Power Series Solution to Sliding Velocity in Three-Dimensional Multibody Systems with Impact and Friction

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***Abstract*—**The system of ordinary nonlinear differential equations describing sliding velocity during impact with friction for a three-dimensional rigid-multibody system is developed. No analytical solutions have been obtained before for this highly nonlinear system. Hence, a power series solution is proposed. Since the validity of this solution is limited to its convergence zone, a suitable time step is chosen and at the end of it a new series solution is constructed. For a case study, the trajectory of the sliding velocity using the proposed method is built using 6 time steps, which coincides with a Runge-Kutta solution using 38 time steps.

***Keywords*—**Impact with friction, nonlinear ordinary differential equations, power series solutions, rough collision.

I. Introduction

In practice, many systems are subjected to impacts during their functional operations and in most cases friction cannot be ignored. These systems have to be modeled as rigid-multibody systems with impact and friction. Examples include part-feeding systems, cooperative manipulators, and automatic assembly of mechanical components. Typically, impact is modeled as sudden event with finite change in velocities and no change in configurations of the system. During the small finite impact period, many scenarios for the sliding velocity could take place. To capture these scenarios and correctly model the impact phenomenon, the equations of motion can be written with normal component of impulse as time-like variable. This analysis results in highly nonlinear system of ordinary differential equation. Numerical solutions are available to this system in literature [1]-[5]. However for the best of authors' knowledge, no analytical solution for this system has been developed yet.

There are different methods used to solve the nonlinearity problem in system of nonlinear differential equations. Linearization is one method, however, as [6] mentioned, many differential equations cannot be linearized or linearized equations no longer represent the model accurately. To evaluate a specific analytical method: (a) it constantly has to produce approximate solutions efficiently and (b) its solutions are acceptable in the whole region of all physical parameters. Perturbation methods are widely used and well understood [7]-[9]; however, they depend upon the existence of a small parameter which is not always available in practical cases. Even with the existence of such parameter, in many cases, no solution could be found. Though the homotopy analysis method (HAM) does not need a small parameter [10]-[12], an initial guess is required which limits the applicability of the method and open the door for serious uncertainties that it can constantly give a solution. Nowadays, there is a need for more satisfactory methods to comply with the two previously mentioned standards. Power series solution, which in fact is a Taylor series solution, is a promising method. For a long time, the method has been used for linear problems. Zhou [13] developed the differential transform method and used it in solving linear and nonlinear initial value problems in electric circuit analysis. The method is an iterative procedure for obtaining power series solution of linear and nonlinear ordinary or partial differential equations [14]. For a system of ordinary differential equations, [15] developed a method that utilizes auxiliary variables to expand the system to larger system that can be solved using power series. Most nonlinear differential equations can be solved using Parker-Sochacki method at the cost of also calculating the coefficients of auxiliary equations [16], [17]. Hence, for a broad range, the method satisfy standard (a). However, the method is ineffective in satisfying standard (b). The power series solution is valid only in the interval of convergence; hence, outside the interval of convergence it is not valid. Some researches tried to develop techniques to extend the convergence interval for the power series solutions. Common techniques are Pade approximation and Laplace-Pade approximation [18].

In this paper, the system of nonlinear differential equations that describe the evolution of the sliding velocity during impact period is considered. Power series solution for the system is built. The procedure neither uses the differential transform method nor uses Parker-Sochacki method to obtain the solution. Hence, no transformed functions or auxiliary variables are introduced. The commercial program Mathematica 9 is used for directly obtaining the coefficients of the power series solutions. Therefore, the cost of calculating the coefficients of auxiliary equations is saved. To overcome the divergence problem, a time step is chosen, within the convergence zone, and the series is evaluated at that time. The process is repeated up to the end of the required time domain.

II. Sliding Velocity

The inertia operator $\mathbf{D} \in R^{3x3}$ for three-dimensional multibody system depends only upon system configuration

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and is a symmetric positive definite matrix which can be written as [3]:

$$\mathbf{D} = \begin{bmatrix} a & \boldsymbol{h}^T \\ \boldsymbol{h} & \boldsymbol{b} \end{bmatrix} \quad (1)$$

where $a \in R^{1\text{x}1}$, $\boldsymbol{h} = \begin{Bmatrix} h_1 \\ h_2 \end{Bmatrix} \in R^{2\text{x}1}$, and $\boldsymbol{b} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in R^{2\times 2}$. For a single point rough collision in three-dimensional rigid-multibody systems subjected to impact with friction; the vector of the evolution of the sliding velocity is given by [3]:

$$\frac{d\mathbf{v_t}}{dp_n} = \mathbf{h} - \mu\mathbf{b}\boldsymbol{\sigma} \quad (2)$$

where $\mathbf{v_t}$ is the tangential component of the velocity of the colliding point, $\mathbf{v_t} \in R^{2\text{x}1}$ (m/s), p_n is the normal component of impulse at collision point (N.s), μ is the coefficient of friction, and σ is the unit vector defining the sliding direction;

$$\boldsymbol{\sigma} = \frac{\mathbf{v}_t}{\|\mathbf{v_t}\|} \quad (3)$$

Equation (2) is a highly nonlinear system of ordinary differential equation which shows that the vector of the evolution of the sliding velocity has a direction that does not coincide, in general, with the direction of the tangential velocity $\mathbf{v_t}$ itself. This means that the tangential velocity could continuously change its direction during the collision interval. Consequently, $\boldsymbol{\sigma}$ is not a constant coefficient, The scalar $p_n$has been selected as a collision parameter, since it is a variable starts with zero at the starting of impact and continuously increases through the collision period until the end of impact. Therefore, it is selected as time-like quantity; in terms of it the progress of impact is expressed. The time has not been chosen because the force-time relation is not known prior to the analysis.

Equation (2) can be written in scalar form instead of vector form as:

$$\frac{dv_{t1}}{dp_n} = h_1 - \mu\left(\frac{b_1 v_{t1}}{\sqrt{(v_{t1})^2 + (v_{t2})^2}} + \frac{b_2 v_{t2}}{\sqrt{(v_{t1})^2 + (v_{t2})^2}}\right) \quad (4)$$

$$\frac{dv_{t2}}{dp_n} = h_2 - \mu\left(\frac{b_3 v_{t1}}{\sqrt{(v_{t1})^2 + (v_{t2})^2}} + \frac{b_4 v_{t2}}{\sqrt{(v_{t1})^2 + (v_{t2})^2}}\right) \quad (5)$$

where $v_{t1}$and v_{t2} are the two perpendicular components of the tangential velocity $\mathbf{v_t}$.

III. Power Series Solution

The method of power series is used to get an approximate solution. The series solution can be written as Taylor expansion:

$$v_{t1} = \sum_{m=0}^{\infty} a_m\, (p_n)^m \quad (6)$$
$$v_{t2} = \sum_{m=0}^{\infty} d_m\, (p_n)^m \quad (7)$$

Substitution in (4) and (5) gives:

$$\sum_{m=0}^{\infty} m a_m\, (p_n)^{m-1} = h_1 - \mu\left(\frac{b_1 \sum_{m=0}^{\infty} a_m\, (p_n)^m}{\sqrt{(\sum_{m=0}^{\infty} a_m\, (p_n)^m)^2 + (\sum_{m=0}^{\infty} d_m\, (p_n)^m)^2}} + \frac{b_2 \sum_{m=0}^{\infty} d_m\, (p_n)^m}{\sqrt{(\sum_{m=0}^{\infty} a_m\, (p_n)^m)^2 + (\sum_{m=0}^{\infty} d_m\, (p_n)^m)^2}}\right) \quad (8)$$

and

$$\sum_{m=0}^{\infty} m d_m\, (p_n)^{m-1} = h_2 - \mu\left(\frac{b_3 \sum_{m=0}^{\infty} a_m\, (p_n)^m}{\sqrt{(\sum_{m=0}^{\infty} a_m\, (p_n)^m)^2 + (\sum_{m=0}^{\infty} d_m\, (p_n)^m)^2}} + \frac{b_4 \sum_{m=0}^{\infty} d_m\, (P_n)^m}{\sqrt{(\sum_{m=0}^{\infty} a_m\, (p_n)^m)^2 + (\sum_{m=0}^{\infty} d_m\, (p_n)^m)^2}}\right) \quad (9)$$

a_0 and d_0 are the initial conditions of v_{t1} and v_{t2} respectively. Equating the coefficients of the same power of p_n in both sides of (8) and (9) specifies the required coefficients of the power series solution. The commercial mathematical program '*Mathematica 9*' is used for performing this task and the coefficients are:

$$a_1 = (h_1 - S) \qquad d_1 = (h_1 - V)$$
$$a_2 = -0.5\,K \qquad d_2 = -0.5\,R$$

$$\begin{aligned} a_3 = -\frac{\mu}{6({a_0}^2 + {d_0}^2)^{5/2}} [&2{a_0}^2 b_2 d_0 {a_1}^2 - 3a_0 b_1 {d_0}^2 {a_1}^2 \\ &- b_2 {d_0}^3 (h_1 - S)^2 \\ &- 2{a_0}^3 b_2 (h_1 - S)(h_2 - V) \\ &+ 4{a_0}^2 b_1 d_0 (h_1 - S)(h_2 - V) \\ &+ 4a_0 b_2 {d_0}^2 (h_1 - S)(h_2 - V) \\ &- 2b_1 {d_0}^3 (h_1 - S)(h_2 - V) \\ &- {a_0}^3 b_1 (h_2 - V)^2 - 3{a_0}^2 b_2 d_0 (h_2 - V)^2 \\ &+ 2a_0 b_1 {d_0}^2 (h_2 - V)^2 + {a_0}^3 b_2 d_0 K \\ &- {a_0}^2 b_1 {d_0}^2 K + a_0 b_2 {d_0}^3 K - b_1 {d_0}^4 K \\ &- {a_0}^4 b_2 R + {a_0}^3 b_1 d_0 R - {a_0}^2 b_2 {d_0}^2 R \\ &+ a_0 b_1 {d_0}^3 R] \end{aligned}$$

$$\begin{aligned} d_3 = -\frac{\mu}{6({a_0}^2 + {d_0}^2)^{5/2}} [&2{a_0}^2 b_4 d_0 (h_1 - S)^2 \\ &- 3a_0 b_3 {d_0}^2 (h_1 - S)^2 \\ &- b_4 {d_0}^3 (h_1 - S)^2 \\ &- 2{a_0}^3 b_4 (h_1 - S)(h_2 - V) \\ &+ 4{a_0}^2 b_3 d_0 (h_1 - S)(h_2 - V) \\ &+ 4a_0 b_4 {d_0}^2 (h_1 - S)(h_2 - V) \\ &- 2b_3 {d_0}^3 (h_1 - S)(h_2 - V) \\ &- {a_0}^3 b_3 (h_2 - V)^2 \\ &- 3{a_0}^2 b_4 d_0 (h_2 - V)^2 \\ &+ 2a_0 b_3 {d_0}^2 (h_2 - V)^2 + {a_0}^3 b_4 d_0 K \\ &- {a_0}^2 b_3 {d_0}^2 K + a_0 b_4 {d_0}^3 K - b_3 {d_0}^4 K \\ &- {a_0}^4 b_4 R + {a_0}^3 b_3 d_0 R - {a_0}^2 b_4 {d_0}^2 R \\ &+ a_0 b_3 {d_0}^3 R] \end{aligned} \quad (10)$$

where

$$K = \mu \frac{-a_0 b_2 d_0 a_1 + b_1 {d_0}^2 a_1 + (h_2 - S)({a_0}^2 b_2 - a_0 b_1 d_0)}{({a_0}^2 + {d_0}^2)^{3/2}}$$

$$S = \frac{a_0 b_1 + b_2 d_0}{\sqrt{{a_0}^2 + {d_0}^2}}\,\mu \quad , \quad V = \frac{a_0 b_3 + b_4 d_0}{\sqrt{{a_0}^2 + {d_0}^2}}\,\mu \quad \text{, and}$$

$$R = \mu \frac{-a_0 b_4 d_0 d_1 + b_3 {d_0}^2 d_1 + (h_2 - S)({a_0}^2 b_4 - a_0 b_3 d_0)}{({a_0}^2 + {d_0}^2)^{3/2}} \quad (11)$$

It can be noticed that all the power series coefficients given in (10) and (11) depends entirely upon the coefficient of friction μ, initial conditions a_0 and d_0, and upon the components of inertia operator **D**. Equations (6) and (7) are the general power series solution for the sliding velocity in a single point rough collision in three-dimensional rigid-multibody systems with impact and friction. Equations (10) and (11) specify the coefficients in (6) and (7) up to the forth term. Though more coefficients have been obtained with same technique they are not presented here because of their extraordinary dimension.

IV. Case Study

A case study of four degrees of freedom spatial rigid-robot that collides with a rough surface is considered. Details of that system are available in [4] and [5]. For that system, the inertia operator is:

$$\mathbf{D} = 10^{-3} \begin{bmatrix} 9.009 & -0.342 & 1.845 \\ -0.342 & 9.307 & 2.26 \\ 1.845 & 2.26 & 3.922 \end{bmatrix}$$

Hence, the mass parameters in (1) are specified. For the case when μ=0.3, this coefficient of friction as well as the mass parameters are substituted in (10) and (11) to calculate the coefficients of the power series solutions for $\boldsymbol{v_{t1}}$ and $\boldsymbol{v_{t2}}$, which are given in (6) and (7). Figs. 1-3 show the relation between the two components of the tangential velocity, $\boldsymbol{v_{t1}}$ and $\boldsymbol{v_{t2}}$, for three different initial conditions. Both the numerical solution and the power series solution are shown in each figure. The convergence interval of the power series solution is the zone where that solution coincides with the numerical solution. Outside that interval, the power series solution diverges from the numerical solution.

V. Improving the Solution's Convergence

Since the power series solution suffers from the problem of divergence of the solution, one could not get a valid solution for the entire solution domain in a single step. To obtain the required valid solution, a variable time step technique is developed. The first time step is chosen within the convergence zone of the power series solution and the solution variables are specified at the end of this time step. A new power series solution is generated started from these specified variables and a new time step is chosen within the convergence zone of this solution. This sequence is repeated up to the end of the required time domain. This technique is applied to the first case, previously presented in Fig. 1. The trajectory is obtained using 6 time steps (Fig. 4) compared to 38 time steps for the numerical solution using fourth order Runge-Kutta scheme.

For each time step, the interval of the normal impulse P_n and the corresponding first seven coefficients of the power series solution for v_{t1} and v_{t2} are shown in Tables I and II, respectively.

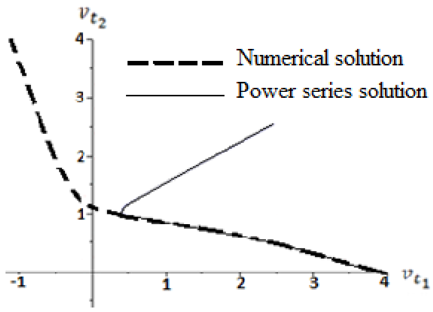

Fig. 1 Solutions for initial condition (4,0)

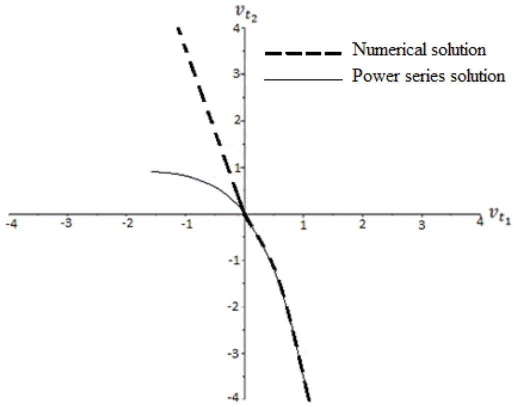

Fig. 2 Solutions for initial condition (1,-4)

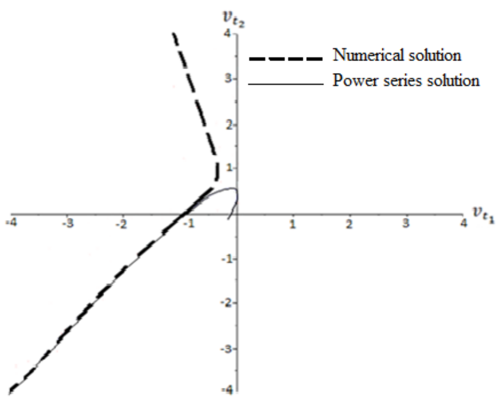

Fig. 3 Solutions for initial condition (-4,-4)

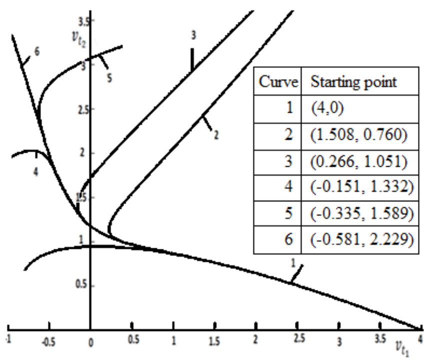

Fig. 4 Power series solution with six time steps

TABLE I
INTERVALS OF THE PERIODS AND THE FIRST SEVEN COEFFICIENTS OF THE SERIES SOLUTION FOR v_{t1}

Interval of the normal impulse	Coefficients						
	a_0	a_1	$a_2 * 10^{-7}$	$a_3 * 10^{-10}$	$a_4 * 10^{-12}$	$a_5 * 10^{-15}$	$a_6 * 10^{-19}$
0-784	4	-0.031342	-0.9890	-0.02356	0.0167142	0.0193681	0.172456
784-1265	1.508	-0.003140	3.95027	9.98373	1.3537	1.07456	-4.0277
1265-1665	0.2662	-0.001684	20.680	-10.518	-2.425	5.362	9.58313
1665-2000	-0.151	-0.0007	6.5021	-7.27	0.64458	-0.3821	-0.4710
2000-2450	-0.335	-0.00043	2.1896	-2.15645	0.18583	-0.14323	0.97602
2450-3000	-0.581	-0.00293	0.3914	-0.2757	0.017297	-0.100297	0.054838

TABLE II
INTERVALS OF THE PERIODS AND THE FIRST SEVEN COEFFICIENTS OF THE SERIES SOLUTION FOR v_{t2}

Interval of the normal impulse	Coefficients						
	d_0	d_1	$d_2 * 10^{-7}$	$d_3 * 10^{-11}$	$d_4 * 10^{-13}$	$d_5 * 10^{-17}$	$d_6 * 10^{-19}$
0-784	0	0.001167	-1.71637	-6.32099	-0.27437	-1.1786	-0.042394
784-1265	0.7603	0.007	-4.518	-7.9562	3.44164	75.699	8.3878
1265-1665	1.051	0.00053	2.9989	0.339004	-12.3161	61.524	34.380
1665-2000	1.332	0.00075	1.8392	-17.1417	0.94314	3.8904	-1.9906
2000-2450	1.589	0.00083	0.69099	-6.42435	0.501825	-3.23826	0.148375
2450-3000	2.229	0.00087	0.1298	-0.902564	0.05544	-0.31163	-1.9906

VI. CONCLUSION

Power series method has been used in this paper to construct the solution for the nonlinear system of differential equations for sliding velocity during impact with friction on 3-dimensional rigid-multibody systems. The power series method does not need a small parameter like the perturbation method. The power series method does not need an initial solution guess like the homotopy analysis method and the homotopy perturbation method. However, power series solution suffers from the problem of limited convergence interval. Hence, one could not get a valid solution for the entire solution domain in one shot. For a specific initial condition and coefficient of friction, the numerical solution is considered the reference solution and the power series solution is compared to it. Whenever the power series solution deviated, from the numerical solution, this analytical solution stops and the values of the variables are marked. Consequently, a new power series solution is generated started from these marked conditions. For the specific case considered, the orbit of the sliding velocity has been obtained with 6 time steps compared to 38 time steps to draw the same orbit for the numerical solution using fourth order Runge-Kutta scheme.

Future work is to explore methods that can increase the convergence zone of the series solution. Power series resummation methods are expected to improve the convergence. Many engineering problems are not solved before using power series; hence, constructing power series solutions to these problems are worth to investigate.

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