Matrix Valued Difference Equations with Spectral Singularities

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Abstract—In this study, we examine some spectral properties of non-selfadjoint matrix-valued difference equations consisting of a polynomial-type Jost solution. The aim of this study is to investigate the eigenvalues and spectral singularities of the difference operator L which is expressed by the above-mentioned difference equation. Firstly, thanks to the representation of polynomial type Jost solution of this equation, we obtain asymptotics and some analytical properties. Then, using the uniqueness theorems of analytic functions, we guarantee that the operator L has a finite number of eigenvalues and spectral singularities.

Keywords—Difference Equations, Jost Functions, Asymptotics, Eigenvalues, Continuous Spectrum, Spectral Singularities

I. INTRODUCTION

WE propose to discuss here the discrete Sturm-Liouville problem on the nonself-adjoint case, investigate some spectral properties of second-order difference operators and their spectrum including both eigenvalues and spectral singularities. The main result on these problems states that, under certain assumptions, the nonself-adjoint difference operator has the finite number of eigenvalues and spectral singularities with finite multiplicity. In spectral theory, it is well known that the spectral singularities are the poles of kernel of the resolvent and belong to the continuous spectrum but they are not the eigenvalues. Spectral singularities are spectral points that spoil the completeness of the eigenfunctions of certain non self-adjoint operators.

Let us shortly give an overwiew on the existing literature of the subject. Study of the spectral analysis of operators with continuous and discrete spectrum was begun by Naimark [1]. Naimark investigated the Sturm-Liouville equation

$$-y'' + q(x)y = \lambda^2 y, \ 0 \le x < \infty, \qquad (1)$$

where q is a real valued function and λ is a spectral parameter. The equation (1) has the bounded solution satisfying the condition

$$\lim_{x \to \infty} y(x, \lambda) e^{-i\lambda x} = 1,$$

where

$$\lambda \in \overline{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C}, \ Im\lambda \ge 0\}.$$

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The solution $e(x, \lambda)$ is called the Jost function of (1). It was shown that the Jost solution has the integral representation

$$e(x, \lambda) = e^{i\lambda x} + \int_x^\infty K(x, t) e^{i\lambda t} dt, \ \lambda \in \overline{\mathbb{C}}_+ \qquad (2)$$

under the condition

$$\int_0^\infty x |q(x)| \, dx < \infty,$$

where $K(x, t)$ is defined by the function q [2]. The representation of Jost function has an remarkable importance on the solutions of direct and inverse problems of quantum scattering theory.

Schwartz studied the spectral singularities of a certain class of abstract linear operators in a Hilbert space and proved that the self-adjoint operators have no spectral singularities [3]. The sets of spectral singularities for closed linear operators on a Banach space was given by Nagy [4]. Nagy shows that the set of spectral singularities defined according to his general definition coincides in the case of differential operators as defined by Naimark and Lyance [1]-[5]. Pavlov established the dependence of the structure of the spectral singularities of Schrödinger operators on the behaviour of the potential function at infinity [6].

Later, the spectral properties of Schrödinger, Dirac and Klein-Gordon operators was intensively studied by various authors [7]-[12]. In [13]-[17], the Schrödinger equations with general point interaction have been investigated in detail. Consequently, it was realized that eigenvalues and spectral singularities have physical meanings in quantum mechanics. A physical interpretation for the spectral singularities that identifies with the energies of scattering states having infinite reflection and transmission coefficients. Recently, spectral analysis of discrete equations have become interesting subject in this field. In particular, in ([20]-[21]) the difference equation

$$a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n \ , \ n \in \mathbb{N} \qquad (3)$$

was analysed where $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$ are complex sequences and $a_n \ne 0$ for all $n \in \mathbb{N}$ under the condition

$$\sum_{n=1}^\infty n (|1 - a_n| + |b_n|) < \infty.$$

Note that, we can write the difference equation (3) in the following Sturm-Liouville form:

$$\Delta(a_{n-1} \Delta y_{n-1}) + q_n y_n = \lambda y_n \ , \ n \in \mathbb{N}$$

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where

$$q_n = a_{n-1} + a_n + b_n$$

and Δ is the forward difference operator.

The modeling of certain problems in engineering, physics, economics, control theory and other areas of study has led to the rapid development of the theory of difference equations. So, there are a lot of studies about selfadjoint and non-selfadjoint difference operators [18]-[22]. In the literature, most of the studies about this kind of difference equations are of scalar coefficients except [23],[24]. But these studies based on matrix coefficients are all about the exponential type Jost solutions of second order. The spectral analysis of matrix-valued difference equations of second order with polynomial-type Jost solution on the nonself-adjoint case has not been investigated yet.

II. JOST SOLUTION AND SPECTRAL PROPERTIES OF L

$l_2(\mathbb{N}, \mathbb{C}_v)$ is a Hilbert space consisting of all matrix sequences in $\mathbb{C}_v$, where $\mathbb{C}_v$ is a v-dimensional ($v < \infty$) Euclidian space. Let L denote the difference operator of second order generated in $l_2(\mathbb{N}, \mathbb{C}_v)$ by the following matrix difference expression

$$A_{n-1}Y_{n-1} + B_nY_n + A_nY_{n+1} = \lambda Y_n \ , \ n \in \mathbb{N}$$

and the boundary condition $Y_0 = 0$, where $\{A_n\}_{n \in \mathbb{N} \cup \{0\}}$ and $\{B_n\}_{n \in \mathbb{N}}$ are linear non-selfadjoint operators (matrices) acting in $\mathbb{C}_v$, $\det A_n \neq 0$ and λ is a spectral parameter. Under this assumption,

$$A_n \neq A_n^\star, (n \in \mathbb{N} \cup \{0\}), B_n \neq B_n^\star \quad (n \in \mathbb{N})$$

and " $\star$ " denotes the adjoint operator. It is clear that the operator L is non-selfadjoint and we can easily obtain the following Jacobi matrix using the operator L.

$$J_{(i,j)} = \begin{cases} B_j & , & i = j \\ A_j & , & i = j+1 \\ A_{j-1} & , & i = j-1 \\ 0 & , & otherwise \end{cases} ; i,j \in \mathbb{N},$$

where 0 is the zero matrix in $\mathbb{C}_v$.

Let us consider the difference equation

$$A_{n-1}Y_{n-1} + B_nY_n + A_nY_{n+1} = (z + z^{-1})Y_n, \ n \in \mathbb{N} \quad (4)$$

and suppose that the matrix sequences $\{A_n\}_{n \in \mathbb{N} \cup \{0\}}$ and $\{B_n\}_{n \in \mathbb{N}}$ satisfy

$$\sum_{n=1}^{\infty} n \left(||I - A_n|| + ||B_n|| \right) < \infty \ , \quad (5)$$

where I denotes the identity matrix in $\mathbb{C}_v$. Furthermore, let $E(z) = \{E_n(z)\}, n \in \mathbb{N} \cup \{0\}$ denotes the matrix solution of the equation (4) satisfying the condition

$$\lim_{n \to \infty} E_n(z)z^{-n} = I, \ z \in D_o := \{z : |z| = 1\}. \quad (6)$$

Note that, the solution $E(z)$ is called the Jost solution of (4). In 2012, the following results were obtained in [24] when the

operator is self-adjoint. The proof of these results are similar with [24] in the non-selfadjoint case of this operator. Under condition (5), equation (4) has unique solution $E(z)$ having representation

$$E_n(z) = T_n z^n \left[I + \sum_{m=1}^{\infty} K_{nm}z^m \right], n \in \mathbb{N} \cup \{0\}, \ z \in D_0, \quad (7)$$

where T_n and K_{nm} are expressed in terms of the matrices $\{A_n\}$ and $\{B_n\}$ as

$$T_n = \sum_{p=n}^{\infty} A_p^{-1}, \quad (8)$$

$$K_{n1} = -\sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p, \quad (9)$$

$$K_{n,2} = -\sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p K_{p1} \\ + \sum_{p=n+1}^{\infty} T_p^{-1} \left(I - A_p^2 \right) T_p, \quad (10)$$

$$K_{n,m+2} = \sum_{p=n+1}^{\infty} T_p^{-1} \left(I - A_p^2 \right) T_p K_{p+1,m} \\ - \sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p K_{p,m+1} + K_{n+1,m} \quad (11)$$

Due to the condition (5), the infinite product and the series are absolutely convergent. Moreover, K_{nm} satisfies

$$||K_{nm}|| \leq C \sum_{p=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty} \left(||I - A_p|| + ||B_p|| \right), \ n \in \mathbb{N}, \quad (12)$$

where $\left[\left|\frac{m}{2}\right|\right]$ is the integer part of $\frac{m}{2}$ and C is a positive constant. Consequently, it follows from (7) and (12) that $E_n(z)$ has analytic continuation from D_0 to

$$D_1 := \{z : |z| < 1\} \setminus \{0\}.$$

$E_n(z)$ also satisfies the following asymptotic equation

$$E_n(z) = z^n [I + o(1)] \ , \quad n \to \infty \quad (13)$$

for $z \in D := \{z : |z| \leq 1\} \setminus \{0\}$.

Theorem 1: If the condition (5) holds, then

$$\sigma_c(L) = [-2, 2],$$

where $\sigma_c(L)$ denotes the continuous spectrum of L.

Proof: Let introduce the difference operators L_0 and L_1 generated in $\ell_2(\mathbb{N}, \mathbb{C}_v)$ by the difference expressions

$$l_0(Y) = Y_{n-1} + Y_{n+1}$$

and

$$l_1(Y) = (A_{n-1} - I)Y_{n-1} + B_nY_n + (A_n - I)Y_{n+1}$$

with the boundary condition $Y_0 = 0$, respectively. We can also define the following Jacobi matrices

$$L_0 = \begin{cases} I & , & i = j+1, \quad i = j-1 \\ 0 & , & otherwise \end{cases}$$

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and

$$L_1 = \begin{cases} B_i & , & i = j \\ A_i - I & , & i = j - 1 \\ A_{i-1} - I & , & i = j + 1 \\ 0 & , & otherwise. \end{cases}$$

corresponding the operators L_0 and L_1, respectively, where $i, j \in \mathbb{N}$. It is obvious that $L = L_0 + L_1$ and L_0 is a self-adjoint operator. It follows from (5) that the operator L_1 is compact in $\ell_2(\mathbb{N}, \mathbb{C}_v)$ (see [25]). Moreover, it is known that $\sigma(L_0) = \sigma_c(L_0) = [-2, 2]$ (see[26]). Then, using the Weyl theorem of a compact perturbation, we obtain

$$\sigma_c(L) = \sigma_c(L_0) = [-2, 2]$$

(see [27]). This completes the proof. ■

Let $g(z) := \det E_0(z)$, $z \in D$, where $E_0(z)$ is the Jost function of L. Now, we will denote the set of all eigenvalues and spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$

$$\begin{aligned} \sigma_d(L) &= \left\{ \lambda : \lambda = z + z^{-1}, z \in D_1, \ g(z) = 0 \right\}, \\ \sigma_{ss}(L) &= \left\{ \lambda : \lambda = z + z^{-1}, z \in D_0, \ g(z) = 0 \right\}, \end{aligned} \tag{14}$$

respectively.

Theorem 2: Under the condition (5),

i) The set of eigenvalues of L is bounded, is no more than countable and its limit points can lie in $[-2, 2]$,

ii) Linear Lebesgue measure of the set σ_{ss} on the real axis is zero and $\sigma_{ss}(L) \subset [-2, 2]$.

Proof: From the definitions of the sets of σ_d and σ_{ss}, we need to discuss the quantitative properties of the zeros of function g in D in order to get the quantitative properties of the eigenvalues and the spectral singularities of the operator L. Using (13) and the definition of $g(z)$, we obtain

$$g(z) = [1 + o(1)] \ , \quad n \to \infty, \tag{15}$$

for all $z \in D_1$. (15) gives the boundedness of the set of zero of g in D_1. Since the analytic function $g \neq 0$ for all $z \in D_1$, we get that the zeros of the function g in D_1 is separated by using the uniqueness theorems given for unit disc [28]. Since the zeros of the set of g in D_1 is bounded and the zeros of the function g in D_1 are separated, g has at most a countable number of zeros in D_1, i.e., the operator L has at most countable number of eigenvalues. By the uniqueness theorem of analytic functions, we find that the the limit points of zeros of g in D_1 can lie only in D_0. Using (14), it can be shown easily that $\sigma_{ss}(L) \subset [-2, 2]$. From Privalov theorem [28], we obtain that the linear Lebesgue measure of the set of zeros of g on real axis is not positive because $g(z) \neq 0$ for all $z \in D_0$, i.e., the linear Lebesgue measure of the $\sigma_{ss}(L)$ is zero. ■

Theorem 3: Assume

$$\sum_{p=0}^{\infty} e^{\varepsilon p}(\|I - A_p\| + \|B_p\|) < \infty, \tag{16}$$

for some $\varepsilon > 0$. Then the operator L has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.

Proof: Using (12) and (16), we find that

$$\|K_{0m}\| \leq Ce^{-\frac{\varepsilon}{6}m}, \quad m \in \mathbb{N}, \tag{17}$$

where C is a constant. It follows from (17) that the function g has an analytic continuation to the set

$$D_3 := \{z : |z| < e^{\frac{\varepsilon}{6}}, \varepsilon > 0\}.$$

Because the series

$$\sum_{m=0}^{\infty} K_{0m} z^m \quad \text{and} \quad \sum_{m=0}^{\infty} m K_{0m} z^{m-1}$$

are uniform convergent if and only if $\ln |z| < \frac{\varepsilon}{6}$. So, the limit points of the zeros of g in D can not lie in D_0, and Theorem 2 shows that, the bounded sets $\sigma_d(L)$ and $\sigma_{ss}(L)$ have no limit points. Finally, the sets $\sigma_d(L)$ and $\sigma_{ss}(L)$ have a finite number of elements from the Bolzano–Weierstraß theorem. From analyticity of g in D_3, we find that all zeros of g in D have a finite multiplicity. Consequently, all eigenvalues and spectral singularities of the operator L have a finite multiplicity. ■

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