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Conservativeness of Probabilistic Constrained Optimal Control Method for Unknown Probability Distribution

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Abstract—In recent decades, probabilistic constrained optimal control problems have attracted much attention in many research fields. Although probabilistic constraints are generally intractable in an optimization problem, several tractable methods haven been proposed to handle probabilistic constraints. In most methods, probabilistic constraints are reduced to deterministic constraints that are tractable in an optimization problem. However, there is a gap between the transformed deterministic constraints in case of known and unknown probability distribution. This paper examines the conservativeness of probabilistic constrained optimization method for unknown probability distribution. The objective of this paper is to provide a quantitative assessment of the conservatism for tractable constraints in probabilistic constrained optimization with unknown probability distribution.

Keywords—Optimal control, stochastic systems, discrete-time systems, probabilistic constraints.

I. Introduction

So far, optimal control methods subject to deterministic constraints have been well developed with a wide range of applications [1]–[14]. On the other hand, probabilistic constrained optimal control problems have attracted much attention in many research fields. Probabilistic constraints are addressed by stochastic optimal control problems where expected values of performance indices, probabilistic constraints and convergence in probability are considered by exploiting the statistical information on the system parameters [15]–[27].

However, probabilistic constraints are generally intractable in an optimization problem. In recent decades, considerable attention has been paid to this difficulty related to the stochastic optimal control problem. Thus, several tractable methods haven been proposed to handle probabilistic constraints.

In [15], [16], a second-order cone approximation method was proposed based on results from robust optimization to solve the stochastic linear-quadratic control problem. In [17], a stochastic optimal control method was proposed while considering the probabilistic polytopic sets instead of the deterministic bounds of uncertain disturbances. Also, the concept of probabilistic invariance was considered for the case of multiplicative uncertainty [18] and the cases of both additive and multiplicative uncertainty [19]. In addition, an alternate method for the convex approximation of probabilistic constraints with polytopic constraint functions was proposed in [20]. In [21], a decomposition method of probabilistic constraints was proposed to obtain a lower bound to the convex optimization problem. Although the aforementioned papers [15]–[21] have achieved tremendous progress in dealing with probabilistic constraints of the stochastic optimization problems, there are several restrictions imposed on the probability distributions of stochastic disturbances such as the normal (Gaussian) distribution, known distribution, finite support, and time invariance.

On the other hand, the method provided here enables us to address unknown arbitrary probability distributions including non-Gaussian, infinitely supported, and time-variant distributions. The sampling methods using scenario approximation [22], [23] and Bernstein approximation [24] are alternative methods for dealing with arbitrary probability distributions. However, the sampling methods usually require heavy computational load. The objective in this study is to provide a stochastic optimal control method for successfully dealing with probabilistic constraints with less computational load. In [25], an ellipsoid approximation method based on Chebyshev's inequality was proposed to handle soft constraints. However, the calculation of the maximum volume inscribed ellipsoid is also computationally demanding. In [26]–[28], a direct component-wise comparison method using the multi-dimensional Chebyshev's inequality was proposed to address probabilistic constraints without using ellipsoid approximation. However, the tractable constraints in [26]–[28] are restricted to the component-wise state constraints. That means affine state constraints cannot be addressed by the method in [26]–[28].

In [29], the Cantelli's inequality, which is a similar concentration inequality to the Chebyshev's inequality, was used to propose a solution method to the stochastic optimal control problems. However, the control law in [29] is restricted to linear state feedback and its feedback gain is computed by solving a convex optimization problem. In contrast, we present here another framework for solving the stochastic optimal control problems in which the control variables are directly optimized by solving a convex optimization problem.

In most methods, probabilistic constraints are reduced to deterministic constraints that are tractable in an optimization problem. However, there is a gap between the transformed deterministic constraints in case of known and unknown probability distribution. In this paper, we

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examine the conservativeness of probabilistic constrained optimization method for unknown probability distribution. The objective of this paper is to provide a quantitative assessment of the conservatism for tractable constraints in probabilistic constrained optimization with unknown probability distribution.

II. Notation and System Model

Let $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers, respectively. Let $\mathbb{R}_+$ denote the set of non-negative real numbers. For matrix A, the transpose and trace of A are denoted by A' and $\mathrm{tr}A$, respectively. Let diag.$\{\cdots\}$ denote a diagonal block matrix. For matrices $A = \{a_{i,j}\}$ and $B = \{b_{i,j}\}$, let the inequalities between A and B, such as $A > B$ and $A \geq B$, indicate that they are component-wise satisfied, i.e., $a_{i,j} > b_{i,j}$ and $a_{i,j} \geq b_{i,j}$ is true for all i and j, respectively. Similarly, let each notation for absolute value $|A|$, square root $\sqrt{A}$, and multiplication $A \otimes B$ indicate that it is true component-wise, i.e., $|A| = \{|a_{i,j}|\}$, $\sqrt{A} = \{\sqrt{a_{i,j}}\}$, and $A \otimes B = \{a_{i,j} \times b_{i,j}\}$ for all i and j.

Let the triple $(\Omega, \mathcal{F}, \mathcal{P})$ denote a probability space where $\Omega \subseteq \mathbb{R}$ is the sampling space, $\mathcal{F}$ is the σ-algebra, and $\mathcal{P}$ is the probability measure [30]. Ω is non-empty and is not necessarily finite. $\mathcal{P}(E)$ denotes the probability that event E occurs. If $\mathcal{P}(E) = 1$, E almost surely occurs. For random variable $z : \Omega \to \mathbb{R}$ defined by $(\Omega, \mathcal{F}, \mathcal{P})$, let the expected value and variance of z be denoted by $\mathcal{E}(z)$ and $\mathcal{V}(z)$, respectively. For a random vector $z = [z_1, \cdots, z_n]'$, where each of its components is a random variable $z_i : \Omega \to \mathbb{R}$ $(i = 1, \cdots, n)$, which is defined on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$, we also adopt the same notations $\mathcal{E}(z)$ and $\mathcal{V}(z)$ to denote $c(z) = [\mathcal{E}(z_1), \cdots, \mathcal{E}(z_n)]'$ and $\mathcal{V}(z) = [\mathcal{V}(z_1), \cdots, \mathcal{V}(z_n)]'$ for notational simplicity. Furthermore, covariance matrix $\mathcal{C}_v(z)$ is defined by $\mathcal{C}_v(z) := \mathcal{E}[\{z - \mathcal{E}(z)\}\{z - \mathcal{E}(z)\}']$.

Throughout this paper, we consider the following linear discrete-time system with stochastic disturbances:

$$x(t+1) = Ax(t) + Bu(t) + Cw(t), \tag{1}$$

where $t \in \mathbb{N}$ is the time step, $x(t) : \mathbb{N} \to \mathbb{R}^n$ is the state, $u(t) : \mathbb{N} \to \mathbb{R}^m$ is the control input, and $w(t) : \mathbb{N} \to \mathbb{R}^\ell$ is the unknown stochastic disturbance. More precisely, for each component $w_i : \mathbb{N} \times \Omega \to \mathbb{R}$ of w, the random sequence $\{w_i(t) : t \in \mathbb{N}\}$ is a collection of random variables in the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with a filtration $\{\mathcal{F}_t : t \in \mathbb{N}\}$ [30]. The system coefficients $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{n\times \ell}$ are all known as constant matrices. The pair (A, B) is assumed to be controllable. We also assume that the initial state $x(0)$ is given and that all components of state $x(t)$ are deterministically observable. Thus, we assume that $\mathcal{E}(x(t)) = x(t)$ and $\mathcal{V}(x(t)) = 0$ at present time t.

Next, we introduce some assumptions about the properties of the stochastic disturbances.

Assumption 1: $w_i(t)$ and $w_j(t)$ are independent of each other for all $i \neq j$ and $t \in \mathbb{N}$. Also, $w_i(t)$ and $w_j(k)$ are independent of each other for all $t \neq k$ and $j \in \{1, \cdots, \ell\}$.

In fact, most previous studies typically assumed that random variables are mutually independent as well as Assumption 1. The case where random variables are mutually correlated requires more complicated analysis than the one provided here because $\mathcal{C}_v(w)$ cannot be neglected.

Assumption 2: $\mathcal{E}(w(t))$ and $\mathcal{V}(w(t))$ are assumed to be known for each time t.

Note that the probability distributions of random variables w_i are not necessarily assumed to be known. However, the probability distributions were assumed to be known in previous studies [15]-[21] to transform the soft constraints into hard constraints. In the present study, the assumption related to known probability distributions is relaxed to include arbitrary unknown probability distributions.

III. Preliminaries

The inequality shown below is known as the Chebyshev's inequality.

Lemma 1 ([31]): For any random variable x and positive constant $\kappa \geq 1$, the following inequality holds:

$$\mathcal{P}\left(|x - \mathcal{E}(x)| \geq \kappa\sqrt{\mathcal{V}(x)}\right) \leq \frac{1}{\kappa^2}. \tag{2}$$

IV. Problem Statement

Hereafter, we formulate the stochastic optimal control problem of a system (1). The control input at each time t is determined to minimize the performance index given by

$$J := \phi[x(t+N)] + \sum_{k=t}^{t+N-1} L[x(k), u(k)]. \tag{3a}$$

Here, $N \in \mathbb{N}$ denotes the length of the prediction horizon. ϕ and L are defined by

$$\phi := \mathcal{E}[x(t+N)'Px(t+N)], \tag{3b}$$

$$L := \mathcal{E}[x(k)'Qx(k)] + u(k)'Ru(k), \tag{3c}$$

where P, Q, and R are positive definite constant matrices. $\phi \in \mathbb{R}_+$ is the terminal cost function, and $L \in \mathbb{R}_+$ is the stage cost function over the prediction horizon.

Let $p(t) = [p_1(t), \cdots, p_n(t)]'$: $\mathbb{N} \to [0\ 1]^n$ denote the probability in vector form, which means that each component $p_i(t)$ belongs to $[0\ 1]$ for each time t.

For notational convenience, let $\mathbf{X} \in \mathbb{R}^{nN}$, $\mathbf{U} \in \mathbb{R}^{mN}$, $\mathbf{W} \in \mathbb{R}^{\ell N}$, $\mathbf{A} \in \mathbb{R}^{nN\times n}$, $\mathbf{B} \in \mathbb{R}^{nN\times mN}$, $\mathbf{C} \in \mathbb{R}^{nN\times \ell N}$, $\mathbf{Q} \in \mathbb{R}^{nN\times nN}$, $\mathbf{R} \in \mathbb{R}^{mN\times mN}$, and $\mathbf{p} \in \mathbb{R}^{nN}$, be defined by

$$\mathbf{X}(t) := \begin{bmatrix} x(t+1) \\ \vdots \\ x(t+N) \end{bmatrix}, \quad \mathbf{U}(t) := \begin{bmatrix} u(t) \\ \vdots \\ u(t+N-1) \end{bmatrix},$$

$$\mathbf{W}(t) := \begin{bmatrix} w(t) \\ \vdots \\ w(t+N-1) \end{bmatrix}, \quad \mathbf{A} := \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix},$$

$$\mathbf{B} := \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix},$$

$$\mathbf{C} := \begin{bmatrix} C & 0 & \cdots & 0 \\ AC & C & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}C & A^{N-2}C & \cdots & C \end{bmatrix},$$

$$\mathbf{Q} := \begin{bmatrix} Q & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & Q & 0 \\ 0 & \cdots & 0 & P \end{bmatrix}, \quad \mathbf{R} := \begin{bmatrix} R & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & R \end{bmatrix},$$

$$\mathbf{p}(t) = \begin{bmatrix} p(t+1) \\ \vdots \\ p(t+N) \end{bmatrix}.$$

Using the aforementioned notation, we rewrite the performance index in (3) as follows:

$$J[x(t), \mathbf{X}(t), \mathbf{U}(t)] = \mathcal{E}[x(t)'Qx(t)] + \mathcal{E}[\mathbf{X}(t)'\mathbf{Q}\mathbf{X}(t)] + \mathbf{U}(t)'\mathbf{R}\mathbf{U}(t), \quad (4)$$

In addition, (1) over the prediction horizon can be rewritten as

$$\mathbf{X}(t) = \mathbf{A}x(t) + \mathbf{B}\mathbf{U}(t) + \mathbf{C}\mathbf{W}(t). \quad (5)$$

We consider the probabilistic component-wise state constraints and unbounded stochastic disturbances with unknown probability distributions. Let $\underline{x}(t)$ and $\overline{x}(t) : \mathbb{N} \to \mathbb{R}^n$ denote the lower and upper bounds of $x(t)$, respectively. Here, we impose the following probabilistic constraint on the optimization problem: for $k = t+1, \cdots, t+N$ and $i = 1, \cdots, n$,

$$\mathcal{P}\left(\underline{x}_i(k) < x_i(k) < \overline{x}_i(k)\right) \geq p_i(k), \quad (6)$$

where $\underline{x}_i(k)$, $\overline{x}_i(k) \in \mathbb{R}$, and $p_i(k) \in [0\ 1]$ for $k = t + 1, \cdots, t+N$ are given constant sequences and their subscript indicates the ith element of the vector. Condition (6) indicates that state x_i over the prediction horizon must remain within the bound $[\underline{x}_i\ \overline{x}_i]$ at least with probability p_i.

Let $\underline{\mathbf{X}} \in \mathbb{R}^{nN}$ and $\overline{\mathbf{X}} \in \mathbb{R}^{nN}$ be defined by

$$\underline{\mathbf{X}}(t) := \begin{bmatrix} \underline{x}(t+1) \\ \vdots \\ \underline{x}(t+N) \end{bmatrix}, \quad \overline{\mathbf{X}}(t) := \begin{bmatrix} \overline{x}(t+1) \\ \vdots \\ \overline{x}(t+N) \end{bmatrix}.$$

Using the above notation, probabilistic constraint (6) is rewritten in vector form as

$$\mathcal{P}\left(\underline{\mathbf{X}}(t) < \mathbf{X}(t) < \overline{\mathbf{X}}(t)\right) \geq \mathbf{p}(t). \quad (7)$$

More precisely, by using the components $\underline{\mathbf{X}}_i, \mathbf{X}_i, \overline{\mathbf{X}}_i \in \mathbb{R}$, and $\mathbf{p}_i \in [0\ 1]$ of the vectors, condition (7) can be described as

$$\bigwedge_{i=1}^{nN} \left\{\mathcal{P}\left(\underline{\mathbf{X}}_i(t) < \mathbf{X}_i(t) < \overline{\mathbf{X}}_i(t)\right) \geq \mathbf{p}_i(t)\right\}, \quad (8)$$

where notation $\wedge$ denotes the logical conjunction.

V. Solution to Stochastic Optimal Control Problem

In this section, we provide the solution to stochastic optimal control problem. First, we transform the minimization problem of (3) subject to (1) into a quadratic programming problem with respect to the sequence of control inputs over the prediction horizon.

From (5), $\mathcal{E}(\mathbf{X}(t))$ and $\mathcal{V}(\mathbf{X}(t))$ are given by

$$\mathcal{E}(\mathbf{X}(t)) = \mathbf{A}x(t) + \mathbf{B}\mathbf{U}(t) + \mathbf{C}\mathcal{E}(\mathbf{W}(t)), \quad (9a)$$

$$\mathcal{V}(\mathbf{X}(t)) = \mathbf{C} \otimes \mathbf{C}\mathcal{V}(\mathbf{W}(t)). \quad (9b)$$

In (9a), we apply $\mathcal{E}(x(t)) = x(t)$ because the present state $x(t)$ is a deterministic vector. Moreover, (4) indicates that

$$J = x(t)'Qx(t) + \mathbf{U}(t)'\mathbf{R}\mathbf{U}(t) \\ \mathrm{tr}[Q\mathcal{C}_v(\mathbf{X}(t))] + \mathcal{E}(\mathbf{X}(t))'\mathbf{Q}\mathcal{E}(\mathbf{X}(t)). \quad (10)$$

Note that covariance matrix $\mathcal{C}_v(\mathbf{X}(t))$ is independent of $\mathbf{U}(t)$:

$$\mathcal{C}_v(\mathbf{X}(t)) = \mathcal{E}\left[\{\mathbf{X}(t) - \mathcal{E}(\mathbf{X}(t))\}\{\mathbf{X}(t) - \mathcal{E}(\mathbf{X}(t))\}'\right] \\ = \mathcal{E}\left[\{\mathbf{C}\mathbf{W}(t) - \mathbf{C}\mathcal{E}(\mathbf{W}(t))\}\{\mathbf{C}\mathbf{W}(t) - \mathbf{C}\mathcal{E}(\mathbf{W}(t))\}'\right].$$

Substituting (9a) into (10) and neglecting the terms that do not contain $\mathbf{U}(t)$, we obtain

$$\min_{\mathbf{U}(t)} J[x(t), \mathbf{X}(t), \mathbf{U}(t)] = \quad (11)$$

$$\min_{\mathbf{U}(t)} \left\{ \begin{array}{l} \mathbf{U}'(t)\left(\mathbf{B}'\mathbf{Q}\mathbf{B} + \mathbf{R}\right)\mathbf{U}(t) \\ \quad +2\left(\mathbf{A}x(t) + \mathbf{C}\mathcal{E}(\mathbf{W}(t))\right)'\mathbf{Q}\mathbf{B}\mathbf{U}(t) \end{array} \right\}.$$

Note that the minimization problem of J in (3) subject to (1) has been reduced to a quadratic programming problem with respect to $\mathbf{U}$.

In general, however, solving the quadratic programming problem with probabilistic constraints is not straightforward. Below, we provide the method for solving stochastic optimal control problems. We can convert the probabilistic constraint into deterministic constraint using the concentration inequality in Lemma 1. The following theorem has been proven in [27].

Theorem 1: Suppose that the following condition holds:

$$\mathbf{U}_{\min}(t) \leq \mathbf{B}\mathbf{U}(t) \leq \mathbf{U}_{\max}(t), \quad (12)$$

where $\mathbf{U}_{\min}$ and $\mathbf{U}_{\max}$ are defined by

$$\mathbf{U}_{\min}(t) := \underline{\mathbf{X}}(t) + \boldsymbol{\kappa}(t) \otimes \sqrt{\mathbf{C} \otimes \mathbf{C}\mathcal{V}(\mathbf{W}(t))} \\ - \mathbf{A}x(t) - \mathbf{C}\mathcal{E}(\mathbf{W}(t)), \quad (13a)$$

$$\mathbf{U}_{\max}(t) := \overline{\mathbf{X}}(t) - \boldsymbol{\kappa}(t) \otimes \sqrt{\mathbf{C} \otimes \mathbf{C}\mathcal{V}(\mathbf{W}(t))} \\ - \mathbf{A}x(t) - \mathbf{C}\mathcal{E}(\mathbf{W}(t)). \quad (13b)$$

$$\boldsymbol{\kappa}(t) := \left[\frac{1}{\sqrt{1 - \mathbf{p}_1(t)}}, \cdots, \frac{1}{\sqrt{1 - \mathbf{p}_{nN}(t)}}\right]'. \quad (13c)$$

Then, the probabilistic condition (7) is fulfilled.

Remark 1: From Theorem 1, the minimization problem of (11) with probabilistic constraint (7) is reduced to a quadratic programming problem with deterministic constraint (12), which can be solved using a conventional algorithm [32].

Remark 2: Suppose that we impose not only probabilistic state constraint (7) but also control input constraint on the optimization problem. Then, the optimization problem can be reduced to a quadratic programming problem (11) subject to the following constraint:

$$\begin{bmatrix} -\mathbf{B} \\ \mathbf{B} \\ \mathbf{F} \end{bmatrix} \mathbf{U}(t) \leq \begin{bmatrix} \mathbf{U}_{\min} \\ \mathbf{U}_{\max} \\ \overline{\mathbf{U}} \end{bmatrix}. \qquad (14)$$

Solving quadratic programming problem (11) subject to constraint (14) is also straightforward using a conventional algorithm [32].

Remark 3: Under the assumption of known probability distribution, the probabilistic constraints can be equivalently transformed into deterministic constraints without conservativeness using a cumulative distribution function. On one hand, the transformation from the probabilistic constraints to the deterministic constraints proposed in this paper yields a certain amount of conservativeness because the Chebyshev's inequality is given through evaluation of loose bounds. Specifically, the proposed method suffers from the disadvantage of conservativeness in transforming the constraints but features the advantage of applicability to arbitrary unknown probability distributions.

Remark 4: We provide a quantitative assessment of the conservatism of inequality (12). Here, we suppose that $w(t)$ is given by the standard normal (Gaussian) distribution with $\mathcal{E}(w(t))$ and $\mathcal{V}(w(t))$. Moreover, we consider the following probabilistic constraint:

$$\mathcal{P}\left(\mathbf{X}_i(t) < \overline{\mathbf{X}}_i(t)\right) \geq \mathbf{p}_i(t). \qquad (15)$$

From (15), we have the following:

$$\mathcal{P}\left((\mathbf{CW})_i < \overline{\mathbf{X}}_i - (\mathbf{A}x)_i - (\mathbf{BU})_i\right) \geq \mathbf{p}_i, \qquad (16)$$

where subscript i denotes the ith element of a vector. Let Φ denote the cumulative distribution function of the standard normal distribution defined by

$$\Phi(\alpha) := \frac{1}{2}\left\{1 + \frac{1}{\sqrt{\pi}}\int_{-\alpha}^{\alpha} e^{-t^2} dt\right\}. \qquad (17)$$

Let μ_i and σ_i be defined by

$$\mu_i := (\mathbf{C}\mathcal{E}(\mathbf{W}))_i, \qquad (18a)$$

$$\sigma_i := (\mathbf{C} \otimes \mathbf{C}\mathcal{V}(\mathbf{W}))_i. \qquad (18b)$$

Let F_i and G_i denote the cumulative distribution functions of the standard normal distribution with mean μ_i and variance σ_i and with zero mean and variance σ_i, respectively.

$$F_i(\alpha) := \Phi\left(\frac{\alpha - \mu_i}{\sigma_i}\right), \qquad (19)$$

$$G_i(\alpha) := \Phi\left(\frac{\alpha}{\sigma_i}\right), \qquad (20)$$

By using F_i in (19), inequality (16) can be rewritten as

$$F_i\left(\overline{\mathbf{X}}_i - (\mathbf{A}x)_i - (\mathbf{BU})_i\right) \geq \mathbf{p}_i. \qquad (21)$$

Then, we have

$$(\mathbf{BU})_i \leq \overline{\mathbf{X}}_i - (\mathbf{A}x)_i - F_i^{-1}(\mathbf{p}_i). \qquad (22)$$

Subtracting (13b) from the right-hand side of (22) yields

$$\left(\boldsymbol{\kappa} \otimes \sqrt{\mathbf{C} \otimes \mathbf{C}\mathcal{V}(\mathbf{W})}\right)_i + (\mathbf{C}\mathcal{E}(\mathbf{W}))_i - F_i^{-1}(\mathbf{p}_i). \qquad (23)$$

By using G_i in (20), (23) can be rewritten as

$$\left(\boldsymbol{\kappa} \otimes \sqrt{\mathbf{C} \otimes \mathbf{C}\mathcal{V}(\mathbf{W})}\right)_i - G_i^{-1}(\mathbf{p}_i). \qquad (24)$$

From (13c) and (18b), we can see that (24) can be rewritten as

$$\frac{\sigma_i}{\sqrt{1 - \mathbf{p}_i}} - G_i^{-1}(\mathbf{p}_i) =: H_i(\sigma_i, \mathbf{p}_i). \qquad (25)$$

Let $H_i(\sigma_i, \mathbf{p}_i)$ be defined as above. The plot of $H_i(\sigma_i, \mathbf{p}_i)$ is shown in Fig. 1. Note that probabilistic constraint (15) is equivalent to deterministic constraint (22), and the gap between (12) and (22) can be evaluated by (25). Thus, $H_i(\sigma_i, \mathbf{p}_i)$ indicates a quantitative assessment of the conservatism of inequality (12).

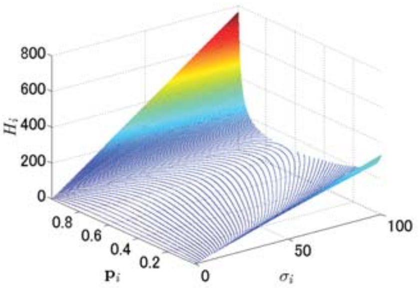

Fig. 1. Plot of $H_i(\sigma_i, \mathbf{p}_i)$.

VI. Conclusion

In this study, we have proposed an optimal control design method for linear discrete-time systems with additive stochastic disturbances under probabilistic constraints. The advantage of the proposed method is its applicability to stochastic disturbances with unknown probability distribution. The Chebyshev's inequality was applied to successfully handle probabilistic constraints with a lower computational load. Thus, the stochastic optimal control problem with probabilistic constraints was reduced to a quadratic programming problem with deterministic constraints, which can be solved using a conventional algorithm. The feasibility and stability analyses based on the proposed method are possible future research areas.

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