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Estimation of the Mean of the Selected Population

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Abstract—Two normal populations with different means and same variance are considered, where the variance is known. The population with the smaller sample mean is selected. Various estimators are constructed for the mean of the selected normal population. Finally, they are compared with respect to the bias and MSE risks by the mehod of Monte-Carlo simulation and their performances are analysed with the help of graphs.

Keywords—Estimation after selection, Brewster-Zidek technique.

I. Introduction

SUPPOSE we have two securities and we choose the security which is less risky. Then what can we say about the estimate of the risk of the selected security? For tackling this kind of situation we may consider the following model. Suppose we have two normal populations with means $\alpha_i,\ i=1,2$ respectively each having the same variance τ^2. A random sample of size n is drawn from each of the populations. Let $X_{11},...,X_{1n}$ and $X_{21},...,X_{2n}$ be the two random samples drawn from the first and second population respectively. Let Y_1 and Y_2 be the sample means of $\{X_{1j}\}$ and $\{X_{2j}\}$, $j=1,...,n$ respectively. Then the expectation of Y_i is α_i and the variance of Y_i is $\frac{\tau^2}{n}$. Now we are interested in selecting the normal population with the smaller mean. For this purpose we select the population with smaller sample mean.The problem corresponding to higher mean was studied in [7]. So, in the current problem the first population is selected if $Y_1 \leq Y_2$ and the second population is selected otherwise. Hence we define I_1 and I_2 as $I_1 = \begin{cases} 1 \text{ if } Y_1 \leq Y_2 \\ 0 \text{ otherwise} \end{cases}$
and I_2 = 1-I_1.

Therefore, the mean of the selected population is

$$M = \alpha_1 I_1 + \alpha_2 I_2.$$

$$\text{where } Y_{min} = min(Y_1, Y_2)$$

II. Derivation of the estimators

In this section we consider the analogous estimators proposed in [7]. The natural estimator of M is Y_{min}.

The bias of Y_{min} as an estimator M is

$$\begin{aligned} B(Y_{min}) &= E[Y_{min} - M] \\ &= E[Y_{min}] - E[M] \end{aligned} \tag{1}$$

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where

$$\begin{aligned} E[Y_{min}] &= E[Y_1 I_1 + Y_2 I_2] \\ &= E[Y_1 I_1] + E[Y_2 I_2] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{y_2} I_1 y_1 f(y_1,y_2) dy_1 dy_2 \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{y_1} I_2 y_2 f(y_1,y_2) dy_1 dy_2 \\ &= A + B \end{aligned} \tag{2}$$

Now,

$A = \int_{-\infty}^{\infty}\int_{-\infty}^{y_2} y_1 \frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}\exp(\frac{-(y_1-\alpha_1)^2}{\frac{2\tau^2}{n}})$
$\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}\exp(\frac{-(y_2-\alpha_2)^2}{\frac{2\tau^2}{n}}) dy_1 dy_2$
$= \int_{-\infty}^{\infty}\int_{-\infty}^{y_2} y_1 \frac{\sqrt{n}}{\tau}\phi(\frac{y_1-\alpha_1}{\frac{\tau}{\sqrt{n}}})\frac{\sqrt{n}}{\tau}\phi(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}) dy_1 dy_2$
$= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}})[\frac{\sqrt{n}}{\tau}\int_{-\infty}^{y_2} y_1 \phi(\frac{y_1-\alpha_1}{\frac{\tau}{\sqrt{n}}}) dy_1] dy_2$
Let $\frac{y_1-\alpha_1}{\frac{\tau}{\sqrt{n}}} = u$ and $dy_1 = \frac{\tau}{\sqrt{n}} du$
$A = \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}})[\int_{-\infty}^{\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}}(u\frac{\tau}{\sqrt{n}}+\alpha_1)\phi(u)du] dy_2$
$= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}})[\frac{\tau}{\sqrt{n}}\int_{-\infty}^{\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}} u\phi(u)du$
$+\alpha_1\int_{-\infty}^{\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}}\phi(u)du] dy_2$
$= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}})$
$[\frac{\tau}{\sqrt{n}}\int_{-\infty}^{\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}} u(\frac{1}{\sqrt{2\pi}}\exp(-\frac{u^2}{2}))du + \alpha_1\Phi(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}})] dy_2$
Let $\frac{u^2}{2} = v$, $udu = dv$

$$\begin{aligned} A &= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}})[\frac{1}{\sqrt{2\pi}}\frac{\tau}{\sqrt{n}}\int_{\infty}^{\frac{1}{2}(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}})^2} \\ & \exp(-v)dv + \alpha_1\Phi(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}})] dy_2 \\ &= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}})[-\frac{1}{\sqrt{2\pi}}\frac{\tau}{\sqrt{n}} \\ & \exp(-\frac{1}{2}(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}})^2) + \alpha_1\Phi(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}})] dy_2 \\ &= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}})[-\frac{\tau}{\sqrt{n}}\phi(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}) \\ &+ \alpha_1\Phi(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}})] dy_2 \end{aligned}$$

Again let $\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}} = u$ and $dy_2 = \frac{\tau}{\sqrt{n}} du$

$$\begin{aligned} A &= -\frac{\tau}{\sqrt{n}}\int_{-\infty}^{\infty}\phi(u)\phi(u+\alpha\frac{\sqrt{n}}{\tau})du \\ & +\alpha_1\int_{-\infty}^{\infty}\phi(u)\Phi(u+\alpha\frac{\sqrt{n}}{\tau})du \end{aligned}$$

where $\alpha = \alpha_2 - \alpha_1$.

We know the following standard results

$$\int_{-\infty}^{\infty} x\Phi(a+bx)\phi(x)dx = \frac{b}{\sqrt{1+b^2}}\phi(\frac{a}{\sqrt{1+b^2}})$$

$$\int_{-\infty}^{\infty} \Phi(a+bx)\phi(x)dx = \Phi(\frac{a}{\sqrt{1+b^2}})$$

and

$$\phi(ax+b)\phi(cx+d)$$

$$= \phi(\frac{(a^2+c^2)x+ab+cd}{\sqrt{a^2+c^2}})\phi(\frac{ad-bc}{\sqrt{a^2+c^2}})$$

A can be evaluated by using the above results as

$$\begin{aligned} A &= -\frac{\tau}{\sqrt{n}}\int_{-\infty}^{\infty}\phi(\sqrt{2}u + \frac{\alpha}{\tau\sqrt{\frac{2}{n}}})\phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}})du \\ &+ \alpha_1\Phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) \end{aligned}$$

$$A = -\frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) + \alpha_1\Phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) \quad (3)$$

where $\alpha = \alpha_2 - \alpha_1$

Similarly, B can be obtained as,

$B = \int_{-\infty}^{\infty}\int_{-\infty}^{y_1} y_2 \frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}exp(\frac{-(y_1-\alpha_1)^2}{\frac{2\tau^2}{n}})$
$\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}exp(\frac{-(y_2-\alpha_2)^2}{\frac{2\tau^2}{n}})dy_1dy_2$

$$B = -\frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) + \alpha_2 - \alpha_2\Phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) \quad (4)$$

Addition of (3)and (4) will give us,

$$\begin{aligned} E(Y_{min}) &= -\frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) + \alpha_1\Phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) + \alpha_2 \\ &- \frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) - \alpha_2\Phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) \quad (5) \end{aligned}$$

The expected value of M
$E[M] = E[\alpha_1 I_1 + \alpha_2 I_2]$
$= E[\alpha_1 I_1] + E[\alpha_2 I_2]$

$$= \alpha_1 P(Y_1 < Y_2) + \alpha_2 P(Y_1 > Y_2) \quad (6)$$

Now, $P(Y_1 < Y_2) =$
$\int_{-\infty}^{\infty}\int_{-\infty}^{y_2} \frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}\exp(\frac{-(y_1-\alpha_1)^2}{2\frac{\tau^2}{n}})$
$\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}\exp(\frac{-(y_2-\alpha_2)^2}{2\frac{\tau^2}{n}})dy_1dy_2$
$= \frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\exp(\frac{-(y_2-\alpha_2)^2}{2\frac{\tau^2}{n}})$
$[\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}\int_{-\infty}^{y_2}\exp(\frac{-(y_1-\alpha_1)^2}{2\frac{\tau^2}{n}})dy_1]dy_2$
$= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}})[\frac{\sqrt{n}}{\tau}\int_{-\infty}^{y_2}\phi(\frac{y_1-\alpha_1}{\frac{\tau}{\sqrt{n}}})dy_1]dy_2$

Let $u = \frac{y_1-\alpha_1}{\frac{\tau}{\sqrt{n}}}$, and $dy_1 = \frac{\tau}{\sqrt{n}}du$

$$\begin{aligned} P(Y_1 < Y_2) &= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}})[\int_{-\infty}^{\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}}\phi(u)du]dy_2 \\ &= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}})\Phi(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}})dy_2 \end{aligned}$$

Again let $u = \frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}$, and $dy_2 = \frac{\tau}{\sqrt{n}}du$

$$\begin{aligned} P(Y_1 < Y_2) &= \int_{-\infty}^{\infty}\phi(u)\Phi(u + \frac{\alpha\sqrt{n}}{\tau})du \\ &= \Phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) \text{ , where } \alpha = \alpha_2 - \alpha_1 \quad (7) \end{aligned}$$

Similarly

$$\begin{aligned} P(Y_2 < Y_1) &= \int_{-\infty}^{\infty}\int_{-\infty}^{y_1}\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}\exp(\frac{-(y_1-\alpha_1)^2}{2\frac{\tau^2}{n}}) \\ &\quad \frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}\exp(\frac{-(y_2-\alpha_2)^2}{2\frac{\tau^2}{n}})dy_1dy_2 \\ &= 1 - \Phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) \quad (8) \end{aligned}$$

Substituting the values of (7) and (8) in (6),we get the expected value of M as,

$$E[M] = \alpha_1\Phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) + \alpha_2 - \alpha_2\Phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) \quad (9)$$

Using the values of (5) and (9), from (1) we get the bias of Y_{min} as,

$$\begin{aligned} B(Y_{min}) &= -\frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) + \alpha_1\Phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) \\ &- \frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) + \alpha_2 - \alpha_2\Phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) \\ &- [\alpha_1\Phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) + \alpha_2 - \alpha_2\Phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}})] \\ &= -\frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) - \frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) \\ &= -2\frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}) \\ &= -\sqrt{2}\frac{\tau}{\sqrt{n}}\phi(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}). \end{aligned}$$

Let $\frac{2\tau^2}{n} = \sigma^2$, Then we get $B(Y_{min}) = -\sigma\phi(\frac{\alpha}{\sigma})$.

Now, we get the estimator $\hat{M}_\lambda$ of the random variable M as

$\hat{M}_\lambda = Y_{min} + \lambda\sigma\phi(\frac{Y}{\sigma})$,where $Y = Y_2 - Y_1$ and $\lambda \geq 0$ is an arbitrary constant.

Note: Y $\sim N(\alpha, \sigma^2)$ and $E[Y_2 - Y_1] = \alpha_2 - \alpha_1 = \alpha, Var(Y) = \frac{2\tau^2}{n} = \sigma^2$

Now , we talk about the estimator $t_c = Y_2 - Y\Phi(\frac{cY}{\sigma}) + c\sigma\phi(\frac{cY}{\sigma})$, where $c > 0$ is an arbitrary constant and $\Phi(u)$ is the standard normal cdf. Now,

$E[t_c] = E[Y_2 - Y\Phi(\frac{cY}{\sigma}) + c\sigma\phi(\frac{cY}{\sigma})]$

$E[t_c] = E[Y_2] - E[Y\Phi(\frac{cY}{\sigma})] + E[c\sigma\phi(\frac{cY}{\sigma})]$

and

$$E[Y_2] = \alpha_2, \tag{10}$$

$E[Y.\Phi(\frac{cY}{\sigma})] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp(-\frac{1}{2}(\frac{y-\alpha}{\sigma})^2)y\Phi(\frac{cy}{\sigma})dy$

$= \frac{1}{\sigma}\int_{-\infty}^{\infty}\phi(\frac{y-\alpha}{\sigma})y\Phi(\frac{cy}{\sigma})dy$

$\Longrightarrow$ Let $\frac{y-\alpha}{\sigma} = u$, then $dy = \sigma du$

$E[Y.\Phi(\frac{cY}{\sigma})] = \int_{-\infty}^{\infty}(\sigma u + \alpha)\phi(u)\Phi(cu + \frac{c\alpha}{\sigma})du$

$= \sigma\int_{-\infty}^{\infty}u\phi(u)\Phi(cu + \frac{c\alpha}{\sigma})du + \alpha\int_{-\infty}^{\infty}\phi(u)\Phi(cu + \frac{c\alpha}{\sigma})$

$$= \frac{\sigma c}{\sqrt{1+c^2}}\phi(\frac{c\alpha}{\sigma\sqrt{1+c^2}}) + \alpha\Phi(\frac{c\alpha}{\sigma\sqrt{1+c^2}}) \tag{11}$$

and

$E[c\sigma\phi(\frac{cY}{\sigma})] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp(-\frac{1}{2}(\frac{y-\alpha}{\sigma})^2)c\sigma\phi(\frac{cy}{\sigma})dy$

$= c\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-\frac{1}{2}(\frac{y-\alpha}{\sigma})^2)\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}(\frac{cy}{\sigma})^2)dy$

$= \frac{c}{2\pi}\int_{-\infty}^{\infty}\exp(-\frac{1}{2\sigma^2}(y^2(1+c^2) + \alpha^2 - 2y\alpha))dy$

$= \frac{c}{2\pi}\int_{-\infty}^{\infty}\exp[-\frac{1}{2\sigma^2}\{(y\sqrt{1+c^2} - \frac{\alpha}{\sqrt{1+c^2}})^2 + \frac{\alpha^2c^2}{1+c^2}\}]dy$

$= \frac{c}{2\pi}\exp[-\frac{1}{2\sigma^2}(\frac{\alpha^2c^2}{1+c^2})]\int_{-\infty}^{\infty}\exp[-\frac{1}{2\sigma^2}\{(y\sqrt{1+c^2} - \frac{\alpha}{\sqrt{1+c^2}})^2\}]dy$

Let $y\sqrt{1+c^2} = u$, and $dy = \frac{1}{\sqrt{1+c^2}}du$

$E[c\sigma\phi(\frac{cY}{\sigma})] = \frac{c}{2\pi\sqrt{1+c^2}}\exp[-\frac{1}{2\sigma^2}(\frac{\alpha^2c^2}{1+c^2})]$
$\int_{-\infty}^{\infty}\exp[-\frac{1}{2\sigma^2}(u - \frac{\alpha}{\sqrt{1+c^2}})^2]du$

$= \frac{c\sigma}{\sqrt{1+c^2}}\phi(\frac{\alpha c}{\sigma\sqrt{1+c^2}})$

$[\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp[-\frac{1}{2\sigma^2}(u - \frac{\alpha}{\sqrt{1+c^2}})^2]du]$

$$= \frac{c\sigma}{\sqrt{1+c^2}}\phi(\frac{\alpha c}{\sigma\sqrt{1+c^2}}) \tag{12}$$

From (10), (11) and (12), we get expected value of t_c as $E[t_c] = \alpha_2 - \alpha\Phi(\frac{c\alpha}{\sigma\sqrt{1+c^2}})$.

So, we can say that estimator t_c is an unbiased estimator of $\alpha_2 - \alpha\Phi(\frac{c\alpha}{\sigma\sqrt{1+c^2}})$, which approaches $E[M] = \alpha_2 - \alpha\Phi(\frac{\alpha}{\sigma})$ for large c. Hence, if c is large then the bias of t_c as an estimator of M is controlled.

Here, we can consider an estimator T of $E[M]$ given by

$$T = Y_2 - Y\Phi(\frac{Y}{\sigma}) \tag{13}$$

and propose to use it to estimate M. The estimator T, actually, is a maximum likelihood estimator of $E[M]$ and its bias will be the same whether we can use it as an estimator of $E[M]$ or M. A more general estimator of this type is given by

$$T_\lambda = T - \lambda[Y\{\phi(\frac{Y}{\sigma}) - \Phi(\frac{Y}{\sqrt{2}\sigma})\} - \frac{\sigma}{\sqrt{2}}\phi(\frac{Y}{\sqrt{2}\sigma})] \tag{14}$$

where $\lambda \geq 0$ is an arbitrary constant. The estimator of T_λ is the same as that of $\hat{M}_\lambda$ and is obtained on subtracting a λ multiple of the estimated bias of T from itself.

Another estimator that we intend to investigate here is given by

$$H_c = \begin{cases} \frac{Y_1+Y_2}{2}\ if & |Y_1 - Y_2| < c\sigma \\ min(Y_1, Y_2), & |Y_1 - Y_2| \geq c\sigma. \end{cases}$$

where $c \geq 0$ is an arbitrary constant. Note that for c=0 we get Y_{min} which is the same as $\hat{M}_\lambda$ for $\lambda = 0$. Then H_c is sometimes called an hybrid estimator. Finally, it may be mentioned that the Bayes estimator of M, for squared error loss with uniform prior distribution on α_1 and α_2, turns out to be $min(Y_1, Y_2)$ which is the same as $\hat{M}_\lambda$ for $\lambda = 0$.

III. Improving upon $\hat{M}_\lambda$

Consider the group $G = g(x)_c = x + c, c \in R$ of affine transformations. Under this transformation :$X_{ij} \to X_{ij} + c, Y_i \to Y_i + c$ and $M \to M + c$. We take the the squared error loss $L(d, M) = (d - M)^2$then estimation problem is invariant.

Theorem 1: Let us consider the class of estimators of the form $\hat{M}_\lambda = Y_{min} + \lambda\sigma\phi(\frac{Y}{\sigma})$. Further, define the estimator $\hat{M}_{\lambda*}$ by

$$\begin{aligned} \hat{M}_{\lambda*} &= \hat{M}_\lambda, \text{if } \lambda \leq \gamma* \\ &= \hat{M}_{\gamma*}, \text{if } \lambda > \gamma*, \end{aligned}$$

where $\gamma* = \frac{\sqrt{3}}{2}$. Then $\hat{M}_{\lambda*}$ improves $\hat{M}_\lambda$ with respect to the squared error loss if $P(\lambda > \gamma*) > 0$ for some $\eta = \{\alpha_1, \alpha_2\}$.

Proof : The mean squared error risk of $\hat{M}_\lambda$ is given by

$$\begin{aligned} \frac{MSE[\hat{M}_\lambda]}{\sigma^2} &= \frac{E[(Y_{min} + \lambda\sigma\phi(\frac{Y}{\sigma}) - M)^2]}{\sigma^2} \\ &= [\frac{1}{2} + \frac{\lambda^2}{\sqrt{6\pi}}\phi(\frac{\gamma\sqrt{2}}{\sqrt{3}}) - \lambda\phi(\frac{\gamma}{\sqrt{2}}) \\ &\quad \{\phi(\frac{\gamma}{\sqrt{2}}) - \frac{3\gamma}{\sqrt{2}}(\Phi(\frac{\gamma}{\sqrt{2}}) - \frac{1}{2})\}] \\ &= \psi(\lambda) \end{aligned}$$

where $\gamma = \frac{\alpha}{\sigma}$.

Now $\psi'(\lambda) = [\frac{2\lambda}{\sqrt{6\pi}}\phi(\frac{\gamma\sqrt{2}}{\sqrt{3}}) - \phi(\frac{\gamma}{\sqrt{2}})\{\phi(\frac{\gamma}{\sqrt{2}}) - \frac{3\gamma}{\sqrt{2}}(\Phi(\frac{\gamma}{\sqrt{2}}) - \frac{1}{2})\}] = 0$ gives

$$\begin{aligned} \lambda &= \frac{\phi(\frac{\gamma}{\sqrt{2}})\{\phi(\frac{\gamma}{\sqrt{2}}) - \frac{3\gamma}{\sqrt{2}}(\Phi(\frac{\gamma}{\sqrt{2}}) - \frac{1}{2})\}}{\frac{2}{\sqrt{6\pi}}\phi(\frac{\gamma\sqrt{2}}{\sqrt{3}})} \\ &= \frac{\sqrt{3}}{2}e^{-\gamma^2 + \frac{2\gamma^2}{3}}\{1 - \frac{\frac{3\gamma}{\sqrt{2}}(\Phi(\frac{\gamma}{\sqrt{2}}) - \frac{1}{2})}{\phi(\frac{\gamma}{\sqrt{2}})}\} \\ &= f(\gamma) \end{aligned}$$

which is the minima of $\psi(\lambda)$. We know $\Phi(0) = \frac{1}{2}$. When γ is positive, $\Phi(\frac{\gamma}{\sqrt{2}})$ is greater than $\frac{1}{2}$, that is $\Phi(\frac{\gamma}{\sqrt{2}}) - \frac{1}{2} > 0$. Thus, $\frac{3\gamma}{\sqrt{2}}\{\Phi(\frac{\gamma}{\sqrt{2}}) - \frac{1}{2}\}$ is positive. If γ is negative, $\frac{3\gamma}{\sqrt{2}}\{\Phi(\frac{\gamma}{\sqrt{2}}) - \frac{1}{2}\}$ is positive. Therefore, $f(\gamma)$ has the maximum value at $\gamma = 0$ and we obtain

$$f(\gamma) \leq \frac{\sqrt{3}}{2}e^{-\gamma^2/3} \leq \frac{\sqrt{3}}{2}.$$

Thus, $\text{Sup}_{\gamma \in R}f(\gamma) = \frac{\sqrt{3}}{2} = \gamma*$. According to Brewster-Zidek technique [4] we can say that if $\lambda > \gamma*$ then $\hat{M}_\lambda$ can be improved by $\hat{M}_{\gamma*}$ as in that case the risk of $\hat{M}_{\gamma*}$ will be less than that of $\hat{M}_\lambda$ and this completes the proof of the theorem.

IV. Comparison of different estimators based on bias and MSE

The biases and mean squared error risks are calculated for the above estimators by the method of Monte-Carlo simulation for different values of $|\gamma|$, where $\gamma = \frac{\alpha_1 - \alpha_2}{\sigma}$. The graphs of these biases and MSE risks are drawn for the above estimators. In these graphs the X axis represnts $|\gamma|$ and Y axis represents the bias or MSE risks. The bias performances of the above estimators are given in Table I, II, III and Table IV. From, these four tables we observe that as $|\gamma|$ increases the value of absolute biases of all the estimators decrease excluding the estimator T_λ. The MSE risks performances are given in Table V, VI, VII and Table VIII. From these four tables it is observed that the risk performances of the estimators become better as the value of $|\gamma|$ increases. It is also observed that in most of the cases the risk of $\hat{M}_\lambda$, for $\lambda = \sqrt{3}/2$ is lower than the other estimators. From the graphs also it is clear that as the value of $|\gamma|$ tends to ∞ the bias and risk values tend to 0. From the Fig. 6 it is observed that the bias as well as risk performance of the estimator T is better than T_λ . In fact, as the value of λ increases the bias and risk values increase. We present below the graphs of biases of different estimators and the graphs of Mean squared error risks of different estimators.

Table I: Bias of Estimator $\hat{M}_\lambda$

$\|\gamma\|$	$\lambda = 0$	$\lambda = \sqrt{3}/2$	$\lambda = 4 - 2\sqrt{2}$	$\lambda = \sqrt{2}$
0.0	-0.5077	-0.2048	-0.0979	-0.0130
0.5	-0.3369	-0.1048	-0.0229	0.0421
1.0	-0.2008	-0.0499	0.0033	0.0455
1.5	-0.0991	-0.00069	0.0340	0.0616
2.0	-0.0251	0.0285	0.0474	0.0625
2.5	-0.0058	0.0148	0.0221	0.0279
3.0	-0.0014	0.0035	0.0053	0.0067
3.5	-0.00056	0.00002	0.00022	0.00038
4.0	-0.0001	-0.00001	0.00002	0.00004
4.5	0.00000	0.00001	0.00001	0.00001
5.0	-0.00002	-0.00002	-0.00002	-0.00002
6.0	0.00001	0.00001	0.00001	0.00001

Table II: Bias of Estimator T_λ

$\|\gamma\|$	T	$\lambda = \sqrt{3}/2$	$\lambda = 1$	$\lambda = 1.25$	$\lambda = 1.5$
0.0	-0.3622	-0.1710	-0.1414	-0.0862	-0.0310
0.5	-0.2230	-0.0783	-0.0559	-0.0142	0.0276
1.0	-0.1190	-0.0313	-0.0177	0.0076	0.0329
1.5	-0.0391	0.0119	0.0198	0.0346	0.0493
2.0	0.0118	0.0333	0.0366	0.0428	0.0490
2.5	0.0098	0.0147	0.0154	0.0168	0.0182
3.0	0.0027	0.0024	0.0024	0.0023	0.0022
3.5	-0.00005	-0.00029	-0.00033	-0.0004	-0.00047
4.0	-0.00002	-0.00012	-0.00013	-0.00016	-0.00019
4.5	0.00001	-0.00001	-0.00001	-0.00002	-0.00002
5.0	-0.00002	-0.00003	-0.00003	-0.00003	-0.00003
6.0	0.00001	0.00001	0.00001	0.00001	0.00001

Table III: Bias of Estimator t_c

$\|\gamma\|$	$c = 1/3$	$c = 1/2$	$c = 1$	$c = \sqrt{3}$
0.0	-0.0059	-0.0078	-0.0124	-0.0163
0.5	0.0762	0.0656	0.0451	0.0336
1.0	0.1676	0.1274	0.0552	0.0183
1.5	0.2535	0.1844	0.0746	0.0255
2.0	0.2967	0.2026	0.0737	0.0295
2.5	0.2096	0.1269	0.0337	0.0109
3.0	0.1016	0.0524	0.0084	0.0014
3.5	0.0229	0.0097	0.00062	-0.00034
4.0	0.0081	0.0028	0.00008	-0.00008
4.5	0.0017	0.00049	0.00002	0.00000
5.0	0.00054	0.0001	-0.00002	-0.00002
6.0	0.0001	0.00002	0.00001	0.00001

Table IV: Bias of Estimator H_c

$\|\gamma\|$	$c = 0.5$	$c = 1$	$c = \sqrt{2}$	$c = 2$
0.0	-0.4500	-0.3103	-0.1897	-0.0743
0.5	-0.2947	-0.1876	-0.0926	0.0203
1.0	-0.1762	-0.1077	-0.0248	0.1056
1.5	-0.0862	-0.0419	0.0286	0.1565
2.0	-0.0207	0.0014	0.0452	0.1581
2.5	-0.0045	0.0020	0.0192	0.0772
3.0	-0.0012	0.00001	0.0038	0.0207
3.5	-0.00055	-0.00047	-0.00004	0.0020
4.0	-0.0001	-0.00009	-0.00004	0.00027
4.5	0.00000	0.00000	0.00000	0.00002
5.0	-0.00002	-0.00002	-0.00002	-0.00002
6.0	0.00001	0.00001	0.00001	0.00001

Table V: Mean Squared Error of Estimator $\hat{M}_\lambda$

$\|\gamma\|$	T	$\lambda = 0$	$\lambda = \sqrt{3}/2$	$\lambda = 4 - 2\sqrt{2}$	$\lambda = \sqrt{2}$
0.0	0.6572	0.7767	0.6659	0.6776	0.7058
0.5	0.4331	0.4866	0.4479	0.4650	0.4901
1.0	0.3136	0.3239	0.3301	0.3472	0.3663
1.5	0.2594	0.2485	0.2759	0.2933	0.3100
2.0	0.1971	0.1804	0.2094	0.2228	0.2345
2.5	0.0858	0.0791	0.0888	0.0929	0.0964
3.0	0.0210	0.0198	0.0213	0.0219	0.0224
3.5	0.0013	0.0013	0.0013	0.0013	0.0014
4.0	0.0002	0.00019	0.0002	0.0002	0.0002
4.5	0.00001	0.00001	0.00001	0.00001	0.00001
5.0	0.00000	0.00000	0.00000	0.00000	0.00000
6.0	0.00000	0.00000	0.00000	0.00000	0.00000

Table VI: Mean Squared Error of Estimator T_λ

$\|\gamma\|$	T	$\lambda = \sqrt{3}/2$	$\lambda = 1$	$\lambda = 1.25$	$\lambda = 1.5$
0.0	0.6572	0.6432	0.6494	0.6671	0.6926
0.5	0.4331	0.4428	0.4494	0.4653	0.4859
1.0	0.3136	0.3364	0.3424	0.3551	0.3701
1.5	0.2594	0.2870	0.2925	0.3035	0.3156
2.0	0.1971	0.2187	0.2224	0.2298	0.2375
2.5	0.0858	0.0916	0.0926	0.0945	0.0965
3.0	0.0210	0.0216	0.0217	0.0219	0.0221
3.5	0.0013	0.0013	0.0013	0.0013	0.0013
4.0	0.0002	0.0002	0.0002	0.0002	0.00019
4.5	0.00001	0.00001	0.00001	0.00001	0.00001
5.0	0.00000	0.00000	0.00000	0.00000	0.00000
6.0	0.00000	0.00000	0.00000	0.00000	0.00000

Table VII: Mean Squared Error of Estimator t_c

$\lvert\gamma\rvert$	$c = 1/3$	$c = 1/2$	$c = 1$	$c = \sqrt{3}$
0.0	0.4249	0.4765	0.6523	0.8718
0.5	0.3326	0.3627	0.4640	0.5818
1.0	0.2932	0.3091	0.3597	0.4012
1.5	0.2950	0.2985	0.3132	0.3139
2.0	0.2665	0.2538	0.2403	0.2250
2.5	0.1291	0.1154	0.0992	0.0902
3.0	0.0334	0.0283	0.0229	0.0211
3.5	0.0021	0.0017	0.0014	0.0013
4.0	0.00031	0.00025	0.0002	0.00019
4.5	0.00002	0.00001	0.00001	0.00001
5.0	0.00000	0.00000	0.00000	0.00000
6.0	0.00000	0.00000	0.00000	0.00000

Table VIII: Mean Squared Error of Estimator H_c

$\lvert\gamma\rvert$	$c = 0.5$	$c = 1$	$c = \sqrt{2}$	$c = 2$
0.0	0.7629	0.6950	0.6082	0.4810
0.5	0.4826	0.4643	0.4372	0.3843
1.0	0.3258	0.3384	0.3557	0.3671
1.5	0.2516	0.2753	0.3185	0.3742
2.0	0.1832	0.2037	0.2398	0.3186
2.5	0.0798	0.0848	0.0983	0.1364
3.0	0.0199	0.0206	0.0224	0.0295
3.5	0.0013	0.0013	0.0013	0.0016
4.0	0.00019	0.00019	0.0002	0.00022
4.5	0.00001	0.00001	0.00001	0.00001
5.0	0.00000	0.00000	0.00000	0.00000
6.0	0.00000	0.00000	0.00000	0.00000

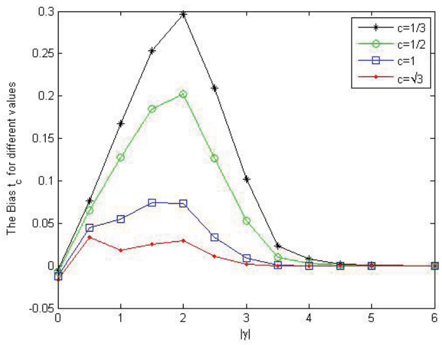

Fig. 3: Bias of T_c

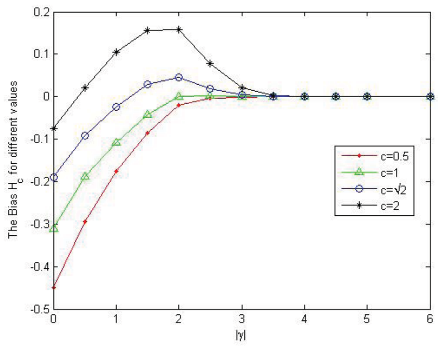

Fig. 4: Bias of H_c

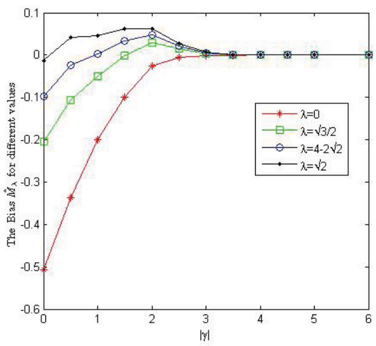

Fig. 1: Bias of $\hat{M}_\lambda$

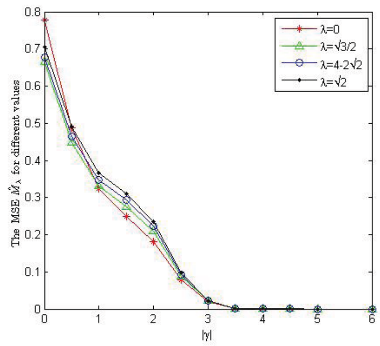

Fig. 5: MSE of $\hat{M}_\lambda$

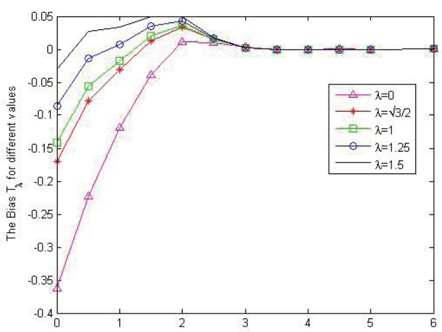

Fig. 2: Bias of T_λ

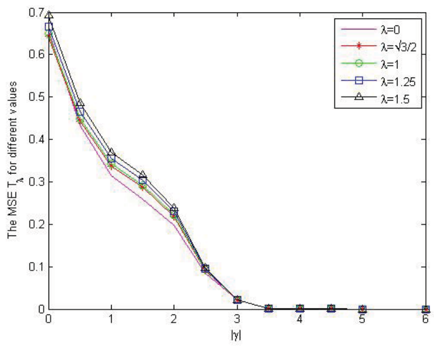

Fig. 6: MSE of T_λ

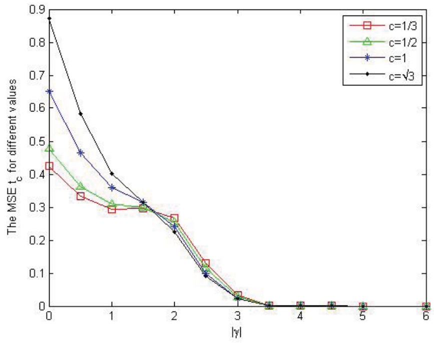

Fig. 7: MSE of T_c

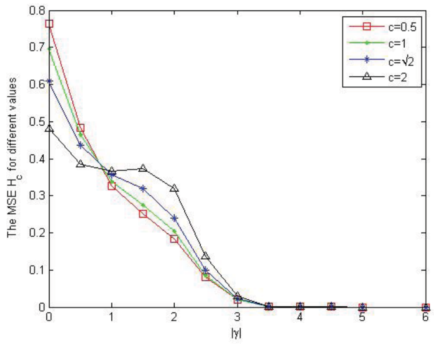

Fig. 8: MSE of H_c

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