to βth component of x_i. The inverse mapping of $vec(\cdot)$ form R^{mn} to $R^{m\times n}$ which is denoted by $mat(\cdot)$ satisfies $mat\left(vec\left(A\right)\right)=A$.

Definition 1. *Let* $A\in SAR^{n\times n}$. A is called the symmetric arrowhead matrix if it has the following form:

$$A=\begin{pmatrix} x_{11} & x_{21} & x_{31} & \cdots & x_{n1}\\ x_{21} & x_{22} & 0 & \cdots & 0\\ x_{31} & 0 & x_{33} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ x_{n1} & 0 & 0 & \cdots & x_{nn}\end{pmatrix}$$

and $vec_i\left(A\right)$ stands for the corresponding vector

$$\left(x_{1:n,1}^T,x_{22},x_{33},\cdots,x_{nn}\right)^T.$$

Symmetric arrowhead matrix has many applications in modern control theory which can represent the parameter matrix of nonlinear control systems. Such a matrix was described as radiationless transition in the isolated molecules and oscillator attached to a Fermi liquid [1]. At present, their potential applications in electromagnetic compatibility have been more important such as the mathematical representation of interference factor.

Based on the study of [2], we consider the matrix equation

$$AXB+CYD=E. \tag{1}$$

Many people have studied the matrix equation above and other constrained matrix equations, see [3], [4], [6], [8], etc. Xu, Wei, and Zhang [2] gave the solution of (1) by making use of the canonical correlation decomposition (CCD). Liao, Bai, and Lei [5] studied the least-squares solution of (1) with the least norm by combining CCD and general singular value decomposition (GSVD).

In this paper, we discuss the least-squares solution of (1) for the symmetric arrowhead matrix. When $A\in R^{m\times n},B\in R^{n\times s}$, $C\in R^{m\times k}$, $D\in R^{k\times s},E\in R^{m\times s}$ and

$$S_w=\left\{\left[X,Y\right]\middle|X\in SAR^{n\times n},Y\in SAR^{k\times k}\right\}$$

finding out $\left[X,Y\right]\in S_w$, such that

$$\min\left\|AXB+CYD-E\right\|_F. \tag{2}$$

In [7], by using Moore-penrose inverse and the Kronecker product, it discussed the best approximation problem (2) and obtained a general expression of solutions.

For $H=diag\left(H_n,H_k\right),P_1=\left(B^T\otimes A\right)H_n,P_2=\left(B^T\otimes A\right)H_k$, and finding $R=\left(I-P_1P_1^+\right)P_2$,

$$\begin{aligned} G&=R^++\left(I-R^+R\right)ZP_2^T\left(P_1^+\right)^TP_1^+\left(I-P_2R^+\right),\\ Z&=\left(I+\left(I-R^+R\right)P_2^T\left(P_1^+\right)^TP_1^+P_2\left(I-R^+R\right)\right)^{-1},\\ K_{11}&=I-P_1^+P_1+P_1^+P_2Z\left(I-R^+R\right)P_2^T\left(P_1^+\right)^T,\\ K_{12}&=-P_1^+P_2\left(I-R^+R\right)Z,\ K_{22}=\left(I-R^+R\right)Z,\end{aligned}$$

then the set of solutions S_W was expressed as

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$$S_W = \left\{ [X,Y] \left| \begin{bmatrix} vec(X) \\ vec(Y) \end{bmatrix} = H \begin{bmatrix} P_1^+ - P_1^+ P_2 G \\ G \end{bmatrix} vec(E) + H \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{13} \end{bmatrix} y \right. \right\}$$

where y is an arbitrary vector with the proper dimension. However, the huge computation cannot be easy to realize in the large scale system which motivates us to find an operable iterative method.

A matrix pair $[X,Y]$ is referred to a minimum norm solution if it minimizes

$$\|X\|_F^2 + \|Y\|_F^2, \tag{3}$$

and a like-minimum norm solution if it minimizes

$$\|tril(X)\|_F^2 + \|tril(Y)\|_F^2, \tag{4}$$

where $tril(X)$ is denoted as lower triangular part of X, that is

$$tril(X) = \begin{pmatrix} x_{11} & 0 & \cdots & 0 \\ x_{21} & x_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}.$$

II. Preliminaries

To study the problem (2), we begin with the following lemma and the classical Algorithm *LSQR* presented for solving least-squares problem [3].

Lemma 1. Let $X \in SAR^{n\times n}$, then $vec(X) = Hvec_i(X)$, where

$$H_n = \begin{pmatrix} e_1 & e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 & 0 \\ 0 & e_1 & 0 & \cdots & 0 & 0 & e_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & e_1 & \cdots & 0 & 0 & 0 & e_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_1 & 0 & 0 & 0 & \cdots & e_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & e_1 & 0 & 0 & \cdots & 0 & e_n \end{pmatrix},$$

$H_n \in R^{n^2\times(2n-1)}$ and $e_i = (\underbrace{0,\cdots,0}_{i-1},1,\underbrace{0,\cdots0}_{n-i})^T$.

First, let us review the Algorithm *LSQR* for solving the least-squares problem:

$$\min_{\varphi\in R^n} \|M\varphi - f\|_2 \tag{5}$$

with given $M \in R^{m\times n}$ and vector $f \in R^m$, whose normal equation is

$$M^T M \varphi = M^T f. \tag{6}$$

The algorithm is summarized as follows.

Algorithm *LSQR*

(1) Initialization

$$\beta_1 u_1 = f, \alpha_1 v_1 = M^T u_1, h_1 = v_1,$$
$$x_0 = 0, \bar{\zeta}_1 = \beta_1, \bar{\rho}_1 = \alpha_1;$$

(2) Iteration. For $i = 1,2\cdots$

(i) bidiagonalization

(a) $\beta_{i+1}u_{i+1} = Mv_i - \alpha_i u_i$,

(b) $\alpha_{i+1}v_{i+1} = M^T u_{i+1} - \beta_{i+1}v_i$;

(ii) construct and use Givens rotation:

$$\rho_i = \sqrt{\bar{\rho}_i^2 + \beta_{i+1}^2},$$
$$c_i = \bar{\rho}_i / \rho_i, s_i = \beta_{i+1} / \rho_i, \theta_{i+1} = s_i\alpha_{i+1},$$
$$\bar{\rho}_{i+1} = -c_i\alpha_{i+1}, \zeta_i = c_i\bar{\zeta}_i, \bar{\zeta}_{i+1} = s_i\bar{\zeta}_i;$$

(iii) update x an h

$$\varphi_i = \varphi_{i-1} + (\zeta_i / \rho_i) h_i,$$
$$h_{i+1} = v_{i+1} - (\theta_{i+1} / \rho_i) h_i;$$

(iv) Check convergence.

We can choose

$$\left\|M^T(f - M\varphi_k)\right\|_2 = \left|\alpha_{k+1}\bar{\zeta}_{k+1}c_k\right| < \tau \tag{7}$$

as convergence criteria, where $\tau > 0$ is a small tolerance. Note that if (5) has a solution $\varphi \in R(M^T M) \in R(M^T)$, then φ which is generated by Algorithm *LSQR* is the minimum norm solution of (5). Then we can have the symmetric arrowhead matrix solution generated by Algorithm *LSQR* which is the like-minimum norm and minimum norm solution of (2).

III. The Matrix-form Algorithm *LSQR* for (2)

A. An Iterative Method for the Like-Minimum Norm Solution of (2)

In this section, we will give some results of this paper and propose iterative methods based on Algorithm *LSQR*.

Theorem 1. Let $X \in SAR^{n\times n}$, and $vec(X) = Hvec_i(X)$, then $H^T HH^\dagger = H^T$.

Proof: It follows Lemma 1,

$$H^T H = \begin{pmatrix} e_1^T & 0 & \cdots & 0 & 0 \\ e_2^T & e_1^T & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ e_{n-1}^T & 0 & \cdots & e_1^T & 0 \\ e_n^T & 0 & \cdots & 0 & e_1^T \\ 0 & e_2^T & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & e_{n-1}^T & 0 \\ 0 & 0 & \cdots & 0 & e_n^T \end{pmatrix} \begin{pmatrix} e_1 & e_2 & \cdots & e_{n-1} & e_n & 0 & \cdots & 0 & 0 \\ 0 & e_1 & \cdots & 0 & 0 & e_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & e_1 & 0 & 0 & \cdots & e_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & e_1 & 0 & \cdots & 0 & e_n \end{pmatrix}$$
$$= \begin{pmatrix} 1 & & \\ & 2I_{n-1} & \\ & & I_{n-1} \end{pmatrix} \in R^{(2n-1)\times(2n-1)},$$

We can obtain matrix H is full column rank, with

$$\left(H^T HH^\dagger\right)^T = \left(HH^\dagger\right)^T \left(H^T\right)^T = HH^\dagger H = H. \ \square$$

Theorem 2. Let $U \in R^{m\times s}$ and $P = A^T UB^T$, $Q = \frac{(P+P^T)}{2}$. Then for the symmetric arrowhead constrained matrix H, we have

$$\left(\left(B^T \otimes A\right)H\right)^T vec(U) = \left(H^T H\right)vec_i\left(E_{11}Q + \left(E_{11}Q\right)^T + diag(Q) - 2E_{11}QE_{11}\right)$$

where $E_{11} = e_1 e_1^T$.

Proof. Notice that

$$\begin{aligned}\left(\left(B^T \otimes A\right)H\right)^T vec(U) &= H^T\left(B \otimes A^T\right)vec(U) = H^T vec\left(A^T UB^T\right)\\ &= \left(H^T H\right)H^\dagger vec\left(A^T UB^T\right) = \left(H^T H\right)H^\dagger vec(P).\end{aligned}$$

From Theorem 1, for any $P \in R^{n\times n}$, we have

$$\begin{aligned}H^\dagger vec(P) &= \left(H^T H\right)^{-1} H^T\left(p_{11},\cdots,p_{n1},p_{12},\cdots,p_{n2},\ \cdots,p_{1n},\cdots,p_{nn}\right)^T\\ &= \left(H^T H\right)^{-1}\left(p_{11},p_{12}+p_{21},p_{13}+p_{31},\cdots,p_{1n}+p_{n1},p_{22},p_{33},\cdots,p_{nn}\right)^T\end{aligned}$$

that is

$$H^\dagger vec(P) = H^\dagger vec\left(P^T\right) = H^\dagger vec\left(\frac{P+P^T}{2}\right).$$

It is easy to obtain that

$$\begin{aligned}H^\dagger vec\left(\frac{P+P^T}{2}\right) &= H^\dagger vec\left(E_{11}Q + \left(E_{11}Q\right)^T + diag(Q) - 2E_{11}QE_{11}\right)\\ &= vec_i\left(E_{11}Q + \left(E_{11}Q\right)^T + diag(Q) - 2E_{11}QE_{11}\right).\end{aligned}$$

From all above, we can have

$$\left(\left(B^T \otimes A\right)H\right)^T vec(U) = \left(H^T H\right)vec_i\left(E_{11}Q + \left(E_{11}Q\right)^T + diag(Q) - 2E_{11}QE_{11}\right). \square$$

Since

$$\begin{aligned}\left\|AXB + CYD - E\right\|_F^2 &= \left\|\left(B^T \otimes A, D^T \otimes C\right)\begin{pmatrix} vec(X)\\ vec(Y)\end{pmatrix} - vec(E)\right\|_F^2\\ &= \left\|\left(\left(B^T \otimes A\right)H_1, \left(D^T \otimes C\right)H_2\right)\begin{pmatrix} vec_i(X)\\ vec_i(Y)\end{pmatrix} - vec(E)\right\|_F^2,\end{aligned}$$

the symmetric arrowhead constrained problem is equivalent to (5) and

$$M = \left(\left(B^T \otimes A\right)H_1, \left(D^T \otimes C\right)H_2\right) \in R^{ms\times(2n+2k-2)}, \tag{8}$$

$$f = vec(E) \in R^{ms}, \varphi = \begin{pmatrix} vec_i(X)\\ vec_i(Y)\end{pmatrix} \in R^{2n+2k-2}, \tag{9}$$

where H_1 and H_2 are the symmetric arrowhead constrained matrices of degree *n* and *k*, respectively. Therefore, the normal equation of (2) is

$$M^T M\varphi = M^T f = M^T vec(E).$$

Now, we will apply Algorithm *LSQR* to (2) and the iterative vector will be transformed into matrix so that Kronecker product and constrained matrix *H* can be released. Then the vector *u* and *v* will be expressed by matrix *U* and *V* respectively so as to transform the matrix-vector product of Mv and $M^T u$ to the matrix-matrix form.

Let $u = vec(U) \in R^{ms}$ with $U \in R^{m\times s}$, $v = \begin{pmatrix} v_1\\ v_2\end{pmatrix} \in R^{2n+2k-2}$, where $v_1 = vec_i(V_1)$ and $v_2 = vec_i(V_2)$ with $V_1 \in SAR^{n\times n}$, $V_2 \in SAR^{k\times k}$. Denoted by

$$W = E_{11}Q + \left(E_{11}Q\right)^T + diag(Q) - 2E_{11}QE_{11},$$

and according to Theorem 2, we have

$$\begin{aligned}mat\left(M^T vec(u)\right) &= mat\left(H^T H vec_i\left(E_{11}Q + \left(E_{11}Q\right)^T + diag(Q) - 2E_{11}QE_{11}\right)\right)\\ &= mat\left(\begin{pmatrix} 1 & & \\ & 2I_{n-1} & \\ & & I_{n-1}\end{pmatrix} vec_i(W)\right) = 2W - diag(W).\end{aligned}$$

Then

$$\begin{aligned}mat\left(M^T u\right) &= mat\begin{pmatrix} H_1^T(B\otimes A^T)vec(U)\\ H_2^T(D\otimes C^T)vec(U)\end{pmatrix} = mat\begin{pmatrix} H_1^T H_1 H_1^\dagger vec(P_1)\\ H_2^T H_2 H_2^\dagger vec(P_2)\end{pmatrix}\\ &= mat\begin{pmatrix} H_1^T H_1 vec_i(W_1)\\ H_2^T H_2 vec_i(W_2)\end{pmatrix} = \begin{pmatrix} 2W_1 - diag(W_1)\\ 2W_2 - diag(W_2)\end{pmatrix}\end{aligned}$$

$$\begin{aligned}mat(Mv) &= mat\left(\left(\left(B^T \otimes A\right)H_1, \left(D^T \otimes C\right)H_2\right)\begin{pmatrix} v_1\\ v_2\end{pmatrix}\right)\\ &= mat\left(\left(B^T \otimes A\right)H_1 vec_i(V_1) + \left(D^T \otimes C\right)H_2 vec_i(V_2)\right)\\ &= mat\left(\left(B^T \otimes A\right)vec(V_1) + \left(D^T \otimes C\right)vec(V_2)\right)\\ &= AV_1B + CV_2D,\end{aligned}$$

where v_1 and v_2 can be formally transformed to symmetric matrices:

$$V_1 = 2W_1 - diag(W_1),$$
$$V_2 = 2W_2 - diag(W_2).$$

Next, we will give the algorithm for the like-minimum norm solution of (2).

Algorithm *LSQR-W.1*

(1) Initialization

$$X_0 = 0\left(\in SAR^{n\times n}\right),\ Y_0 = 0\left(\in SAR^{k\times k}\right),$$

$$\beta_1 = \|E\|_F,\ U_1 = \frac{E}{\beta_1},$$

$$P_1^{(1)} = A^T U_1 B^T,\ P_1^{(2)} = C^T U_1 D^T,$$

$$Q_1^{(1)} = \frac{P_1^{(1)} + P_1^{(1)^T}}{2},\ Q_1^{(2)} = \frac{P_1^{(2)} + P_1^{(2)^T}}{2},$$

$$
\begin{aligned}
&W_1^{(1)} = E_{11}Q_1^{(1)} + \left(E_{11}Q_1^{(1)}\right)^T + diag\left(Q_1^{(1)}\right) - 2E_{11}Q_1^{(1)}E_{11},\\
&W_1^{(2)} = E_{11}Q_1^{(2)} + \left(E_{11}Q_1^{(2)}\right)^T + diag\left(Q_1^{(2)}\right) - 2E_{11}Q_1^{(2)}E_{11},\\
&\bar{V}_1^{(1)} = 2W_1^{(1)} - diag\left(W_1^{(1)}\right),\\
&\bar{V}_1^{(2)} = 2W_1^{(2)} - diag\left(W_1^{(2)}\right),\\
&\alpha_1 = \sqrt{\left\|tril\left(\bar{V}_1^{(1)}\right)\right\|_F^2 + \left\|tril\left(\bar{V}_1^{(2)}\right)\right\|_F^2},\\
&V_1^{(1)} = \frac{\bar{V}_1^{(1)}}{\alpha_1},\ V_1^{(2)} = \frac{\bar{V}_1^{(2)}}{\alpha_1},\\
&Z_1^{(1)} = V_1^{(1)},\ Z_1^{(2)} = V_1^{(2)},\\
&\bar{\xi}_1 = \beta_1,\ \bar{\rho}_1 = \alpha_1.
\end{aligned}
$$

(2) Iteration. For $i = 1, 2, \cdots$

(i) Compute U_{i+1}:

$$\bar{U}_{i+1} = AV_i^{(1)}B + CV_i^{(2)}D - \alpha_i U_i,$$

$$\beta_{i+1} = \left\|\bar{U}_{i+1}\right\|_F,\ U_{i+1} = \frac{\bar{U}_{i+1}}{\beta_{i+1}},$$

(ii) Compute V_{i+1}:

$$
\begin{aligned}
&P_{i+1}^{(1)} = A^T U_{i+1} B^T,\ P_{i+1}^{(2)} = C^T U_{i+1} D^T,\\
&Q_{i+1}^{(1)} = \frac{P_{i+1}^{(1)} + P_{i+1}^{(1)^T}}{2},\ Q_{i+1}^{(2)} = \frac{P_{i+1}^{(2)} + P_{i+1}^{(2)^T}}{2},\\
&W_{i+1}^{(1)} = E_{11}Q_{i+1}^{(1)} + \left(E_{11}Q_{i+1}^{(1)}\right)^T + diag\left(Q_{i+1}^{(1)}\right) - 2E_{11}Q_{i+1}^{(1)}E_{11},\\
&W_{i+1}^{(2)} = E_{11}Q_1^{(2)} + \left(E_{11}Q_{i+1}^{(2)}\right)^T + diag\left(Q_{i+1}^{(2)}\right) - 2E_{11}Q_{i+1}^{(2)}E_{11},\\
&\bar{V}_{i+1}^{(1)} = 2W_{i+1}^{(1)} - diag\left(W_{i+1}^{(1)}\right) - \beta_{i+1}V_i^{(1)},\\
&\bar{V}_{i+1}^{(2)} = 2W_{i+1}^{(2)} - diag\left(W_{i+1}^{(2)}\right) - \beta_{i+1}V_i^{(2)},\\
&\alpha_{i+1} = \sqrt{\left\|tril\left(\bar{V}_{i+1}^{(1)}\right)\right\|_F^2 + \left\|tril\left(\bar{V}_{i+1}^{(2)}\right)\right\|_F^2},\\
&V_{i+1}^{(1)} = \frac{\bar{V}_{i+1}^{(1)}}{\alpha_{i+1}},\ V_{i+1}^{(2)} = \frac{\bar{V}_{i+1}^{(2)}}{\alpha_{i+1}},
\end{aligned}
$$

(iii) Compute Givens rotation:

$$
\begin{aligned}
&\rho_i = \sqrt{\bar{\rho}_i^2 + \beta_{i+1}^2},\\
&c_i = \frac{\bar{\rho}_i}{\rho_i},\ s_i = \frac{\beta_{i+1}}{\rho_i},\ \theta_{i+1} = s_i\alpha_{i+1},\\
&\bar{\rho}_{i+1} = -c_i\alpha_{i+1},\ \xi_i = c_i\bar{\xi}_i,\ \bar{\xi}_{i+1} = s_i\bar{\xi}_i,
\end{aligned}
$$

(iv) Update X_i, Y_i and Z_i:

$$
\begin{aligned}
&X_i = X_{i-1} + \left(\xi_i/\rho_i\right)Z_i^{(1)},\\
&Y_i = Y_{i-1} + \left(\xi_i/\rho_i\right)Z_i^{(2)},\\
&Z_{i+1}^{(1)} = V_{i+1}^{(1)} - \left(\theta_i/\rho_i\right)Z_i^{(1)},\\
&Z_{i+1}^{(2)} = V_{i+1}^{(2)} - \left(\theta_i/\rho_i\right)Z_i^{(2)}.
\end{aligned}
$$

(3) Check convergence.

Algorithm *LSQR-W.1* can compute the like-minimum norm solution $\varphi = \begin{pmatrix} vec_i(X) \\ vec_i(Y) \end{pmatrix}$ of (5). So we have the following result.

Theorem 3. The symmetric arrowhead solution generated by Algorithm LSQR-W.1 is the like-minimum norm solution of (2).

B. An Iterative Method for the Minimum Norm Solution of (2)

In Section *A*, the like-minimum solution generated by Algorithm *LSQR-W.1* is not the best approximation result and non-unique. In this section, we will give an iterative method for the minimum norm solution of (2).

For $X \in R^{n\times n}$, define $v\bar{e}c_i(X) = Svec_i(X)$, and add the weight values to elements x_{ij} , that is

$$
S = \begin{array}{c}
\begin{array}{ccc} \phantom{\sqrt{2}/2} & \overbrace{\qquad\qquad}^{2,\cdots,n} & \overbrace{\qquad\qquad}^{n+1,\cdots,2n-1} \end{array}\\
\left(\begin{array}{c|ccc|ccc}
\sqrt{2}/2 & & & & & & \\ \hline
 & 1 & & & & & \\
 & & \ddots & & & & \\
 & & & 1 & & & \\ \hline
 & & & & \sqrt{2}/2 & & \\
 & & & & & \ddots & \\
 & & & & & & \sqrt{2}/2
\end{array}\right)
\end{array}
$$

with $S \in R^{(2n-1)\times(2n-1)}$. Obviously, there is one to one linear mapping from $v\bar{e}c_i(X)$ to $vec(X)$. Let $\bar{H}$ denote the minimum norm constrained matrix with

$$vec(X) = \bar{H}v\bar{e}c_i(X).$$

It readily follows form Theorem 1 that

Theorem 4. Suppose $\bar{H}$ is the symmetric arrowhead constrained matrix, then

$$\bar{H} = HS^{-1}$$

and

$$\bar{H}^T H = 2I_{2n-1}.$$

Since

$$\|AXB + CYD - E\|_F^2 = \left\|\left(\left(B^T \otimes A\right)\bar{H}_1, \left(D^T \otimes C\right)\bar{H}_2\right)\begin{pmatrix} vec_i(X) \\ vec_i(Y) \end{pmatrix} - vec(E)\right\|_F^2,$$

problem (2) is equivalent to (5) and

$$M = \left(\left(B^T \otimes A\right)\bar{H}_1, \left(D^T \otimes C\right)\bar{H}_2\right) \in R^{ms\times(2n+2k-2)},$$

$$f = vec(E) \in R^{ms}, x = \begin{pmatrix} v\bar{e}c_i(X) \\ v\bar{e}c_i(X) \end{pmatrix} \in R^{2n+2k-2},$$

where $\bar{H}_1$ and $\bar{H}_2$ are the new symmetric arrowhead constrained matrices of degree *n* and *k*, respectively.

For any $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in R^{2n+2k-2}$, $v_1 = vec_i(V_1)$ and $v_2 = vec_i(V_2)$ where $V_1 \in SAR^{n\times n}$, $V_2 \in SAR^{k\times k}$, we have

$$\tilde{V} = V + \left(\sqrt{2} - 1\right)diag(V),$$

and

$$\begin{aligned} mat(Mv) &= mat\left(\left(B^T\otimes A\right)\bar{H}_1 vec_i(V_1)+\left(D^T\otimes C\right)\bar{H}_2 vec_i(V_2)\right)\\ &= mat\left(\left(B^T\otimes A\right)H_1 S_1^{-1}v\bar{e}c_i(V_1)+\left(D^T\otimes C\right)H_2 S_2^{-1}v\bar{e}c_i(V_2)\right)\\ &= mat\left(\left(B^T\otimes A\right)H_1 vec_i(\tilde{V}_1)+\left(D^T\otimes C\right)H_2 vec_i(\tilde{V}_2)\right)\\ &= mat\left(\left(B^T\otimes A\right)vec(\tilde{V}_1)+\left(D^T\otimes C\right)vec(\tilde{V}_2)\right)\\ &= A\tilde{V}_1 B + C\tilde{V}_2 D, \end{aligned}$$

For any $u = vec(U)\in R^{ms}$ with $U\in R^{m\times s}$, denoted by

$$W = E_{11}Q + (E_{11}Q)^T + diag(Q) - 2E_{11}QE_{11},$$

according to Theorem 2, we have

$$\begin{aligned} mat(M^T u) &= mat\begin{pmatrix}\bar{H}_1^T(B\otimes A^T)vec(U)\\ \bar{H}_2^T(D\otimes C^T)vec(U)\end{pmatrix} = mat\begin{pmatrix}\bar{H}_1^T\bar{H}_1\bar{H}_1^\dagger vec(P_1)\\ \bar{H}_2^T\bar{H}_2\bar{H}_2^\dagger vec(P_2)\end{pmatrix}\\ &= mat\begin{pmatrix}2I_n SH_1^\dagger vec(W)\\ 2I_k SH_2^\dagger vec(W)\end{pmatrix} mat\begin{pmatrix}2Svec_i(W_1)\\ 2Svec_i(W_2)\end{pmatrix}\\ &= \begin{pmatrix}2W_1-\left(2-\sqrt{2}\right)diag(W_1)\\ 2W_2-\left(2-\sqrt{2}\right)diag(W_2)\end{pmatrix}, \end{aligned}$$

where v_1 and v_2 can be formally transformed to symmetric matrices

$$V_1 = 2W_1 - \left(2-\sqrt{2}\right)diag(W_1),$$
$$V_2 = 2W_2 - \left(2-\sqrt{2}\right)diag(W_2).$$

Next, we give the algorithm for the minimum norm solution of (2).

Algorithm *LSQR-W.2*

(1) Initialization

$$X_0 = 0\left(\in SAR^{n\times n}\right),\ Y_0 = 0\left(\in SAR^{k\times k}\right),$$
$$\beta_1 = \|E\|_F,\ U_1 = \frac{E}{\beta_1},$$
$$P_1^{(1)} = A^T U_1 B^T,\ P_1^{(2)} = C^T U_1 D^T,$$
$$Q_1^{(1)} = \frac{P_1^{(1)} + P_1^{(1)^T}}{2},\ Q_1^{(2)} = \frac{P_1^{(2)} + P_1^{(2)^T}}{2},$$
$$W_1^{(1)} = E_{11}Q_1^{(1)} + \left(E_{11}Q_1^{(1)}\right)^T + diag\left(Q_1^{(1)}\right) - 2E_{11}Q_1^{(1)}E_{11},$$
$$W_1^{(2)} = E_{11}Q_1^{(2)} + \left(E_{11}Q_1^{(2)}\right)^T + diag\left(Q_1^{(2)}\right) - 2E_{11}Q_1^{(2)}E_{11},$$
$$\bar{V}_1^{(1)} = 2W_1^{(1)} - \left(2-\sqrt{2}\right)diag\left(W_1^{(1)}\right),$$
$$\bar{V}_1^{(2)} = 2W_1^{(2)} - \left(2-\sqrt{2}\right)diag\left(W_1^{(2)}\right),$$
$$\alpha_1 = \sqrt{\left\|tril\left(\bar{V}_1^{(1)}\right)\right\|_F^2 + \left\|tril\left(\bar{V}_1^{(2)}\right)\right\|_F^2},$$
$$V_1^{(1)} = \frac{\bar{V}_1^{(1)}}{\alpha_1},\ V_1^{(2)} = \frac{\bar{V}_1^{(2)}}{\alpha_1},$$
$$Z_1^{(1)} = V_1^{(1)},\ Z_1^{(2)} = V_1^{(2)},$$
$$\bar{\xi}_1 = \beta_1,\ \bar{\rho}_1 = \alpha_1;$$

(2) Iteration. For $i = 1, 2, \cdots$

(i) Compute U_{i+1}:

$$\tilde{V}_i^{(1)} = V_i^{(1)} + \left(\sqrt{2}-1\right)diag\left(V_i^{(1)}\right),$$
$$\tilde{V}_i^{(2)} = V_i^{(2)} + \left(\sqrt{2}-1\right)diag\left(V_i^{(2)}\right),$$
$$\bar{U}_{i+1} = A\tilde{V}_i^{(1)}B + C\tilde{V}_i^{(2)}D - \alpha_i U_i,$$
$$\beta_{i+1} = \left\|\bar{U}_{i+1}\right\|_F,\ U_{i+1} = \bar{U}_{i+1}/\beta_{i+1},$$

(ii) Compute V_{i+1}:

$$P_{i+1}^{(1)} = A^T U_{i+1} B^T,\ P_{i+1}^{(2)} = C^T U_{i+1} D^T,$$
$$Q_{i+1}^{(1)} = \frac{P_{i+1}^{(1)} + P_{i+1}^{(1)^T}}{2},\ Q_{i+1}^{(2)} = \frac{P_{i+1}^{(2)} + P_{i+1}^{(2)^T}}{2},$$
$$W_{i+1}^{(1)} = E_{11}Q_{i+1}^{(1)} + \left(E_{11}Q_{i+1}^{(1)}\right)^T + diag\left(Q_{i+1}^{(1)}\right) - 2E_{11}Q_{i+1}^{(1)}E_{11},$$
$$W_{i+1}^{(2)} = E_{11}Q_1^{(2)} + \left(E_{11}Q_{i+1}^{(2)}\right)^T + diag\left(Q_{i+1}^{(2)}\right) - 2E_{11}Q_{i+1}^{(2)}E_{11},$$
$$\bar{V}_{i+1}^{(1)} = 2W_{i+1}^{(1)} - \left(2-\sqrt{2}\right)diag\left(W_{i+1}^{(1)}\right) - \beta_{i+1}V_i^{(1)},$$
$$\bar{V}_{i+1}^{(2)} = 2W_{i+1}^{(2)} - \left(2-\sqrt{2}\right)diag\left(W_{i+1}^{(2)}\right) - \beta_{i+1}V_i^{(2)},$$
$$\alpha_{i+1} = \sqrt{\left\|tril\left(\bar{V}_{i+1}^{(1)}\right)\right\|_F^2 + \left\|tril\left(\bar{V}_{i+1}^{(2)}\right)\right\|_F^2},$$
$$V_{i+1}^{(1)} = \frac{\bar{V}_{i+1}^{(1)}}{\alpha_{i+1}},\ V_{i+1}^{(2)} = \frac{\bar{V}_{i+1}^{(2)}}{\alpha_{i+1}},$$

(iii) Compute Givens rotation:

$$\rho_i = \sqrt{\bar{\rho}_i^2 + \beta_{i+1}^2},$$
$$c_i = \frac{\bar{\rho}_i}{\rho_i},\ s_i = \frac{\beta_{i+1}}{\rho_i},\ \theta_{i+1} = s_i\alpha_{i+1},$$
$$\bar{\rho}_{i+1} = -c_i\alpha_{i+1},\ \xi_i = c_i\bar{\xi}_i,\ \bar{\xi}_{i+1} = s_i\bar{\xi}_i,$$

(iv) Update X_i, Y_i and Z_i:

$$x_i = x_{i-1} + \left(\xi_i/\rho_i\right)Z_i^{(1)},$$
$$y_i = y_{i-1} + \left(\xi_i/\rho_i\right)Z_i^{(2)},$$
$$X_i = x_i + \left(\sqrt{2}-1\right)diag(x_i),$$
$$Y_i = y_i + \left(\sqrt{2}-1\right)diag(y_i).$$
$$Z_{i+1}^{(1)} = V_{i+1}^{(1)} - \left(\theta_i/\rho_i\right)Z_i^{(1)},$$
$$Z_{i+1}^{(2)} = V_{i+1}^{(2)} - \left(\theta_i/\rho_i\right)Z_i^{(2)}.$$

(3) Check convergence.

From above translation process and the algorithm, we can obtain the approximate solution $\left[X_i, Y_i\right]$ with

$$vec_i(x_i) = v\bar{e}c_i(X_i),\ vec_i(y_i) = v\bar{e}c_i(Y_i).$$

Thus, we have

$$X_i = x_i + \left(\sqrt{2}-1\right)diag(x_i),$$
$$Y_i = y_i + \left(\sqrt{2}-1\right)diag(y_i).$$

Because of

$$\|X\|_F^2 + \|Y\|_F^2 = 2\left(\left\|v\bar{e}c_i(X)\right\|_2^2 + \left\|v\bar{e}c_i(Y)\right\|_2^2\right),$$

The Algorithm *LSQR-W.2* can compute the minimum norm solution of (5). So we have the following result.

Theorem 5. The symmetric arrowhead solution generated by Algorithm LSQR-W.2 is the minimum norm solution of (2).

IV. Numerical Examples

In this section, we reported three numerical examples to illustrate the efficiency of the algorithms we proposed. First, we present the Example 1 in [7] with the results generated by Algorithm *LSQR-W.1* and Algorithm *LSQR-W.2*.

Example 1. Considering the following matrix equation $AXB + CYD = E$ with $A = B = C = D = I_4$ and

$$E = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

i.e.,

$$I_4 X I_4 + I_4 Y I_4 = E,$$

where I_4 is the unit matrix of order 4. We can also obtain the minimum solution by Algorithm *LSQR-W.1* and Algorithm *LSQR-W.2* which should be

$$X = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{bmatrix}, Y = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{bmatrix}.$$

The exact result could be computed by our methods in no more than two iterations. Fig. 1 plots the relation between error

$$\gamma_k = \log 10\left(\left\| AX_k B + CY_k D - E \right\|_F\right)$$

and iterative number *K*. Next we present two other matrix equations which show the methods numerically reliable in various circumstances.

Example 2. Given

$$A = \begin{bmatrix} hilb(5) & zeros(5,3) \\ eye(5) & ones(5,3) \end{bmatrix}, B = \begin{bmatrix} ones(3,7) & zeros(3,5) \\ zeros(5,7) & pascal(5) \end{bmatrix},$$
$$C = \begin{bmatrix} magic(6) \\ ones(4,6) \end{bmatrix}, \quad D = \begin{bmatrix} hankel(1:4) & zeros(4,8) \\ zeros(2,4) & ones(2,8) \end{bmatrix},$$
$$X = ones(8,8),\ Y = ones(6,6) \text{ and } E = AXB + CYD.$$

Notice that problem (1) is consistent and has a symmetric arrowhead solution. For *M* and φ defined by (8) and (9), we can choose residual error

$$\left\| AX_k B + CY_k D - E \right\|_F = \left\| Mx_k - f \right\|_2 = \overline{\xi}_{k+1}.$$

From (3) and (4), we can compute by Algorithm *LSQR-W.1* and

$$\|X\|_F^2 + \|Y\|_F^2 = 38, \|tril(X)\|_F^2 + \|tril(Y)\|_F^2 = 26,$$
$$\|X_{83}\|_F^2 + \|Y_{83}\|_F^2 = 38.9580, \|tril(X_{83})\|_F^2 + \|tril(Y_{83})\|_F^2 = 25.5309.$$

Thus, we can obtain the like-minimum norm solution of (2) and Fig. 2 plots the relation between error

$$\varepsilon_k = \log 10\left(\left\| AX_k B + CY_k D - E \right\|_F\right)$$

and iterative number *K* which shows the favorable efficiency of Algorithm *LSQR-W.1*. Next, we also compute the result by Algorithm *LSQR-W.2* and obtain that

$$\|X_{83}\|_F^2 + \|Y_{83}\|_F^2 = 38, \|tril(X_{83})\|_F^2 + \|tril(Y_{83})\|_F^2 = 26.$$

Thus, we obtain the minimum norm solution of (2) and Fig. 3 plots the relation between error

$$\mu_k = \log 10\left(\left\| AX_k B + CY_k D - E \right\|_F\right)$$

and iterative number *K* of the Algorithm *LSQR-W.2*.

Example 3. Given

$$A = \begin{bmatrix} hilb(5) & zeros(5,3) \\ eye(5) & ones(5,3) \end{bmatrix}, B = \begin{bmatrix} ones(3,7) & zeros(3,5) \\ zeros(5,7) & pascal(5) \end{bmatrix},$$
$$C = \begin{bmatrix} magic(6) \\ ones(4,6) \end{bmatrix}, \quad D = \begin{bmatrix} hankel(1:4) & zeros(4,8) \\ zeros(2,4) & ones(2,8) \end{bmatrix},$$
$$E = \begin{bmatrix} toeplitz(1:10) & ones(10,2) \end{bmatrix}.$$

Notice that the problem (1) is not consistent. For *M* and φ defined by (8) and (9), we choose residual error

$$\left\| M^T M x_k - M^T f \right\|_2 = \left| \alpha_{k+1} \overline{\xi}_{k+1} c_k \right|.$$

Then Figs. 4 and 5 plot the relation between error

$$\eta_k, \delta_k = \log 10\left(\left\| M^T \left(Mx_k - f \right) \right\|_2\right)$$

and iterative number *K* by Algorithm *LSQR-W.1* and Algorithm *LSQR-W.2,* respectively.

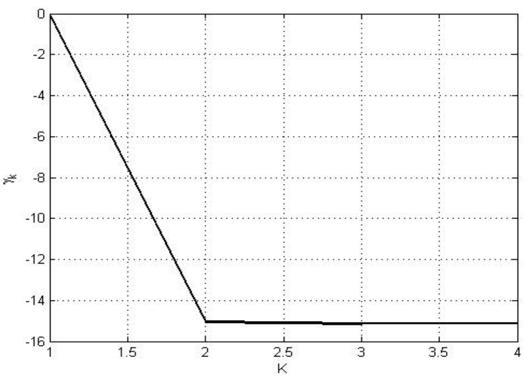

Fig. 1 The relation between error γ_k and iterative number K

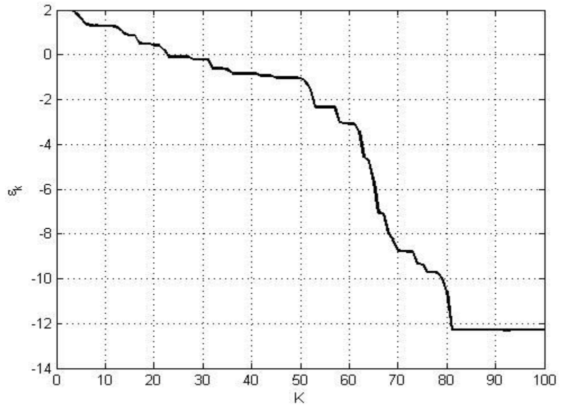

Fig. 2 The relation between error ε_k and iterative number K

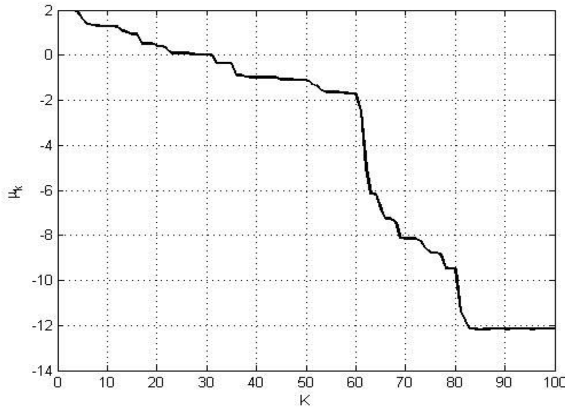

Fig. 3 The relation between error μ_k and iterative number K

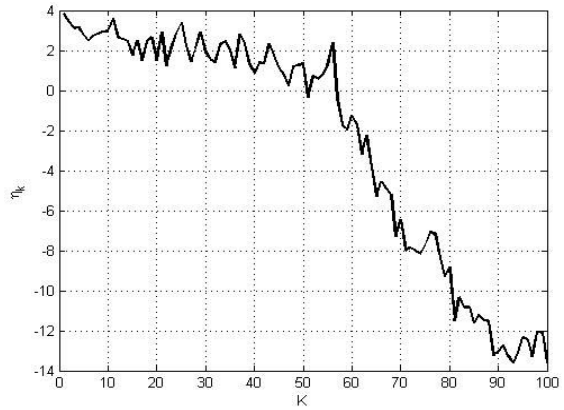

Fig. 4 The relation between error η_k and iterative number K

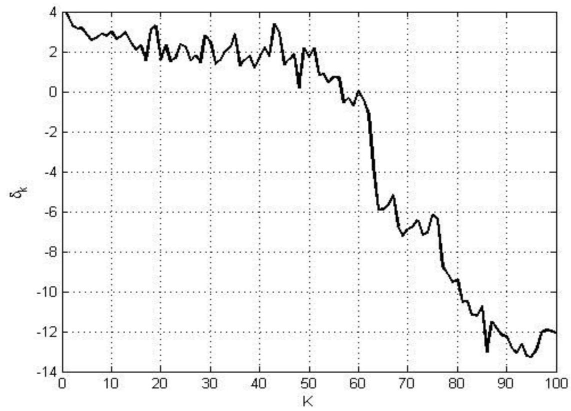

Fig. 5 The relation between error δ_k and iterative number K

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