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A Boundary Backstepping Control Design for 2-D, 3-D and N-D Heat Equation

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Abstract—We consider the problem of stabilization of an unstable heat equation in a 2-D, 3-D and generally n-D domain by deriving a generalized backstepping boundary control design methodology. To stabilize the systems, we design boundary backstepping controllers inspired by the 1-D unstable heat equation stabilization procedure. We assume that one side of the boundary is hinged and the other side is controlled for each direction of the domain. Thus, controllers act on two boundaries for 2-D domain, three boundaries for 3-D domain and ”n” boundaries for n-D domain. The main idea of the design is to derive ”n” controllers for each of the dimensions by using ”n” kernel functions. Thus, we obtain ”n” controllers for the ”n” dimensional case. We use a transformation to change the system into an exponentially stable ”n” dimensional heat equation. The transformation used in this paper is a generalized Volterra/Fredholm type with ”n” kernel functions for n-D domain instead of the one kernel function of 1-D design.

Keywords—Backstepping, boundary control, 2-D, 3-D, n-D heat equation, distributed parameter systems.

I. Introduction

We consider a problem of stabilization of two, three and in generally ”n” dimensional unstable heat equation, which is controlled from one end and rigid from the other end for all dimensions. Thus, we get ”n” controllers for ”n” dimensional case. The backstepping boundary control has been successful in one dimensional Partial differential equations(PDEs). Many studies are conducted the control of one-dimensional heat PDEs in recent years. In [1], backstepping controller designs are obtained for four different case of unstable heat equations in one dimensional case. In [1]–[5], many control designs are defined for different PDE models in one dimensional domain[6]. In [7], another approach for ordinary differential equations with actuator delay is achieved by modeling time delay by a first-order hyperbolic PDE. In [8], an explicit feedback law is presented for a multi-input LTI system which compensates the wave PDE dynamics in its input and stabilizes the overall system by using the backstepping method for PDEs. Many more studies such as [9]–[20] can be useful in this area. However, as far as the author can find out no studies concerned of the boundary control of PDEs in 2 and more dimensional case by using this approach. In only [3], in two dimensional domain, another approach can be seen by using an estimator. Inspired by these control design methodologies of one dimensional case, this study has control designs by using transformations and stability analysis by using Lyapunov method for two, three and ”n” dimensional unstable heat equations. Hence the paper includes an expanded case of the studies for the boundary backstepping control of one dimensional heat equations. By deriving the backstepping method in more than one dimensional cases, it is promising to use this method for distributed time delay systems for further studies. Since there are many studies on this topic such as [8], [13]. However, this topic should be tackled in further study.

We consider an unstable reaction-advection-diffusion system given by

$$v_t(x^*,y^*,t) = \mu[v_{x^*x^*}(x^*,y^*,t) + v_{y^*y^*}(x^*,y^*,t)]$$
$$+av_{x^*}(x^*,y^*,t) + bv_{y^*}(x^*,y^*,t) + dv(x^*,y^*,t), \tag{1}$$
$$v(x^*,0,t) = v(0,y^*,t) = 0, \tag{2}$$
$$v(1,y^*,t) = V_1(y^*,t), \tag{3}$$
$$v(x^*,1,t) = V_2(x^*,t), \tag{4}$$

where a, b, and d are positive constants. Defining another function as $u(x^*,y^*,t) = v(x^*,y^*,t)e^{\frac{a}{2\mu}x^* + \frac{b}{2\mu}y^*}$ and changing variables as $x = \sqrt{\mu}x^*$, and $y = \sqrt{\mu}y^*$ the system (1)–(4) gets into another system as follows. With an assumption as $\mu = 1$, the investigated original system is given by

$$u_t(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t) + \lambda u(x,y,t), \tag{5}$$
$$u(x,0,t) = u(0,y,t) = 0, \tag{6}$$
$$u(\sqrt{\mu} = 1,y,t) = U_1(y,t), \tag{7}$$
$$u(x,\sqrt{\mu} = 1,t) = U_2(x,t), \tag{8}$$

where $\lambda = d - \left(\frac{a^2+b^2}{4\mu}\right)$, which is assumed as a positive constant in this study, and makes the system unstable. $U_1(y,t)$ and $U_2(x,t)$ are the controllers, which maps the system (5)–(8) into an exponentially stable system. Fistly, this study presents control designs for the systems (1)–(4), and (5)–(8). Secondly, in three and ”n” dimensional domains with similar definitions for heat equation as two dimensional case. Also, the stability analyses of the target systems in all domains are tackled in this study.

The paper is organized as follows. In Section II, control designs and transformations for the two dimensional unstable heat equation are obtained. Two controllers for two dimensions are given explicitly. Section III presents the control design and transformations for the unstable heat equation in three dimensional domain. In Section IV, the general case boundary backstepping controllers can be seen for ”nD” unstable heat equation. Stability analysis of 2D, 3D, and nD target systems are given in Section V by using the Lyapunov methodology

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which shows the target systems are exponentially stable. In Section VI, the inverse transformations are given for the used transformations used in previous sections for all investigated systems.

II. Backstepping Design For 2-D Unstable Heat Equation

To get a stable system from (5)-(8), we use the transformation

$$w(x,y,t) = u(x,y,t) - \int_0^x k(x,\xi)u(\xi,y,t)d\xi - \int_0^y l(y,\eta)u(x,\eta,t)d\eta, \tag{9}$$

where the gains $k(x,\xi)$, and $l(y,\eta)$ are to be determined in this section. Compared with the transformations for one dimensional PDEs, there is only one additional term in (9)–the third term in (9). Because of the increasing of the dimensions, we need this term and it is in fact the main conceptual novelty of the paper, different then the other approach [3]. Our approach is to map the system (5)-(8) into the following target system

$$w_t(x,y,t) = w_{xx}(x,y,t) + w_{yy}(x,y,t), \tag{10}$$
$$w(x,0,t) = w(0,y,t) = 0, \tag{11}$$
$$w(x,1,t) = w(1,y,t) = 0, \tag{12}$$

which is exponentially stable and proved in Section V. In [4], the stabilization analysis of the system (10)-(12) is tackled in one dimensional domain. By using the transformation (9) we define the derivative of transformation by time. By applying integration by parts two times and using the boundary conditions, the derivative of transformation by time is given by

$$\begin{aligned} w_t(x,y,t) &= u_{xx}(x,y,t) + u_{yy}(x,y,t) + \lambda u(x,y,t) \\ &- k(x,x)u_x(x,y,t) + k(x,0)u_x(0,y,t) \\ &+ k_\xi(x,x)u(x,y,t) - \int_0^x k_{\xi\xi}(x,\xi)u(\xi,y,t)d\xi \\ &- \int_0^x k(x,\xi)\left[u_{yy}(\xi,y,t) + \lambda u(\xi,y,t)\right]d\xi \\ &- l(y,y)u_y(x,y,t) + l(y,0)u_y(x,0,t) \\ &+ l_\eta(y,y)u(x,y,t) - \int_0^y l_{\eta\eta}(y,\eta)u(x,\eta,t)d\eta \\ &- \int_0^y l(y,\eta)\left[u_{xx}(x,\eta,t) + \lambda u(x,\eta,t)\right]d\eta \end{aligned} \tag{13}$$

Similarly, by using (9), we define the second derivative of the transformation by dimension x by applying Leibnitz differentiation rule as

$$\begin{aligned} w_{xx}(x,y,t) &= u_{xx}(x,y,t) - \frac{d}{dx}\left[k(x,x)\right]u(x,y,t) \\ &- k(x,x)u_x(x,y,t) - k_x(x,x)u(x,y,t) \\ &- \int_0^x k_{xx}(x,\xi)u(\xi,y,t)d\xi \\ &- \int_0^y l(y,\eta)u_{xx}(x,\eta,t)d\eta, \end{aligned} \tag{14}$$

and by the dimension y as

$$\begin{aligned} w_{yy}(x,y,t) &= u_{yy}(x,y,t) - \int_0^x k(x,\xi)u_{yy}(\xi,y,t)d\xi \\ &- \frac{d}{dy}\left[l(y,y)\right]u(x,y,t) - l(y,y)u_y(x,y,t) \\ &- l_y(y,y)u(x,y,t) - \int_0^y l_{yy}(y,\eta)u(x,\eta,t)d\eta. \end{aligned} \tag{15}$$

With the help of (10), and the boundary conditions of the original system (5)–(8), along the solutions of (13), (14), (15); (10) is given by

$$\begin{aligned} & u(x,y,t)\left[\lambda + 2\frac{d}{dx}\left[k(x,x)\right] + 2\frac{d}{dy}\left[l(y,y)\right]\right] \\ & + u_x(0,y,t)k(x,0) + u_y(x,0,t)l(y,0) \\ & + \int_0^x u(\xi,y,t)\left[k_{xx} - k_{\xi\xi} - k(x,\xi)\lambda\right]d\xi \\ & + \int_0^y u(x,\xi,t)\left[l_{yy} - l_{\xi\xi} - l(y,\xi)\lambda\right]d\xi = 0, \end{aligned} \tag{16}$$

where in order to equalize (16) to zero we choose

$$2\frac{d}{dx}\left[k(x,x)\right] + 2\frac{d}{dy}\left[l(y,y)\right] = -\lambda, \tag{17}$$

where the summation of a function of x and a function of y is equal to a constant. So both of the terms have to be constant. Let's take one of them $-\frac{c}{2}$, where c is a positive definite constant, and choose the following two partial differential equations systems to equalize the equation to zero as

$$l_{yy}(y,\eta) - l_{\eta\eta}(y,\eta) = \lambda l(y,\eta), \tag{18}$$
$$l(y,y) = -\frac{\lambda - c}{2}y, \tag{19}$$
$$l(y,0) = 0, \tag{20}$$

and the solution of this system is given in [2] as

$$l(y,\eta) = -(\lambda - c)\eta\frac{I_1\left[\sqrt{\lambda\left(y^2-\eta^2\right)}\right]}{\sqrt{\lambda\left(y^2-\eta^2\right)}}. \tag{21}$$

Similarly;

$$k_{xx}(x,\xi) - k_{\xi\xi}(x,\xi) = \lambda k(x,\xi), \tag{22}$$
$$k(x,x) = -\frac{c}{2}x, \tag{23}$$
$$k(x,0) = 0, \tag{24}$$

where the solution of this system is given in [2] as

$$k(x,\xi) = -c\xi\frac{I_1\left[\sqrt{\lambda\left(x^2-\xi^2\right)}\right]}{\sqrt{\lambda\left(x^2-\xi^2\right)}}. \tag{25}$$

By using the boundary conditions of the system (10)–(12), and the system (5)–(8), the controllers of the system (5)–(8) are given by

$$U_1(y,t) = \int_0^1 k(1,\xi)u(\xi,y,t)d\xi + \int_0^y l(y,\eta)u(1,\eta,t)d\eta, \tag{26}$$

and

$$U_2(x,t) = \int_0^x k(x,\xi)u(\xi,1,t)d\xi + \int_0^1 l(1,\eta)u(x,\eta,t)d\eta, \quad (27)$$

where the Kernel functions of the system are defined. Thus, if these controllers applied for the unstable system (5)–(8), the system will behave as an exponentially stable system given in (10)–(12), where its stability analysis is given in Section (V). An the controllers for the system (1)–(4) are defined as

$$V_1(y^*,t) = \int_0^{\sqrt{\mu}} k(\sqrt{\mu},\xi)v(\xi,y^*,t)e^{\frac{a}{2\mu}\xi+\frac{b}{2\mu}y^*}d\xi + \int_0^{y^*} l(y^*,\eta)v(\sqrt{\mu},\eta,t)e^{\frac{a}{2\sqrt{\mu}}+\frac{b}{2\mu}\eta}d\eta, \quad (28)$$

and

$$V_2(x^*,t) = \int_0^{x^*} k(x^*,\xi)v(\xi,\sqrt{\mu},t)e^{\frac{a}{2\mu}\xi+\frac{b}{2\sqrt{\mu}}}d\xi + \int_0^{\sqrt{\mu}} l(\sqrt{\mu},\eta)u(x^*,\eta,t)e^{\frac{a}{2\sqrt{\mu}}+\frac{b}{2\mu}\eta}d\eta. \quad (29)$$

In the next section a similar design methodology is given in three dimensional domain for unstable heat equation.

III. Backstepping Design For 3-D Unstable Heat Equation

We consider a general unstable 3-D heat equation given by

$$\begin{aligned} v_t(x^*,y^*,z^*,t) &= v_{x^*x^*}(x^*,y^*,z^*,t) + v_{y^*y^*}(x^*,y^*,z^*,t) \\ &+ v_{z^*z^*}(x^*,y^*,z^*,t) + \alpha v_{x^*}(x^*,y^*,z^*,t) \\ &+ \beta v_{y^*}(x^*,y^*,z^*,t) + \gamma v_{z^*}(x^*,y^*,z^*,t) \\ &+ dv(x^*,y^*,z^*,t), \quad (30) \end{aligned}$$

$$v(0,y^*,z^*,t) = v(x^*,0,z^*,t) = v(x^*,y^*,0,t) = 0, \quad (31)$$
$$v(1,y^*,z^*,t) = V_1(y^*,z^*,t), \quad (32)$$
$$v(x^*,1,z^*,t) = V_2(x^*,z^*,t), \quad (33)$$
$$v(x^*,y^*,1,t) = V_3(x^*,y^*,t). \quad (34)$$

By making similar changes of variables as in two dimensional design and defining the function $u(x^*,y^*,z^*,t) = v(x^*,y^*,z^*,t)e^{\frac{\alpha}{2\mu}x^*+\frac{\beta}{2\mu}y^*+\frac{\gamma}{2\mu}z^*}$, and the variables $x = x^*$, $y = y^*$, and $z = z^*$, and taking $\sqrt{\mu} = 1$

$$\begin{aligned} u_t(x,y,z,t) &= u_{xx}(x,y,z,t) + u_{yy}(x,y,z,t) \\ &+ u_{zz}(x,y,z,t) + \lambda u(x,y,z,t), \quad (35) \end{aligned}$$

$$u(0,y,z,t) = u(x,0,z,t) = u(x,y,0,t) = 0, \quad (36)$$
$$u(\sqrt{\mu}=1,y,z,t) = U_1(y,z,t), \quad (37)$$
$$u(x,\sqrt{\mu}=1,z,t) = U_2(x,z,t), \quad (38)$$
$$u(x,y,\sqrt{\mu}=1,t) = U_3(x,y,t). \quad (39)$$

By using the transformation

$$w(x,y,z,t) = u(x,y,z,t) - \int_0^x k(x,\xi)u(\xi,y,z,t)d\xi - \int_0^y l(y,\eta)u(x,\eta,z,t)d\eta - \int_0^z s(z,\zeta)u(x,y,\zeta,t)d\zeta, \quad (40)$$

where the gains $k(x,\xi)$, $l(y,\eta)$, and $s(z,\zeta)$ are to be determined. Compared with the transformations for one dimensional PDEs, there are two additional term in (40)–the third and fourth terms in (40). This transformation maps the system (35)–(39) into the stable target system

$$w_t(x,y,z,t) = w_{xx}(x,y,z,t) + w_{yy}(x,y,z,t) + w_{zz}(x,y,z,t), \quad (41)$$
$$w(0,y,z,t) = w(x,0,z,t) = w(x,y,0,t) = 0, \quad (42)$$
$$w(1,y,z,t) = w(x,1,z,t) = w(x,y,1,t) = 0, \quad (43)$$

where this system is exponentially stable and the the stability analysis of this system is given in Section V. By making similar calculations as in two dimensional design we get

$$\begin{aligned} &u(x,y,z,t)\left[\lambda + 2\frac{d}{dx}[k(x,x)] + 2\frac{d}{dy}[l(y,y)] + 2\frac{d}{dz}[s(z,z)]\right] \\ &+ u_x(0,y,z,t)k(x,0) + u_y(x,0,z,t)l(y,0) + u_z(x,y,0,t)s(z,0) \\ &+ \int_0^x u(\xi,y,z,t)\left[k_{xx}(x,\xi) - k_{\xi\xi}(x,\xi) - k(x,\xi)\lambda\right]d\xi \\ &+ \int_0^y u(x,\eta,z,t)\left[l_{yy}(y,\eta) - l_{\eta\eta}(y,\eta) - l(y,\eta)\lambda\right]d\eta \\ &+ \int_0^z u(x,y,\zeta,t)\left[s_{zz}(z,\zeta) - s_{\zeta\zeta}(z,\zeta) - s(z,\zeta)\lambda\right]d\zeta = 0. \quad (44) \end{aligned}$$

In order to equalize (44) to zero, we choose each of term equals zero.

$$2\frac{d}{dx}[k(x,x)] + 2\frac{d}{dy}[l(y,y)] + 2\frac{d}{dz}[s(z,z)] = -\lambda, \quad (45)$$

where the summation of a function of x, a function of y and a function of z is equal to a constant. So all of the terms have to be constant. Assuming one of them equals $-\frac{c_1}{2}$ and the other one equals $-\frac{c_2}{2}$. So, the PDE for the Kernel function $l(y,\eta)$ is

$$l_{yy}(y,\eta) - l_{\eta\eta}(y,\xi) = \lambda l(y,\eta), \quad (46)$$
$$l(y,y) = -\frac{\lambda - c_1 - c_2}{2}y, \quad (47)$$
$$l(y,0) = 0, \quad (48)$$

where c_1 and c_2 is arbitrary control parameters and the solution of the PDE is given in [2]

$$l(y,\eta) = -(\lambda - c_1 - c_2)\eta\frac{I_1\left[\sqrt{\lambda\,(y^2-\eta^2)}\right]}{\sqrt{\lambda\,(y^2-\eta^2)}}. \quad (49)$$

Similarly, for the kernel function $k(x,\xi)$

$$k_{xx}(x,\xi) - k_{\xi\xi}(x,\xi) = \lambda k(x,\xi), \quad (50)$$
$$k(x,x) = -\frac{c_1}{2}x, \quad (51)$$
$$k(x,0) = 0, \quad (52)$$

where the solution is given in [2]

$$k(x,\xi) = -c_1\xi\frac{I_1\left[\sqrt{\lambda\,(x^2-\xi^2)}\right]}{\sqrt{\lambda\,(x^2-\xi^2)}}. \quad (53)$$

Finally for the last Kernel function $s(z,\zeta)$, the PDE is derived as

$$s_{zz}(z,\zeta) - s_{\zeta\zeta}(x,\xi) = \lambda s(x,\zeta), \quad (54)$$
$$s(x,x) = -\frac{c_2}{2}x, \quad (55)$$
$$s(x,0) = 0, \quad (56)$$

and the solution is given in [2]

$$s(z,\zeta) = -c_2\xi \frac{I_1\left[\sqrt{\lambda\left(z^2-\zeta^2\right)}\right]}{\sqrt{\lambda\left(z^2-\zeta^2\right)}}. \tag{57}$$

Now the controller can be obtained by using the transformation (40) and plugging the determined gains and setting $x=1$, $y=1$, $z=1$, consequently. The controllers of the system (35)-(39) are given by

$$U_1(y,z,t) = \int_0^1 k(1,\xi)u(\xi,y,z,t)d\xi + \int_0^y l(y,\eta)u(1,\eta,z,t)d\eta + \int_0^z s(z,\zeta)u(1,y,\zeta,t)d\zeta, \tag{58}$$

$$U_2(x,z,t) = \int_0^x k(x,\xi)u(\xi,1,z,t)d\xi + \int_0^1 l(1,\eta)u(x,\eta,z,t)d\eta + \int_0^z s(z,\zeta)u(x,1,\zeta,t)d\zeta, \tag{59}$$

$$U_3(x,y,t) = \int_0^x k(x,\xi)u(\xi,y,1,t)d\xi + \int_0^y l(y,\eta)u(x,\eta,1,t)d\eta + \int_0^1 s(1,\zeta)u(x,y,\zeta,t)d\zeta. \tag{60}$$

Also the three adaptive controllers for the system (30)–(34) in three dimensional case

$$V_1(y^*,z^*,t) = \int_0^{\sqrt{\mu}} k(\sqrt{\mu},\xi)v(\xi,y^*,z^*,t)e^{\frac{\alpha}{2\mu}\xi+\frac{\beta}{2\mu}y^*+\frac{\gamma}{2\mu}z^*}d\xi + \int_0^{y^*} l(y^*,\eta)v(\sqrt{\mu},\eta,z^*,t)e^{\frac{\alpha}{2\sqrt{\mu}}+\frac{\beta}{2\mu}\eta+\frac{\gamma}{2\mu}z^*}d\eta + \int_0^{z^*} s(z^*,\zeta)v(\sqrt{\mu},y^*,\zeta,t)e^{\frac{\alpha}{2\sqrt{\mu}}+\frac{\beta}{2\mu}y^*+\frac{\gamma}{2\mu}\zeta}d\zeta, \tag{61}$$

$$V_2(x^*,z^*,t) = \int_0^{x^*} k(x^*,\xi)v(\xi,\sqrt{\mu},z^*,t)e^{\frac{\alpha}{2\mu}\xi+\frac{\beta}{2\sqrt{\mu}}+\frac{\gamma}{2\mu}z^*}d\xi + \int_0^{\sqrt{\mu}} l(\sqrt{\mu},\eta)v(x^*,\eta,z^*,t)e^{\frac{\alpha}{2\mu}x^*+\frac{\beta}{2\mu}\eta+\frac{\gamma}{2\mu}z^*}d\eta + \int_0^{z^*} s(z^*,\zeta)v(x^*,\sqrt{\mu},\zeta,t)e^{\frac{\alpha}{2\mu}x^*+\frac{\beta}{2\sqrt{\mu}}+\frac{\gamma}{2\mu}\zeta}d\zeta, \tag{62}$$

$$V_3(x^*,y^*,t) = \int_0^{x^*} k(x^*,\xi)v(\xi,y^*,\sqrt{\mu},t)e^{\frac{\alpha}{2\mu}\xi+\frac{\beta}{2\mu}y^*+\frac{\gamma}{2\sqrt{\mu}}}d\xi + \int_0^{y^*} l(y^*,\eta)v(x^*,\eta,\sqrt{\mu},t)e^{\frac{\alpha}{2\mu}x^*+\frac{\beta}{2\mu}\eta+\frac{\gamma}{2\sqrt{\mu}}}d\eta + \int_0^{\sqrt{\mu}} s(\sqrt{\mu},\zeta)v(x^*,y^*,\zeta,t)e^{\frac{\alpha}{2\mu}x^*+\frac{\beta}{2\mu}y^*+\frac{\gamma}{2\mu}\zeta}d\zeta. \tag{63}$$

In next section the controllers for a general "n" dimensional unstable heat equation are defined.

IV. General Case

Consider an "n" dimensional unstable heat equation by positive stiffness constant as

$$u_t(\mathbf{x},t) = \sum_{i=1}^{n} u_{x_ix_i}(\mathbf{x},t) + \lambda u(\mathbf{x},t), \tag{64}$$

$$u(0,\mathbf{x_k},t) = 0, \tag{65}$$

$$u(1,\mathbf{x_k},t) = U_i(\mathbf{x_k},t), \tag{66}$$

where $\mathbf{x} = [x_1x_2x_3...x_n]$, $\mathbf{x_k} = [x_1x_2x_3...x_{i-1}x_{i+1}...x_n]$, $k = \{1,2..,n\}\setminus\{i\}$. By using the same methodology to map the transformation

$$w(\mathbf{x},t) = u(\mathbf{x},t) - \sum_{i=1}^{n}\int_0^{x_i} k_i(x_i,\xi_i)u(\xi_i,\mathbf{x_k},t)d\xi_i, \tag{67}$$

which maps the system (64)–(66) into the following system

$$w_t(\mathbf{x},t) = \sum_{i=1}^{n} w_{x_ix_i}(\mathbf{x},t), w(0,\mathbf{x_k},t) = 0, w(1,\mathbf{x_k},t) = 0. \tag{68}$$

The controllers for the system (64)–(66) are defined as

$$U_i(\mathbf{x_k},t) = \int_0^1 k_i(1,\xi_i)u(\xi_i,\mathbf{x_k},t)d\xi_i + \sum_{j=1}^{n-1}\int_0^{x_j} k_j(1,\mathbf{x_k})u(\xi_j,\mathbf{x_k},t)d\xi_j, \tag{69}$$

where the Kernel functions are defined as

$$k_i(x_i,\xi_i) = -c_i\xi_i \frac{I_1\left[\sqrt{\lambda\left(x_i^2-\xi_i^2\right)}\right]}{\sqrt{\lambda\left(x_i^2-\xi_i^2\right)}}, \tag{70}$$

where c_i are design parameters and chosen arbitrarily with the constraint $\sum_{i=1}^{i=n} c_i = \lambda$.

V. Stability Analysis

A. In 2-D Domain

The system (10)–(12) is exponentially stable in L_2. Consider a Lyapunov Function

$$V = \frac{1}{2}\int_0^1\int_0^1 w^2(x,y,t)dxdy. \tag{71}$$

Deriving the selected Lyapunov function by time

$$\begin{aligned}\dot{V} &= \int_0^1\int_0^1 ww_t dxdy = \int_0^1\int_0^1 w(w_{xx}+w_{yy})dxdy \\ &= \int_0^1\int_0^1 ww_{xx}dxdy + \int_0^1\int_0^1 ww_{yy}dxdy \\ &= \int_0^1\left[ww_x|_0^1 - \int_0^1 w_x^2dx\right]dy + \int_0^1\left[ww_y|_0^1 - \int_0^1 w_y^2dy\right]dx \\ &= \int_0^1\left[-\int_0^1 w_x^2dx\right]dy + \int_0^1\left[-\int_0^1 w_y^2dy\right]dx, \end{aligned}\tag{72}$$

$$\dot{V} = -\int_0^1\int_0^1\left[w_x^2+w_y^2\right]dxdy. \tag{73}$$

The time derivative of V shows that is bounded. But, it doesn't depends on w. Thus, it isn't clear if V goes to zero or not. Poincare's Inequality[15]:

$$\lambda_0\int_\Omega f^2dx \le \int_\Omega |\nabla f|^2dx, \tag{74}$$

where Ω is a bounded region in $\mathbb{R}^2$ or $\mathbb{R}^3$. $f\in C^1(\Omega)$, $f=0$ on the boundary on Ω and λ_0 is the smallest eigenvalue of the problem

$$\nabla f + \lambda f = 0 \quad in \quad \Omega, \tag{75}$$

$$f = 0 \quad on \quad \partial\Omega. \tag{76}$$

Using (73) and Poincare's Inequality along with the time derivative of V, we get

$$\dot{V} = -\int_0^1\int_0^1 \left[w_x^2 + w_y^2\right] dxdy$$
$$\Rightarrow \dot{V} \le -\frac{1}{\lambda_0}\int_0^1\int_0^1 w^2 dxdy \le -\frac{2}{\lambda_0}V, \tag{77}$$

where the smallest eigenvalue of (76) for two dimensional case $\lambda_0 = \sqrt{2}\pi > 0$.

B. In 3-D Domain

Similarly, with the help of the same Lyapunov function in 3D case, by using the same methodology it can be proved that (41)–(43) is also exponentially stable is exponentially stable in L_2. Consider a Lyapunov Function

$$V = \frac{1}{2}\int_0^1\int_0^1\int_0^1 w^2(x,y,z,t)dxdydz. \tag{78}$$

With making same calculations as in two dimensional domain the time derivative of the selected Lyapunov function is

$$\dot{V} = -\int_0^1\int_0^1\int_0^1 \left[w_x^2 + w_y^2 + w_z^2\right] dxdydz. \tag{79}$$

The time derivative of V shows that is bounded. But, it doesn't depends on w. Thus, it isn't clear if V goes to zero or not. By using similar methodology one can show that

$$\dot{V} \le -\frac{1}{\lambda_0}\int_0^1\int_0^1\int_0^1 w^2 dxdydz \le -\frac{2}{\lambda_0}V, \tag{80}$$

where $\lambda_0 = \sqrt{3}\pi > 0$ and inequality (80) shows that (41)–(43) is exponentially stable.

C. In N-D Domain

With the help of the same Lyapunov function in nD case, by using the same approach it can be proved that (68) is also exponentially stable. (68) is exponentially stable in L_2. Consider a Lyapunov Function

$$V = \frac{1}{2}\int_0^1\int_0^1 \dots \int_0^1\int_0^1 w^2(\mathbf{x},t)d\mathbf{x}. \tag{81}$$

With making same calculations as in two dimensional domain

$$\dot{V} = -\int_0^1\int_0^1 \dots \int_0^1\int_0^1 \sum_{i=1}^{n} x_i d\mathbf{x}. \tag{82}$$

By using similar methodology one can show that

$$\dot{V} \le -\frac{1}{\lambda_0}\int_0^1\int_0^1 \dots \int_0^1\int_0^1 w^2 d\mathbf{x} \le -\frac{2}{\lambda_0}V, \tag{83}$$

where $\lambda_0 = \sqrt{n}\pi$, and this shows the system (68) is exponentially stable.

VI. Inverse Transformations

A. In 2-D Domain

By using the following transformation to obtain the inverse kernel functions for 2-D Heat Equation.

$$u(x,y,t) = w(x,y,t) + \int_0^x s(x,\xi)w(\xi,y,t)d\xi + \int_0^y r(y,\eta)w(x,\eta,t)d\eta, \tag{84}$$

Differentiating (84) with respect to time, by using integration by part and (10), we get

$$\begin{aligned} u_t(x,y,t) &= w_{xx}(x,y,t) + w_{yy}(x,y,t) + s(x,x)w_x(x,y,t) \\ &+ \int_0^x s(x,\xi)w_{yy}(\xi,y,t)d\xi - s_\xi(x,x)w(x,y,t) \\ &- s(x,0)w_x(0,y,t) + r(y,y)w_y(x,y,t) - r_\eta(y,y)w(x,y,t) \\ &- r(y,0)w_y(x,0,t) + \int_0^y r_{\eta\eta}(y,\eta)w(x,\eta,t)d\eta \\ &+ \int_0^x s_{\xi\xi}(x,\xi)w(\xi,y,t)d\xi + \int_0^y r(y,\eta)w_{xx}(x,\eta,t)d\eta \end{aligned} \tag{85}$$

Similarly, differentiating twice with respect to x

$$\begin{aligned} u_{xx}(x,y,t) &= w_{xx}(x,y,t) + \frac{d}{dx}\left[s(x,x)\right]w(x,y,t) \\ &+ s(x,x)w_x(x,y,t) + s_x(x,x)w(x,y,t) \\ &+ \int_0^x s_{xx}(x,\xi)w(\xi,y,t)d\xi + \int_0^y r(y,\eta)w_{xx}(x,\eta,t)d\eta, \end{aligned} \tag{86}$$

and respect to the dimension y

$$\begin{aligned} u_{yy}(x,y,t) &= w_{yy}(x,y,t) + \int_0^x s(x,\xi)u_{yy}(\xi,y,t)d\xi \\ &+ \frac{d}{dy}\left[r(y,y)\right]w(x,y,t) + s(y,y)w_y(x,y,t) \\ &+ r_y(y,y)w(x,y,t) + \int_0^y r_{yy}(y,\eta)w(x,\eta,t)d\eta. \end{aligned} \tag{87}$$

With the help of (5), along the solutions and we follow the same procedure. Thus we get the following conditions for the kernel functions $s(x,\xi)$ and $r(y,\eta)$

$$r_{yy}(y,\eta) - r_{\eta\eta}(y,\eta) = -\lambda r(y,\eta), \tag{88}$$
$$r(y,y) = -\frac{\lambda - c_i}{2}y, \tag{89}$$
$$r(y,0) = 0, \tag{90}$$

where c_i is a positive control parameter and the solution of this system is given in [2] as

$$r(y,\eta) = -(\lambda - c_i)\eta\frac{J_1\left[\sqrt{\lambda\left(y^2-\eta^2\right)}\right]}{\sqrt{\lambda\left(y^2-\eta^2\right)}}. \tag{91}$$

Similarly;

$$s_{xx}(x,\xi) - s_{\xi\xi}(x,\xi) = -\lambda s(x,\xi), \tag{92}$$
$$s(x,x) = -\frac{c_i}{2}x, \tag{93}$$
$$s(x,0) = 0, \tag{94}$$

where the solution of this system is given in [2] as

$$s(x,\xi) = -c_i\xi\frac{J_1\left[\sqrt{\lambda\left(x^2-\xi^2\right)}\right]}{\sqrt{\lambda\left(x^2-\xi^2\right)}}. \tag{95}$$

B. In 3-D Domain

To define the inverse Kernel functions for the systems in 3 dimensional case, following transformation can be used.

$$u(x,y,z,t) = w(x,y,z,t) + \int_0^x s(x,\xi)w(\xi,y,z,t)d\xi + \int_0^y r(y,\eta)w(x,\eta,z,t)d\eta + \int_0^z p(y,\eta)w(x,y,\zeta,t)d\zeta. \quad (96)$$

By making the same calculations the Kernel can be obtained as follows

$$s(x,\xi) = -c_{1i}\xi \frac{J_1\left[\sqrt{\lambda\,(x^2-\xi^2)}\right]}{\sqrt{\lambda\,(x^2-\xi^2)}}, \quad (97)$$

$$r(y,\eta) = -c_{2i}\xi \frac{J_1\left[\sqrt{\lambda\,(x^2-\xi^2)}\right]}{\sqrt{\lambda\,(x^2-\xi^2)}}, \quad (98)$$

$$p(z,\zeta) = -\left(\lambda - c_{1i} - c_{2i}\right)\xi \frac{J_1\left[\sqrt{\lambda\,(x^2-\xi^2)}\right]}{\sqrt{\lambda\,(x^2-\xi^2)}}, \quad (99)$$

where c_{1i}, and c_{2i} are arbitrary parameters.

C. In N-D Domain

By using the transformation

$$u(\mathbf{x},t) = w(\mathbf{x},t) + \sum_{i=1}^{n}\int_0^{x_i} r_i(x_i,\xi_i)u(\xi_i,\mathbf{x_k},t), \quad (100)$$

where the Kernel functions are defined as

$$r_i(x_i,\xi_i) = -c_i\xi_i \frac{J_1\left[\sqrt{\lambda\left(x_i^2-\xi_i^2\right)}\right]}{\sqrt{\lambda\left(x_i^2-\xi_i^2\right)}}, \quad (101)$$

where c_i are design parameters and chosen arbitrarily with the constraint $\sum_{j=1}^{j=n} c_j = \lambda$.

VII. Conclusions

This paper presents an expanded backstepping boundary control design methodology for unstable heat equations in two, three and ”n” dimensional cases, and their stability analysis to reach an exponentially stable target systems. We introduce new integral transformations and use them to obtain explicit controllers for heat equations to map an unstable system into an exponentially stable system by using ”n” controllers in generally ”n” dimensional domain. At the end of the study we define the controllers, the transformations, and the inverse transformations used in design methodology. Finally, by using this method one can define any other conrollers for different cases of the heat equations in 2D, 3D or nD domain to reach an exponentially stable system from an unstable system for more than one dimensional systems, if he can obtain the system as a heat equation with boundaries.

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