World Academy of Science, Engineering and Technology
International Journal of Aerospace and Mechanical Engineering
Vol:9, No:2, 2015

On a New Inverse Polynomial Numerical Scheme for the Solution of Initial Value Problems in Ordinary Differential Equations

R. B. Ogunrinde

Abstract**—**This paper presents the development, analysis and implementation of an inverse polynomial numerical method which is well suitable for solving initial value problems in first order ordinary differential equations with applications to sample problems. We also present some basic concepts and fundamental theories which are vital to the analysis of the scheme. We analyzed the consistency, convergence, and stability properties of the scheme. Numerical experiments were carried out and the results compared with the theoretical or exact solution and the algorithm was later coded using MATLAB programming language.

Keywords**—**Differential equations, Numerical, Initial value problem, Polynomials.

I. Introduction

IN the years past, a large number of methods suitable for solving ordinary differential equations have been proposed, A major impetus to developing numerical procedures was the invention of the calculus by Newton and Leibnitz as this led to accurate mathematical models for physical reality such as Sciences, Engineering, Medicine and Business. These mathematical models cannot be usually solved explicitly and numerical method to obtain approximate solutions is needed. Up to the late 1800's it appears that most mathematicians were quite broad in their interest. Many researchers have developed numerical methods while some try to improve the accuracy of some methods. Generally, the efficiency of any of these methods depends on the stability and accuracy properties. Accuracy properties of different methods are usually compared by considering the order of convergence as well as the truncation error coefficient of the various methods. The major sources of motivation for this work are those of [1]-[4], [11], [13]. In this paper we shall consider the initial value problem of the form;

$$y^1 = f(x,y);\, y(a) = \eta \tag{1}$$

We developed an algorithm which can effectively solve initial value problems in ordinary differential equation.

Ogunrinde Roseline Bosede is currently a Senior Lecturer in the Department of Mathematical Sciences, Ekiti State University, Ado Ekiti, Nigeria (phone: +2347031343029; e-mail: roseline.ogunrinde@eksu.edu.ng)

II. Some Basic Concepts

We shall consider the following basic concepts which are very vital to the development of the new scheme based on [5]-[7], [10]

A. Stability

A numerical method is said to be stable if the difference between the numerical solution and the exact solution can be made as small as possible, that is if there is exists two positive e_0 and K such that the following holds. [8], [9], [14]

$$\|y_n - y(x_n)\| \le K\|e_0\| \tag{2}$$

B. Consistency

A numerical scheme with an increment function $\phi(x_{n,}, h, y)$ is said to be consistent with the initial problem consideration if

$$\phi(x_{n,}, h, y) = f(x,y) \tag{3}$$

C. Convergence

A numerical method is said to be convergent if for all initial value problem satisfying the hypothesis of Lipschitz condition given by

$$\left| f(x,y) - f(x,y^*) \right| \le L\left| y - y^* \right| \tag{4}$$

where the Lipschitz condition is denoted by L. The necessary and sufficient conditions for convergence are the stability and consistency.

D. Round off Error

This can be defined as the error due to computing device. They arise because it is possible to represent all real numbers exactly on a finite-state machine. It can be represented mathematically as

$$R_{n+1} = y_{n+1} - q_{n+1} \tag{5}$$

where y_{n+1} is the approximate solution and q_{n+1} is the computer/machine output.

III. Development of the New Scheme

Let the numerical approximation $y_n + h$ be evaluated at $x = x_n + k$ to the exact solution $y(x_n + h)$ to the first order ordinary differential equation be represented as

$$y(x_{n+1}) = y_n[e^p + \sum_{j=1}^{k} b_j x_n{}^j]^{-1} \tag{6}$$

where e^p is the exponential of p and the parameters $b_j s'$ are to be determined.

$$y_{n+1} = y_n[e^p + \sum_{j=1}^{k} b_j x_n{}^j]^{-1} \tag{7}$$

$$y_{n+1} = \frac{y_n}{(e^p + \sum_{j=1}^{k} b_j x_n{}^j)}$$

Setting k=3 (order three)

$$y_n + h = y_{n+1} = \frac{y_n}{(e^p + \sum_{j=1}^{3} b_j x_n{}^j)} \tag{8}$$

$$y_{n+1} = \frac{y_n}{e^p + b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3} \tag{9}$$

Applying the Taylor's series expansion on the L.H.S. of (9)

$$y_{n+1} = y_n + h = y_n + h y_n{}' + \frac{h^2 y_n{}''}{2!} + \frac{h^3 y_n{}'''}{3!} + \ldots \tag{10}$$

$$y_n[e^p + b_1 x_n + b_2 x_n + b_3 x_n{}^3]^{-1}$$

where

$$e^p = 1 + p + \frac{p^2}{2!} + \frac{p^3}{3!} + \ldots \tag{11}$$

But at $p=0$ (a constant), we have that

$$e^p = 1 + 0 + 0 + 0 + \ldots = 1$$

Hence

$$e^0 = 1 \tag{12}$$

So that,

$$y_{n+1} = y_n + h = \frac{y_n}{1 + b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3} \tag{13}$$

$$y_n + h y_n{}' + \frac{h^2 y_n{}''}{2!} + \frac{h^3 y_n{}'''}{3!} + \ldots = \frac{y_n}{1 + b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3} \tag{14}$$

which can also be written as

$$y_n + h y_n{}' + \frac{h^2 y_n{}''}{2!} + \frac{h^3 y_n{}'''}{3!} + \ldots = y_n[1 + b_1 x_n + b_2 x_n + b_3 x_n{}^3]^{-1} \tag{15}$$

Using Binomial series expansion on the R.H.S. of (15)

$$y_n[1 + (b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3)]^{-1}$$

$$\left.\begin{aligned} &= y_n[1 + (-1)(b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3) + \frac{(-1)(-2)}{2!}(b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3)^2 \\ &+ \frac{(-1)(-2)(-3)}{3!}(b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3)^3 + \ldots] \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} &= y_n[1 + (-1)(b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3) + \\ &(b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3)^2 - \\ &(b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3)^3 + \ldots] \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} &= y_n[1 - (b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3) + (b_1{}^2 x_n{}^2 + 2b_1 b_2 x_n{}^3 + 2b_1 b_2 x_n{}^3) \\ &-(b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3)^3 + \ldots] \end{aligned}\right\}$$

$$\left.\begin{aligned} &y_n[1 - (b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3) + b_1{}^2 x_n{}^2 + 2b_1 b_2 x_n{}^3 + \\ &2b_1 b_3 x_n{}^4 + b_2{}^2 x_b{}^4 + 2b_2 b_3 x_n{}^5 + b_3{}^2 x_n{}^6 + \ldots \\ &-(b_1{}^3 x_n{}^3 + b_1{}^2 b_2 x_n{}^4 + b_1{}^2 b_3 x_n{}^5 + 2b_1{}^2 b_2 x_n{}^4) - \\ &(2b_1 b_2{}^2 x_n{}^5 + 2b_1{}^2 b_3 x_n{}^5 + b_1 b_2{}^2 x_n{}^5) + \ldots \end{aligned}\right\}$$

Firstly, we expand

$$(b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3)^2$$

$$(b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3)\ (b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3)$$

$$b_1{}^2 x_n{}^2 + b_1 b_2 x_n{}^3 + b_1 b_3 x_n{}^4 + b_2 b_1 x_n{}^3 + b_2{}^2 x_n{}^4$$

$$+ b_2 b_3 x_n{}^5 + b_3 b_1 x_n{}^4 + b_2 b_3 x_n{}^5 + b_3{}^2 x_n{}^6 + \ldots$$

$$(b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3)^2$$

$$b_1{}^2 x_n{}^2 + 2b_1 b_2 x_n{}^3 + (2b_1 b_3 + b_2{}^2) x_n{}^4 + \ldots \tag{18}$$

Now, expanding

$$(b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3)^3 =$$

$$(b_1{}^2 x_n{}^2 + 2b_1 b_2 x_n{}^3 + (2b_1 b_3 + b_2{}^2) x_n{}^4 + \ldots)(b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3)$$

$$b_1{}^3 x_n{}^3 + b_1{}^2 b_3 x_n{}^5 + 2b_1{}^2 b_2 x_n{}^4 + b_1{}^2 b_3 x_n{}^5 + 2b_1 b_2{}^2 x_n{}^5 + 2b_1 b_2 b_3 x_n{}^6$$

$$+(2b_1 b_3 + b_2{}^2) b_1 x_n{}^5 + (2b_1 b_3 + b_2{}^2) b_2 x_n{}^6 + \ldots)]$$

$$(b_1 x_n + b_2 x_n{}^2 + b_3 x_n{}^3)^3 =$$

$$b_1{}^3 x_n{}^3 + b_1{}^2 b_3 x_n{}^5 + 2b_1{}^2 b_2 x_n{}^4 + b_1{}^2 b_3 x_n{}^5 + 2b_1 b_2{}^2 x_n{}^5 + 2b_1 b_2 b_3 x_n{}^6 \tag{19}$$

$$+(2b_1 b_3 + b_2{}^2) b_1 x_n{}^5 + (2b_1 b_3 + b_2{}^2) b_2 x_n{}^6 + \ldots)]$$

Substituting (18) & (19) into (17)

$$y_n[1 - b_1 x_n - b_2 x_n{}^2 - b_3 x_n{}^3 + b_1{}^2 x_n{}^2 + 2b_1 b_2 x_n{}^3 + (2b_1 b_3 + b_2{}^2) x_n{}^4 - b_1{}^3 x_n{}^3 - b_1{}^2 b_2 x_n{}^4 - b_1{}^2 b_3 x_n{}^5$$

$$- 2b_1{}^2 b_2 x_n{}^4 - 2b_1 b_2{}^2 x_n{}^5 - 2b_1 b_2 b_3 x_n{}^6 - (2b_1 b_3 + b_2{}^2) b_1 x_n{}^5 - (2b_1 b_3 + b_2{}^2) b_2 x_n{}^6 + \ldots$$

Hence,

$$y_n[1 - b_1x_n + (b_1^2 - b_2)x_n^2 + (2b_1b_2 - b_3 - b_1^3)x_n^3 + (2b_1b_3 - 3b_1^2b_2 + b_2^2)x_n^4 = y_n - b_1x_ny_n + (b_1^2 - b_2)x_n^2y_n + (2b_1b_2 - b_3 - b_1^3)x_n^3y_n + \cdots$$

$$y_n + hy_n' + \frac{h^2y_n''}{2!} + \frac{h^3y_n'''}{3!} + \ldots = \tag{20}$$

We must ensure that the expansion on the L.H.S. of (20) agrees term by term with that on the R.H.S.

$$hy_n' = -b_1x_ny_n$$

$$b_1 = -\frac{hy_n'}{x_ny_n} \tag{21}$$

$$\frac{h^2y_n''}{2!} = (b_1^2 - b_2)x_n^2y_n$$

$$(b_1^2 - b_2) = \frac{h^2y_n''}{2x_n^2y_n} \tag{22}$$

Substituting (21) into (22), we have

$$b_2 = \frac{h^2(y_n')^2}{x_n^2y_n^2} - \frac{h^2y_n''}{2x_n^2y_n}$$

$$b_2 = \frac{2h^2(y_n')^2 - h^2y_n''y_n}{2x_n^2y_n^2} \tag{23}$$

$$\frac{h^3y_n'''}{3!} = (2b_1b_2 - b_3 - b_1^3)x_n^3y_n$$

$$b_3 = 2b_1b_2 - b_1^3 - \frac{h^3y_n'''}{6x_n^3y_n} \tag{24}$$

Substitute (21) and (23) into (24)

$$b_3 = 2[-\frac{hy_n'}{x_ny_n}][\frac{2h^2(y_n')^2 - h^2y_n''y_n}{2x_n^2y_n^2}] + [\frac{h^3(y_n')^3}{x_n^3y_n^3}] - [\frac{h^3y_n'''}{6x_n^3y_n}]$$

$$= -\frac{h^3y_n'[2(y_n')^2 - y_ny_n'']}{x_n^3y_n^3} + \frac{h^3(y_n')^3}{x_n^3y_n^3} - \frac{h^3y_n'''}{6x_n^3y_n}$$

$$= \frac{[-12h^3(y_n')^3 + 6h^3y_ny_n'y_n'' + 6h^3(y_n')^3 - h^3y_n^2y_n''']}{6x_n^3y_n^3}$$

$$b_3 = \frac{[6h^3y_ny_n'y_n'' - 6h^3(y_n')^3 - h^3y_n^2y_n''']}{6x_n^3y_n^3} \tag{25}$$

Substituting (21), (23) and (25) in to (13)

$$y_{n+1} = \frac{y_n}{1 - \left[\frac{hy_n'}{x_ny_n}\right]x_n + \left[\frac{2h^2(y'n)^2 - h^2y_ny_n''}{2x_n^2y_n^2}\right]x_n^2 + \left[\frac{[(6h^3y_ny_n'y_n'' - 6h^3(y_n')^3 - h^3y_n'''y_n^2)]}{6x_n^3y_n^3}\right]x_n^3}$$

$$y_{n+1} = \frac{y_n}{1 - \left[\frac{hy_n'}{y_n}\right] + \left[\frac{2h^2(y'n)^2 - h^2y_ny_n''}{2y_n^2}\right] + \left[\frac{[(6h^3y_ny_n'y_n'' - 6h^3(y_n')^3 - h^3y_n'''y_n^2)]}{6y_n^3}\right]} \tag{26}$$

$$y_{n+1} = \frac{6y_n^4}{6y_n^3 - 6hy_n^2y_n' + 6h^2y_n(y_n')^2 - 3h^2y_n^2y_n'' + 6h^3y_ny_n'y_n'' - 6h^3(y'n)^3 - h^3y_n'''y_n^2} \tag{27}$$

Hence, the scheme is

$$y_{n+1} = \frac{6y_n^4}{6y_n^3 - 6hy_n^2y_n' + 3h^2y_n[2(y_n')^2 - y_ny_n''] + h^3[6y_ny_n'y_n'' - 6(y'n)^3 - y_n'''y_n^2]} \tag{28}$$

IV. ALGORITHM

An approximate solution to the IVP $y' = f(x, y)\,; y(x_0) = y_0$ at the points $x_0, x_1, \ldots, x_n$ is given by $y_{n+1} = y_n[e^p + \sum_{j=1}^{3} b_jx_n^j]^{-1}$ where p=0 (a constant) with parameters

$$b_1 = -\frac{hy_n'}{x_ny_n}$$

$$b_2 = \frac{2h^2(y_n')^2 - h^2y_n''y_n}{2x_n^2y_n^2}$$

$$b_3 = \frac{[6h^3y_ny_n'y_n'' - 6h^3(y_n')^3 - h^3y_n^2y_n''']}{6x_n^3y_n^3}$$

$$x_{n+1} = x_0 + (n+1)h,$$

$n = 0,1,\ldots,N$ *where* N, x_0, y_0 *are given and*

$h = \frac{x_n - x_0}{N}$ *is the step size.*

A. The Consistency Property of the Scheme

The conventional one step integrator for the Initial value problem is generally described according to [12] as

$$y_{n+1} = y_n + h\emptyset(x_n, y_n, h) \tag{29}$$

By subtracting y_n from both sides of (28) we obtain

$$y_{n+1} - y_n = h\emptyset(x_n, y_n, h) \tag{30}$$

Divide both sides of (30) by h, we have

$$\frac{y_{n+1} - y_n}{h} = \frac{h\emptyset(x_n, y_n, h)}{h}$$

$$\frac{y_{n+1} - y_n}{h} = \emptyset(x_n, y_n, h) \tag{31}$$

If

$$\emptyset(x_n, y_n, h) = f(x, y)\,, \tag{32}$$

We then say that the given integrator formula (scheme) is consistent with the Initial value problem under consideration.

Now to show this with respect to the scheme derived above, subtract y_n from both sides of (28)

$$y_{n+1}-y_n=\frac{6y_n^4}{6y_n^3-6hy_n^2y_n'+6h^2y_n(y_n')^2-3h^2y_n^2y_n''+6h^3y_n'y_ny_n''-6h^3(y_n')^3-h^3y_n'''y_n^2}-\frac{y_n}{1} \quad (33)$$

$$=\frac{6y_n^4-6y_n^4+6hy_n^3y_n'-6h^2y_n^2(y_n')^2+3h^2y_n^3y_n''-6h^3y_n^2y_n'y_n''+6h^3y_n^3(y_n')+h^3y_n'''y_n^3}{6y_n^3-6hy_n^2y_n'+6h^2y_n(y_n')^2-3h^2y_n^2y_n''+6h^3y_n'y_ny_n''-6h^3(y_n')^3-h^3y_n'''y_n^2}$$

$$y_{n+1}-y_n=\frac{6y_n^4-6y_n^4+6hy_n^3y_n'-6h^2y_n^2(y_n')^2+3h^2y_n^3y_n''-6h^3y_n^2y_n'y_n''+6h^3y_n^3(y_n')+h^3y_n'''y_n^3}{6y_n^3-6hy_n^2y_n'+6h^2y_n(y_n')^2-3h^2y_n^2y_n''+6h^3y_n'y_ny_n''-6h^3(y_n')^3-h^3y_n'''y_n^2} \quad (34)$$

$$y_{n+1}-y_n=\frac{h[6y_n^3y_n'-6hy_n^2(y_n')^2+3hy_n^3y_n''-6h^2y_n^2y_n'y_n''+6h^2y_n^3(y_n')+h^2y_n'''y_n^3]}{6y_n^3-6hy_n^2y_n'+6h^2y_n(y_n')^2-3h^2y_n^2y_n''+6h^3y_ny_n'y_n''-6h^3(y_n')-h^3y_n'''y_n^2} \quad (35)$$

Divide both sides of (35) by h,

$$\frac{y_{n+1}-y_n}{h}=\frac{6y_n^3y_n'-6hy_n^2(y_n')^2+3hy_n^3y_n''-6h^2y_n^2y_n'y_n''+6h^2y_n^3(y_n')+h^2y_n'''y_n^3}{6y_n^3-6hy_n^2y_n'+6h^2y_n(y_n')^2-3h^2y_n^2y_n''+6h^3y_ny_n'y_n''-6h^3(y_n')-h^3y_n'''y_n^2} \quad (36)$$

As h tends to zero yields

$$\frac{y_{n+1}-y_n}{h}=\frac{6y_n^3y_n'}{6y_n^3}$$
$$\frac{y_{n+1}-y_n}{h}=y_n' \quad (37)$$

which implies that (32) is satisfied and thus the scheme (28) is consistent.

B. The Stability Property of the Scheme

We use the general form of the scheme to investigate the stability property

$$y_{n+h}=\frac{y_{n+h-1}}{(e^p+\sum_{j=1}^k b_jx_{n+h}^j)} \quad (38)$$

The theoretical solution y(x) is given as

$$y(x_{n+h})=\frac{y(x_{n+h-1})}{(e^p+\sum_{j=1}^k b_jx_{n+h}^j)}+T_{n+h} \quad (39)$$

By subtracting (38) from (39), we obtain

$$y(x_{n+h})-y_{n+h}=\frac{y(x_{n+h-1})}{(e^p+\sum_{j=1}^k b_jx_{n+h}^j)}-\frac{y_{n+h-1}}{(e^p+\sum_{j=1}^k b_jx_{n+h}^j)}+T_{n+h} \quad (40)$$

But the globalization error associated with general one-step scheme (28) is given by

$$e_{n+h}=y_{n+h}-y(_{n+h})$$

Now by adopting (40) on (39) and simplifying, we obtain

$$e_{n+h}=\frac{e_{n+h-1}}{(e^p+\sum_{j=1}^k b_jx_{n+h}^j)}+T_{n+h} \quad (41)$$

But since p=0 is a constant, then e^p=1. Hence,

$$e_{n+h}=\frac{e_{n+h-1}}{1+\sum_{j=1}^k b_jx_{n+h}^j}+T_{n+h} \quad (42)$$

Taking the modulus of both sides, yields

$$\left|\frac{1}{1+\sum_{j=1}^k b_jx_{n+h}^j}\right|=\frac{1}{1+\sum_{j=1}^k b_jx_{n+h}^j}$$

By setting, Q=$|1+\sum_{j=1}^k b_jx_{n+h}^j|$, we have

$$\left|\frac{1}{1+\sum_{j=1}^k b_jx_{n+h}^j}\right|=\frac{1}{Q}=M \quad (43)$$

Then,

$$|e_{n+h}|\le M|e_{n+h}|+|T_{n+h}|$$

Let T= sup(T_{n+h}) and M<1 similarly by setting

$$E_{n+h}=\sup e_{n+h,}$$

Then, the inequality modifies into 0<n<∞

$$E_{n+h}\le ME_{n+h-1}+T$$

Hence for h=1, we have

$$E_{n+1}\le ME_n+T$$

For h=2,

$$E_{n+2}\le M^2E_n+MT+T$$

By following this trend, it could be seen that

$$E_{n+h}\le M^kE_{n+h-1}+\sum_{r=0}M^rT \quad (44)$$

Since M<1, then as n tends to infinity, $E_{n+h}\longrightarrow 0$

C. Convergence Property of the Scheme

Having tested for the consistency and stability of the method, we can conclude that the convergence property is also satisfied.

Theorem: Let $y_n=y(x_n)$ and $l_n=l(x_n)$ denote two different numerical solutions of differential equation (1) with initial conditions specified as $y(x_0)=\mu$ *and* $l(x_0)=\mu*$ respectively, such that $|\mu-\mu*|<\varepsilon, \varepsilon>0$. If the two numerical estimates are generated by the integration scheme (28)

We have

$$y_{n+1}=y_n+h\emptyset(x_n,y_n,h)$$
$$l_{n+1}=l_n+h\emptyset(x_n,l_n,h)$$

The condition $|y_{n+1} - l_{n+1}| \leq k|\mu - \mu *|$ is the necessary and sufficient condition that the method/scheme is stable and convergent.

Proof: From (34)

$$y_{n+1} - y_n = \frac{6y_n^4 - 6y_n^4 + 6hy_n^3y_n' - 6h^2y_n^2(y_n')^2 + 3h^2y_n^3y_n'' - 6h^3y_n^2y_n'y_n'' + 6h^3y_n(y_n') + h^3y_n'''y_n^3}{6y_n^3 - 6hy_n^2y_n' + 6h^2y_n(y_n')^2 - 3h^2y_n^2y_n'' + 6h^3y_n'y_ny_n'' - 6h^3(y_n')^3 - h^3y_n'''y_n^2}$$

Adding y_n to both sides, we have that,

$$y_{n+1} = y_n + \frac{6y_n^4 - 6y_n^4 + 6hy_n^3y_n' - 6h^2y_n^2(y_n')^2 + 3h^2y_n^3y_n'' - 6h^3y_n^2y_n'y_n'' + 6h^3y_n(y_n') + h^3y_n'''y_n^3}{6y_n^3 - 6hy_n^2y_n' + 6h^2y_n(y_n')^2 - 3h^2y_n^2y_n'' + 6h^3y_n'y_ny_n'' - 6h^3(y_n')^3 - h^3y_n'''y_n^2}$$

$$y_{n+1} = \frac{6y_n^4 - 6h^3y_n^3y_n' + 6h^2y_n^2\left(y_n'\right)^2 - 3h^2y_n^3y_n'' + 6h^3y_n^2y_n'y_n'' - 6h^3y_n\left(y_n'\right) - h^3y_n'''y_n^3 + h[6y_n^3y_n' - 6hy_n^2\left(y_n'\right)^2 + 3hy_n^3y_n'' - 6h^2y_n^2y_n'y_n'' + 6h^2y_n\left(y_n'\right) + h^3y_n'''y_n^3}{6y_n^3 - 6hy_n^2y_n' + 6h^2y_n\left(y_n'\right)^2 - 3h^2y_n^2y_n'' + 6h^3y_n'y_ny_n'' - 6h^3\left(y_n'\right) - h^3y_n'''y_n^2}$$

$$= \frac{6y_n^4 - 6h^3y_n^3y_n' + 6h^2y_n^2(y_n')^2 - 3h^2y_n^3y_n'' + 6h^3y_n^2y_n'y_n'' - 6h^3y_n(y_n') - h^3y_n'''y_n^3 + h[6y_n^3y_n' - 6hy_n^2(y_n')^2 + 3hy_n^3y_n'' - 6h^2y_n^2y_n'y_n'' + 6h^2y_n(y_n') + h^3y_n'''y_n^3}{6y_n^3 - 6h^3y_n^2y_n' + 6h^2y_n(y_n')^2 - 3h^2y_n^2y_n'' + 6h^3y_n'y_ny_n'' - 6h^3(y_n') - h^3y_n'''y_n^2}$$

$$y_{n+1} = \frac{6y_n^4 - 6h^3y_n^3y_n' - 6h^3y_n(y_n') - h^3y_n'''y_n^3 + 6hy_n^3y_n' + 6h^3(y_n')^3}{6y_n^3 - 6h^3y_n^2y_n' + 6h^2y_n(y_n')^2 - 3h^2y_n^2y_n'' + 6h^3y_n'y_ny_n'' - 6h^3(y_n') - h^3y_n'''y_n^2}$$

Similarly, from (29)

$$l_{n+1} = l_n + h\emptyset(x_n, l_n, h)$$
$$l_{n+1} - l_n = h\emptyset(x_n, l_n, h)$$
$$l_{n+1} - l_n = \frac{6l_n^4 - 6l_n^4 + 6hl_n^3l_n' - 6h^2l_n^2(l_n')^2 + 3h^2l_n^3l_n'' - 6h^3l_n^2l_n'l_n'' + 6h^3l_n(l_n') + h^3l_n'''l_n^3}{6l_n^3 - 6hl_n^2l_n' + 6h^2l_n(l_n')^2 - 3h^2l_n^2l_n'' + 6h^3l_n'l_nl_n'' - 6h^3(l_n')^3 - h^3l_n'''l_n^2}$$

Adding l_n to both sides we have that,

$$l_{n+1} = \frac{l_n}{1} + \frac{h[6l_n^3l_n' - 6hl_n^2(l_n')^2 + 3hl_n^3l_n'' - 6h^2l_n^2l_n'l_n'' + 6h^2l_n(l_n') + h^3l_n'''l_n^3}{6l_n^3 - 6hl_n^2l_n' + 6h^2l_n(l_n')^2 - 3h^2l_n^2l_n'' + 6h^3l_n'l_nl_n'' - 6h^3(l_n') - h^3l_n'''l_n^2}$$

$$l_{n+1} = \frac{6l_n^4 - 6h^3l_n^3l_n' + 6h^2l_n^2\left(l_n'\right)^2 - 3h^2l_n^3l_n'' + 6h^3l_n^2l_n'l_n'' - 6h^3l_n\left(l_n'\right) - h^3l_n'''l_n^3 + h[6l_n^3l_n' - 6hl_n^2\left(l_n'\right)^2 + 3hl_n^3l_n'' - 6h^2l_n^2l_n'l_n'' + 6h^2l_n\left(l_n'\right) + h^3l_n'''l_n^3}{6l_n^3 - 6hl_n^2l_n' + 6h^2l_n\left(l_n'\right)^2 - 3h^2l_n^2l_n'' + 6h^3l_n'l_nl_n'' - 6h^3\left(l_n'\right) - h^3l_n'''l_n^2}$$

$$= \frac{6l_n^4 - 6h^3l_n^3l_n' + 6h^2l_n^2(l_n')^2 - 3h^2l_n^3l_n'' + 6h^3l_n^2l_n'l_n'' - 6h^3l_n(l_n') - h^3l_n'''l_n^3 + h[6l_n^3l_n' - 6hl_n^2(l_n')^2 + 3hl_n^3l_n'' - 6h^2l_n^2l_n'l_n'' + 6h^2l_n(l_n') + h^3l_n'''l_n^3}{6l_n^3 - 6hl_n^2l_n' + 6h^2l_n(l_n')^2 - 3h^2l_n^2l_n'' + 6h^3l_n'l_nl_n'' - 6h^3(l_n') - h^3l_n'''l_n^2}$$

$$l_{n+1} = \frac{6l_n^4 - 6h^3l_n^3l_n' - 6h^3l_n(l_n') - h^3l_n'''l_n^3 + 6hl_n^3l_n' + 6h^3(l_n')^3}{6l_n^3 - 6h^3l_n^2y_n' + 6h^2l_n(l_n')^2 - 3h^2l_n^2l_n'' + 6h^3l_n'l_nl_n'' - 6h^3(l_n') - h^3l_n'''l_n^2}$$

$$y_{n+1} - l_{n+1} = \frac{6y_n^4 - 6h^3y_n^3y_n' - 6h^3y_n(y_n') - h^3y_n'''y_n^3 + 6hy_n^3y_n' + 6h^3(y_n')^3}{6y_n^3 - 6h^3y_n^2y_n' + 6h^2y_n(y_n')^2 - 3h^2y_n^2y_n'' + 6h^3y_n'y_ny_n'' - 6h^3(y_n') - h^3y_n'''y_n^2} -$$

$$\frac{6l_n^4 - 6h^3l_n^3l_n' - 6h^3l_n(l_n') - h^3l_n'''l_n^3 + 6hl_n^3l_n' + 6h^3(l_n')^3}{6l_n^3 - 6h^3l_n^2y_n' + 6h^2l_n(l_n')^2 - 3h^2l_n^2l_n'' + 6h^3l_n'l_nl_n'' - 6h^3(l_n') - h^3l_n'''l_n^2}$$

As $h \to 0, we\ have$

$$y_{n+1} - l_{n+1} = \frac{6y_n^4}{6y_n^3} - \frac{6l_n^4}{6l_n^3}$$

$$|y_{n+1} - l_{n+1}| = |y_n - l_n|$$

$$|y_{n+1} - l_{n+1}| = 1|y_n - l_n|$$

But since the initial conditions are given as $y(x_0) = \mu\ and\ l(x_0) = \mu *$ respectively, such that $|\mu - \mu *| < \varepsilon$, $\varepsilon > 0$. Hence,

$$|y_{n+1} - l_{n+1}| = 1|\mu - \mu *| \quad (45)$$

D.Local Truncation Error and Order of the Scheme

We define the local truncation error for the Inverse polynomial method as the error committed in the most recent integration step, on a single integration step. The local truncation error in pth order of the inverse polynomial method is (ch^{p+1}) where c is some constant bound on c for p=2,3,4 also exist but since we only focus on the third step of the method, we can say this method is of third order and with local truncation error $O(h^4)$

V. Implementation, Results, Recommendation and Conclusion

A. Implementation and Numerical Results

It is always necessary to demonstrate the applicability and suitability of every newly proposed numerical method. Thus, to do this, the method was rewritten in an algorithm form and test runs were made with scheme for problems listed below using MATLAB programming language and it was implemented. However, the numerical solutions contained here are therefore compared with the corresponding theoretical solutions.

Example 1: We consider the IVP,

$$y' = 1 + y^2;\ y(0) = 1 \tag{46}$$

whose exact solution is $y(x) = \tan\left(x + \frac{\pi}{4}\right)$, in the interval $0 \le x \le 1$.

The numerical results are shown below in the table at h = *0.025.*

TABLE I
RESULTS GENERATED FROM THE NEW SCHEME

x_n	New Scheme	Exact Solution	Error
0.0000	1.0000	1.0000	0.0000
0.0250	1.0512	1.0520	0.0008
0.0500	1.1051	1.1061	0.0010
0.0750	1.1621	1.1632	0.0011
0.1000	1.2225	1.2238	0.0013
0.1250	1.2867	1.2883	0.0016
0.1500	1.3551	1.3570	0.0019
0.1750	1.4283	1.4305	0.0022
0.2000	1.5069	1.5095	0.0026
0.2250	1.5917	1.5947	0.0030

Example 2: We consider IVP called the test function

$$y' = y;\ y(0) = 1 \tag{47}$$

whose exact solution is given as: $y(x) = e^x$ in the interval $0 \le x \le 1$.

The numerical results is shown below in the table at h=0.25

TABLE II
COMPARATIVE RESULTS ANALYSIS

x_n	RK-1	RK-2	New scheme	$y(x_n)$
0.0000	1.0000	1.0000	1.0000	1.0000
0.2500	1.2500	1.2813	1.2843	1.2840
0.5000	1.5625	1.6416	1.6494	1.6487
0.7500	1.9531	2.1033	2.1183	2.1170
1.0000	2.4414	2.6949	2.7205	2.7183

Example 3: We consider the IVP

$$y' = \frac{y^2}{x^2+1};\ y(0)=1 \tag{48}$$

whose exact solution is given as:

$y(x) = -\frac{1}{(\tan^{-1} x) - 1}$ in the interval $0 \le x \le 1$ at h=*0.1*

TABLE III
RESULTS GENERATED FROM THE NEW SCHEME

x_n	New Scheme	Exact Solution	Error
0.0000	1.0000	1.0000	0.0000
0.1000	1.1103	1.1107	0.0004
0.2000	1.2443	1.2459	0.0016
0.3000	1.4070	1.4113	0.0043
0.4000	1.6047	1.6142	0.0095
0.5000	1.8458	1.8644	0.0186
0.6000	2.1414	2.1759	0.0345
0.7000	2.5071	2.5689	0.0618
0.8000	2.9652	3.0745	0.1093
0.9000	3.5487	3.7427	0.1940

B. Discussion of Results and Conclusion

We have successfully developed an inverse polynomial method for the numerical solution of first order ordinary differential equations. Analysis of the basic properties of the method showed that it is consistent, convergence and absolutely stable, confirming that the method is suitable for the numerical solution of non-stiff, stiff, and ordinary equation with singularites. Numerical results of the method compared favourably with the existing methods like the Modified Euler's Method and Runge-Kutta method of order 4. On a general note, the results show a measure of convergence towards the theoretical solution.

It is to be noted that the 1st and 2nd stage of polynomial method by [13] have derivative advantages over our new scheme in the sense that fewer derivative and also there is no much rigorous analysis of the Taylor's series method when compared with this new 3rd stage scheme. However, it enjoys higher order advantages over that of [13] first and second stage which are of order one and two respectively while this method is of order three.

References

[1] R. A Ademiluyi, "New Hybrid methods for system of ODE, Ph.D. Thesis, UNIBEN Nigeria, 1987.
[2] D. Barton, "On Taylor series and stiff equations", ACM transaction on mathematical software, 1980, 6, 280-294.
[3] D. Barton, I.M Willers and R.V.M Zahar "Taylor series method for ODE - an evaluation", Mathematical Software. New York: Academic Press, 1971b, 369-390.
[4] G. F. Corlis and Y. F. Chang, "Solving ODE using Taylor's series", ACM, Transactions on Mathematical Software, 1982, 114-144.
[5] G. Dahlquist, and A. Bjorck, Numerical methods, Englewood cliffs, New Jersey; Prentice Hall, 1974.
[6] S. O Fatunla, "A new algorithm for the Numerical solution of ODEs'', Computers and Mathematics with Applications, 1976, 2, 247-2531.
[7] S. O. Fatunla, "An Impact Two-Point Numerical Integration Formula for linear and nonlinear stiff systems of ODEs", Maths of Computation 32, 1978a.
[8] S. O. Fatunla, "A variable order one-step scheme for numerical solutions of ODEs," Computer and Mathematics with Application, 1978b, 4, 33-41.
[9] S. O. Fatunla, "Recent advances in stiff ODE solvers" UNIBEN, Nigeria, 1981e, 25-29.
[10] C. W. Gear, "Numerical IVPs in ODE", Englewood cliffs, New Jersey; Prentice Hall, 1971b.
[11] E. A. Ibijola and R. B. Ogunrinde, "On a new numerical scheme for the solution of IVPs, Australian Journal of Basic and Applied science, 2010.
[12] J. D. Lambert, "Computational methods in ODEs", New York, U.K, 1973a
[13] K. O. Okosun, "Kth order inverse polynomial methods for the integration of ordinary differential equations with singularities'', An M.

Tech in Industrial Mathematics and Computer Department of Federal University of Technology. Akure, Nigeria, 2003.

[14] C. S. Wallace and G. K. Gupta, (1973), "General linear multi step methods to solve ODEs", The Australian Computer Journal, 1973, 5, 62-69.