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Influence of Internal Heat Source on Thermal Instability in a Horizontal Porous Layer with Mass Flow and Inclined Temperature Gradient

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Abstract—An investigation has been presented to analyze the effect of internal heat source on the onset of Hadley-Prats flow in a horizontal fluid saturated porous medium. We examine a better understanding of the combined influence of the heat source and mass flow effect by using linear stability analysis. The resultant eigenvalue problem is solved by using shooting and Runga-Kutta methods for evaluate critical thermal Rayleigh number with respect to various flow governing parameters. It is identified that the flow is switch from stabilizing to destabilizing as the horizontal thermal Rayleigh number is enhanced. The heat source and mass flow increases resulting a stronger destabilizing effect.

Keywords—Linear stability analysis, heat source, porous medium, mass flow.

I. Introduction

In the last few decades, the study on thermal convection driven by an internal heat source has been attracted by many researchers due to its importance in real life applications. The present problem is also in connection with the above study induced by horizontal mass flow. It has many practical applications such as underground energy transport, cooling of nuclear reactors, geophysical and environmental problems etc. Specific important areas are like the food processing, oil recovery, underground storage of waste products and thermal convection in clouds [1]. The mechanism of thermal convection has a great importance in environmental problem processes [2].

Some of the authors reported on convection by internal heat sources. Few papers are concerning to the experimental investigation by Schwiderski et al. [3] and Tritton et al. [4]. Roberts [5] and Thirlby [6] done the theoretical analysis on the above experimental results. Parthiban and Patil [7] investigated the thermal convection due to non-uniform heating boundaries with inclined thermal gradients in the presence of internal heat source, followed by the extension of anisotropic porous layer studied by Parthiban and Patil [8]. The effect of internal heat source with inclined porous layer for various flow parameters are analyzed by Barletta et al. [9], where both boundaries are isothermal and keep them at same temperature. Rionero and Straughan [10] investigated the linear and nonlinear instability in presence of heat generation and variable gravity. Extensive reviews of the theory and applications can be seen in the articles of Alex and Patil [11]. Hill [12] reported on a fluid-saturated porous layer with concentration based internal heat generation, in that, he studied the linear and energy stability analysis of thermosolutal convection. Chamka [13] analyzed the influence of an internal heat source or sink for hydromagnetic simultaneous heat and mass transfer by using similarity solutions. Thermosolutal convection in a saturated anisotropic porous medium with internal heat source is reported by Bhadauria [14]. Borujerdi et al. [15] examine the study state heat conduction with uniform heat source where solid and fluid phases are at different temperature. Then after, Borujerdi et al. [16] studied the influence of Darcy number on the critical thermal Rayleigh number in onset of convection with uniform internal heating. A collection of comprehensive theories and experiments of thermal convection in porous media on real life problems were surveyed in the recent book of Nield and Bejan [17].

The purpose of this theoretical study is to analyze the situation in which both the effects of heat source and mass flow are present simultaneously. The governing equations have been transformed into eigenvalue problem, and it is solved numerically by using Shooting and Runga-Kutta method for various modes of instability. We organize the paper in the following steps. Section II deals with the governing equations of the model considered and section III followed by basic state solution, linear analysis and numerical scheme described in section IV and section V. Results are analyzed in section VI.

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II. Mathematical Analysis

An infinite shallow horizontal fluid saturated porous medium with thickness d is considered. z^* -axis is taken vertically upward and there is a net flow along the direction of x^* -axis with magnitude M^*. The vertical thermal differences along the boundaries is $\nabla\theta$. Further, we imposed the horizontal thermal gradient vector as $(\beta_{\theta_x},\ \beta_{\theta_y})$ and the internal heat source is $Q^{'}$. The linear Boussinesq approximation is applicable. The flow in porous layer is formed by the Darcy law and the governing equations in dimensional form are

$$\nabla^{'} \cdot q^{'} = 0\,, \tag{1}$$

$$\frac{\mu}{K} q^{'} + \nabla^{'} P^{'} = \rho_0 \left[1 - \gamma_\theta \left(\theta^{'} - \theta_0 \right) \right] g\mathbf{k} \,, \tag{2}$$

$$(\rho c)_m \left(\frac{\partial \theta^{'}}{\partial t^{'}} \right) + (\rho c_p)_f \, q^{'} \cdot \nabla^{'} \theta^{'} = k_m \nabla^{'2} \theta^{'} + Q^{'} \,, \tag{3}$$

with the following boundary conditions:

$$w^{'} = 0, \;\; \theta^{'} = \theta_0 - \frac{1}{2} \left(\pm \triangle \theta \right) - \beta_{\theta_x} x^{'} - \beta_{\theta_y} y^{'} \;\; \text{at} \;\; z = \pm \frac{1}{2}$$

Here, the Darcy velocity notated as $q^{'} = (u^{'}, v^{'}, w^{'})$, **k** is a unit vector along the vertical direction. $P^{'}$ is the pressure and $\theta^{'}$ is temperature. Where the subscripts f and m are referred to fluid and porous medium, respectively. Here K is the permeability of the porous layer. Also c, ρ, μ and k_m denote the specific heat, density, viscosity and thermal conductivity, respectively. Also γ_θ is the thermal expansion coefficients in the porous medium.

The following dimensionless variables were used to non-dimensionalize the governing equations.

$$(x, y, z) = \frac{1}{d} \left(x^{'}, y^{'}, z^{'} \right), \; t = \frac{\alpha_m t^{'}}{a d^2}, \; q = \frac{d q^{'}}{\alpha_m}, \; M = \frac{d M^{'}}{\alpha_m},$$

$$P = \frac{K \left(P^{'} + \rho_0 g z^{'} \right)}{\mu \alpha_m}, \; \theta = \frac{R_z \left(\theta^{'} - \theta_0 \right)}{\triangle \theta}, \; Q = \frac{d^2 Q^{'}}{k_m \triangle \theta}.$$

Where

$$\alpha_m = \frac{k_m}{(\rho c_p)_f}, \quad a = \frac{(\rho c)_m}{(\rho c_p)_f}, \quad R_z = \frac{\rho_0 g \gamma_\theta K d \triangle \theta}{\mu \alpha_m}.$$

Here, R_z denote the vertical thermal Rayleigh number and the horizontal thermal Rayleigh numbers are referred as follows

$$R_x = \frac{\rho_0 g \gamma_\theta K d^2 \beta_{\theta_x}}{\mu \alpha_m}, \quad R_y = \frac{\rho_0 g \gamma_\theta K d^2 \beta_{\theta_y}}{\mu \alpha_m}.$$

The above scaling for dimensional variables and the horizontal thermal Rayleigh numbers was introduced by Weber [18] and used extensively by Nield [19]. Under these dimensionless variables, the governing equations (1) - (3) take the form

$$\nabla \cdot q = 0 \,, \tag{4}$$

$$q + \nabla P = \theta \mathbf{k} \,, \tag{5}$$

$$\frac{\partial \theta}{\partial t} + q \cdot \nabla \theta = \nabla^2 \theta + Q R_z \,, \tag{6}$$

and the conditions of the plates become

$$w = 0, \quad \theta = -\frac{1}{2} \left(\pm R_z \right) - R_x x - R_y y \quad \text{at} \quad z = \pm \frac{1}{2} \,. \tag{7}$$

From (4) - (6); we observe that all the thermal Rayleigh numbers involved in the boundary condition (7).

III. Steady-state Solution

The flow of governing equations (4) - (6), subject to (7) has a basic state solution as follows

$$u_s = u(z) \;, \quad v_s = v(z) \;, \quad w_s = 0,$$

$$P_s = P(x, y, z) \;, \quad \theta_s = \widetilde{\theta}(z) - R_x x - R_y y \,, \tag{8}$$

with

$$u = -\frac{\partial P}{\partial x} \,, \quad v = -\frac{\partial P}{\partial y} \,,$$

$$0 = -\frac{\partial P}{\partial z} + \widetilde{\theta}(z) - R_x x - R_y y \,,$$

$$D^2 \widetilde{\theta} = -u R_x - v R_y - Q R_z \,. \tag{9}$$

Where $D = \frac{d}{dz}$, we have a net flow (M) in the horizontal direction, then $\int_{-1/2}^{1/2} u(z) dz = M$ and $\int_{-1/2}^{1/2} v(z) dz = 0$. We obtain the basic state solution in the form of flow velocity, temperature in the medium.

$$u_s = R_x z + M \,, \quad v_s = R_y z \,, \quad w_s = 0 \tag{10}$$

$$\widetilde{\theta} = -R_z z - \frac{\lambda}{24} \left(4z^3 - z \right) - \left(M R_x + Q R_z \right) \left(\frac{z^2}{2} - \frac{1}{8} \right) \,, \tag{11}$$

where $\lambda = {R_x}^2 + {R_y}^2$.

IV. Linear Stability Analysis

We assume the disturbance quantities in the form of $q = q_s + \overline{q}$, $\theta = \theta_s + \overline{\theta}$ and $P = P_s + \overline{P}$. By substituting these perturbations in dimensionless governing equations (4) - (6), further, we get a linear system by neglecting the products of perturbations

$$\nabla \cdot \overline{q} = 0 \,, \tag{12}$$

$$\overline{q} = -\nabla \overline{P} + \overline{\theta} \mathbf{k} \,, \tag{13}$$

$$\frac{\partial\overline{\theta}}{\partial t}+q_s\cdot\nabla\overline{\theta}+\overline{q}\cdot\nabla\theta_s=\nabla^2\overline{\theta}\ , \tag{14}$$

where

$$\nabla\theta_s=-\left(R_x,R_y,R_z-\frac{\lambda}{24}\left[1-12z^2\right]+\left(MR_x+QR_z\right)z\right).$$

The conditions at the plates are

$$\overline{w}=0,\quad \overline{\theta}=0\quad \text{at}\quad z=\pm\frac{1}{2}\ . \tag{15}$$

These conditions in (15) says that, there is zero perturbation in velocity and temperature at the plates. We are looking for a solution of (12) - (14) in the form of normal modes

$$\left[\overline{q},\overline{\theta},\overline{P}\right]=\left[q\left(z\right),\theta\left(z\right),P\left(z\right)\right]\exp\left\{i\left[kx+ly-\sigma t\right]\right\}\ , \tag{16}$$

further eliminate P from (13), we get

$$\left(D^2-\alpha^2\right)w+\alpha^2\theta=0, \tag{17}$$

$$\begin{aligned}&\left(D^2-\alpha^2+i(\sigma-Mk-ku_s-lv_s)\right)\theta\\&\quad+\frac{i}{\alpha^2}\left(kR_x+lR_y\right)Dw-\left(D\widetilde{\theta}\right)w=0,\end{aligned} \tag{18}$$

where

$$D\widetilde{\theta}=-R_z-\frac{\lambda}{24}\left(12z^2-1\right)-\left(MR_x+QR_z\right)z\ . \tag{19}$$

The above (17) - (18) subject to $w=\theta=0$ at both the plates $z=\frac{1}{2}$ and $z=-\frac{1}{2}$ constitute an eigenvalue problem for vertical thermal Rayleigh number R_z with a, R_x, R_y, k and l as parameters. In the above, $\alpha=\sqrt{k^2+l^2}$ is the overall wave number.

V. Numerical Solution

To find an accurate solution to above (17) - (18) are solved based on Shooting and Runge-Kutta method. This scheme is necessarily applied by converting the boundary value problem into an initial value problem. Thus, the boundary conditions on $w(z)$ and $\theta(z)$ have been replaced by the set of initial conditions

$$w(0)=0,\quad Dw(0)=1,\quad \theta(0)=0,\quad D\theta(0)=\eta \tag{20}$$

here the extra condition on $Dw(0)$ is utilizes the indeterminate scale factor of the solution $w(z)$. The value of η may be either a real or complex constant. The software package *Matlab R2012b* gives the builtin function ode45, which provides a good advantage to implement the explicit Runge Kutta method.

TABLE I
Critical thermal Rayleigh number at $M=0$.

Q	R_x	0	10	20	30	40	50
0	R_z	39.478	42.007	49.548	61.956	78.966	100.116
	α	3.139	3.1399	3.149	3.159	3.219	3.339
1	R_z	39.236	41.729	49.146	61.275	77.702	97.534
	α	3.159	3.1599	3.169	3.209	3.309	3.599
5	R_z	34.594	36.468	41.779	49.531	57.957	65.044
	α	3.419	3.4599	3.609	3.969	4.599	5.400

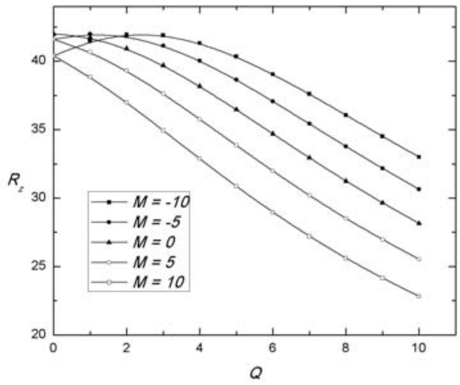

Fig. 1. Variation of R_z with Q at $R_x=10$.

VI. Results and Discussion

The onset of thermal convection in a fluid-saturated porous medium in the presence of mass flow and an internal heat generation is analyzed by applying the linear theory. The instability analysis is studied by using the classical normal mode technique. Here, the critical vertical thermal Rayleigh number (R_z) is defined as the minimum of all R_z values as wave number (α) is varied. The vector of wave number is defined as $\alpha=(k,l,0)$. To achieve the stationary convection, we set $\sigma=0$ as shown by Nield [20]. We also set $(R_x,R_y)\cdot(k,l)=0$. The term longitudinal disturbances are characterized by $k=0$. From the Table I, it is observed that when $Q=0$ and $M=0$, the present results are very good agreement with earlier published results in the literature by Nield [21]. An increase the magnitude of Q from 0 to 5, the critical value of R_z is reduced seen in Table I. Hence, the heat flow parameter (Q) causes destabilization in the medium.

A comparison of critical value of R_z as a function of Q for different values of mass flow rate (M) is shown in Fig. 1 at $R_x=10$ and $R_y=0$. It is interesting to noted that, at negative value of $M=-10$, R_z is increasing with the increasing the values of Q upto $Q<3$, there after, it is decreasing smoothly. It means for small magnitudes of Q the flow is stabilized and higher magnitudes of Q destabilizes the flow and convection commences. But, as M increases -10 to 10, at higher value of M, the critical R_z value is lower than at lower magnitudes of M. From Fig. 1, it is to summarize that for both positive and negative magnitudes of M, R_z is decreasing with the increasing the values of Q. It indicates that, increasing the heat source has a destabilizing effect is as seen in Fig. 1.

The response of R_z with varying R_x is shown in Fig.

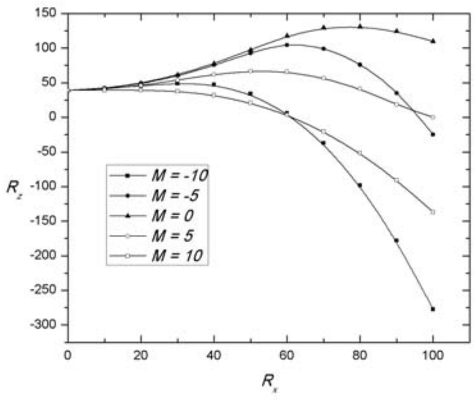

Fig. 2. Variation of R_z with R_x at $Q = 1$.

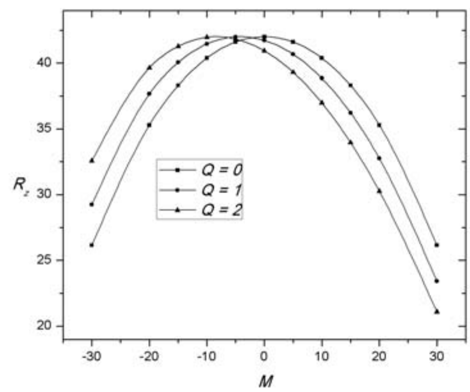

Fig. 3. Variation of R_z with M at $R_x = 10$.

2 for positive and negative magnitudes of M at $R_y = 0$, respectively. It is noted that, at $M = 0$, the critical value of R_z is higher than the remaining all magnitudes of M. An increase in R_x, reduces the onset of convection, so that in this range, the system is least stable when $M = -10$. It is also interesting to notified that, as R_x increases then the critical value of R_z is also increased up to certain value of R_x, there after, R_z value is decreased for all M values as seen in Fig. 2. It means that, the flow rate is strongly destabilizing at higher values of R_x for irrespective of the value of M.

Fig. 3 shows that, the response of R_z with mass flow rate (M) in the presence of different heat source values of $Q = 0, 1, 2$ and $R_x = 10$ at $R_y = 0$. As M is increasing then R_z also increasing up to certain values of M there after decreasing for all values of Q as seen in Fig. 3. Thus, in presence and absence of heat source, the thermal flow has strongly stabilizing effect up to certain value of M, then after strongly destabilize the thermal flow. It is notified that, for positive values of M, the flow is destabilize earlier at higher value of Q as compared to lower values of Q. It concluded that, in the presence or absence of internal heat source, creates a dual role (for negative M flow is stabilizing and for positive M flow is destabilizing) on the thermal instability of the system.

VII. Conclusion

We have analyzed the instability of thermal convection in a Hadley-Prats flow subject to internal heat source and mass flow by using linear stability analysis. The critical value of R_z is evaluated for different combinations of the flow governing parameters. It has been concluded from the table and graphs that the following results can be obtained:

- As heat source and mass flow increases causes the strong destabilization.
- In the presence of heat generation, the flow is destabilizing at higher horizontal Rayleigh numbers irrespective of mass flow.
- It is clear that the qualitative changes appear in critical Rayleigh number subject to heat source and mass flow.

Nomenclature

d	height of porous layer
g	acceleration due to gravity
K	permeability
k_m	thermal conductivity
M	dimensionless mass flow
P	dimensionless pressure
q	dimensionless velocity
Q	dimensionless heat source
R_x, R_y	horizontal thermal Rayleigh number
R_z	vertical thermal Rayleigh number
t	dimensionless time
u, v, w	x, y, z component of dimensionless velocities
Greek symbols	
α	dimensionless overall wave number
α_m	thermal diffusivity
$(\beta_{\theta_x}, \beta_{\theta_y})$	horizontal thermal gradient vector
γ_θ	thermal expansion coefficients
θ	dimensionless temperature
ν	kinematic viscosity
ρ	density
Φ	porosity
Subscripts	
f	fluid medium
m	porous medium
s	study state
Superscripts	
$'$	dimensional variables

– disturbance quantities

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